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Existence of a Cauchy Surface and Compactness of Causally Convex Hulls in a Spacetime *

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ABSTRACT: In this paper it is shown that, if a spacetime contains a Cauchy surface then the causally convex hulls of compact sets are compact. The converse is not true in general, however if the spacetime is causal then the compactness of causally convex hulls of compact sets of the spacetime ascertain the existence of a Cauchy surface. Also, the convex hull operator is defined and some results regarding it have been explored.

Key Words: Causality, globally hyperbolic spacetime, Cauchy surface, causally convex hull, time function.

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1. Introduction

Global hyperbolicity condition is the strongest and most reasonable causal condition in spacetime. The causality condition which occurs in the standard definition of globally hyperbolic spacetime is of great importance in the study of spacetime causation and hence in general relativity. There are many equivalent concepts and notions of global hyperbolicity among which the existence of Cauchy surface deserves a special attention of its own. In last few decades, scientists have been devoted to explore many aspects of it, including the improvements of its definition and finding its equivalent conditions. It is always reasonable and natural to design a definition of global hyperbolicity using minimum conditions and hypothesis. Many scientists are endeavouring to do so and as a consequence, some of its beautiful results have given the study of this subject a great leap. Although many facts regarding this are still to be explored to have a comprehensive knowledge about the geometry of the universe we live in. The present study is devoted to cast a little light thatward. In [2], Minguzzi reformulates three equivalent conditions for closed causal diamonds in a spacetime which actually lead to another important equivalent condition regarding the closedness of causally convex hulls in that series. Very recently in [1], Hounnonkpe and Minguzzi discussed what important role of causally convex hulls has to play with the causal simplicity and global hyperbolicity condition in a spacetime. They also showed that there is an inherent relationship between the compactness of causal diamonds and the Cauchy convex hulls of compact sets. Particularly this work is the main motivation for the present study and this is destined to investigate if there is any implicit correlation between the notion of Cauchy convex hulls of compact sets and the existence of Cauchy surfaces in a spacetime. Finally, the convex hull operator is defined in an arbitrary spacetime which preserves the union and intersections of any two subsets in a spacetime. Also it is shown that the causally convex hull operator shifts the epigraph [21] of a time function into its causal past. Convex hull operator is very useful to introduce many important mathematical ideas into causal theory of spacetime.

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2. Preliminary definitions and examples

In the Lorentzian causality theory, the definition of globally hyperbolic spacetime [3,6] is formulated in terms of causal curves which is used very occasionally in the study of general causality theory. Before getting into the subject, we may recall the basics of causality theory [7]. Any point p in a spacetime (M, g) is called an event. In any spacetime, the event x is called chronologically, (resp. strictly causally) related to another event y if there exists a future directed timelike (resp. causal) curve connecting x to y and is denoted by $x << y$(resp. $x < y$). If $x < y$ or $x = y$ then we say x is causally related to y and right $p \leq q$. We denote $I^+(p) = \{q \in M : p << q\}$ and $J^+(p) = \{q \in M : p \leq q\}$. $I^-(p), J^-(p)$ can be defined in the time-dual manner. The readers are reffered to have a glance over [7] for more details on causality and different aspects of spacetime geometry used in this article.

• **Globally hyperbolic spacetimes:** A strongly causal spacetime having all the causal diamonds compact is called globally hyperbolic spacetime, [1,3,5,20].

• **Achronal sets:** A subset S of M is said to be achronal if there do not exist $p, q \in S$ such that $q \in I^+(p)$, i.e., achronal sets do not crossed more than once by any timelike curve, [7,18].

• **Cauchy surfaces:** A Cauchy surface in a spacetime (M, g) is a subset S that met exactly once by every inextendible timelike curve in M [4]. In fact, in a globally hyperbolic spacetime, every compact, connected, locally acusal hypersurface S is a Cauchy surface, [2].

• **Domains of dependence (Cauchy development):** The future domain of dependence (resp. past domain of dependence) of an achronal set S in a spacetime (M, g) is denoted by $D^+(S)$(resp. $D^-(S)$) and defined by: $D^+(S) = \{s \in M$:every past directed inextendible casual curve from s intersectsS$\}$ (resp.) $D^-(S) = \{s \in M$:every future directed inextendible casual curve from s intersectsS$\}$. Now the domain of dependence of S is $D(S) = D^+(S) \cup D^-(S)$, in [19]. Domain of dependence, in different literature has also been defined using timelike curve.
Cauchy surface in a spacetime is sometimes defined as an achronal set S for which $D(S) = M$, [18].

• **Causally convex hulls:** The causally convex hull of a set $S \subset M$ in a spacetime (M, g) is the set $J^+(S) \cap J^-(S)$, where $J^{\pm}(S) = \bigcup_{x \in S} J^{\pm}(x)$. Causally convex hull of a set $S \subset M$ is actually collection of those causal curves which are originated and limited themselves in S.

3. The main result

In the following, we state and prove a theorem for the sake of completeness, which is already established by [1].

Theorem 3.1. *In a spacetime (M, g) the following two conditions are equivalent:*

1. *Causal diamonds are compact,*
2. *Causally convex hulls of compact sets are compact.*

Proof. Let the causally convex hulls of compact sets are compact in (M, g). To show $J^+(p) \cap J^-(q)$ is compact for any points $p, q \in M$. Now if $q \leq p$ then $J^+(q) \cap J^-(p) = J^+(K) \cap J^-(K)$ where $K = \{p, q\}$. Now the set $J^+(K) \cap J^-(K)$, being a casually convex hull of compact set K, is compact. Hence the causal diamonds are compact in (M, g).

Now conversely, let the causal diamonds are compact in (M, g). We are to show that $J^+(K) \cap J^-(K)$ is compact for an arbitrary compact set K in (M, g). Now let K and $\hat{K}$ be two compact sets such that $K \subseteq int\hat{K}$. So for each $p \in K$ we can have $q, r \in \hat{K}$ such that $p \in I^+(p) \cap I^-(r)$ so we can find a finite covering of K, which can explicitely be of the form $I^+(q_i) \cap I^-(r_i)$ for $i = 1, 2, \ldots, n$. Then $J^+(K) \cap J^-(K) \subset \bigcup_{i,j} J^+(q_i) \cap J^-(r_i)$, which is again compact. The set $J^+(K) \cap J^-(K)$, being a closed subset of a compact set in the spacetime (M, g) is compact.

□

Remark 3.1. Causally convex hull operator respects the compactness in a globally hyperbolic spacetime.

We again describe a theorem which is stated in [1].

Theorem 3.2. *If a spacetime (M, g) is globally hyperbolic, then the following are equivalent:*

1. *(M, g) is causal and all causal diamonds $\{J^+(p) \cap J^-(q), \forall p, q \in M\}$ is compact,*
2. *(M, g) is non-totally imprisoning and $\overline{J^+(p) \cap J^-(q)}, \forall p, q \in M$ is compact,*
3. *(M, g) is stably causal and $J_S^+(p) \cap J_S^-(q)$ is compact,*
4. *(M, g) contains a Cauchy surface.*

Each equivalent condition mentioned above has their own importance over the others. As the first condition helps to find the position of the global hyperbolicity condition in the causal hierarchy [5,7,9,17]. Second condition asserts that sufficiently small perturbations on the causal cones does not affect the global hyperbolicity [10,11]. Third condition explores a beautiful fact that globally hyperbolicity is intrinsically related to the Seifert relation [10]. The fourth condition provide us a way to decompose a spacetime topologically as $M = S \times \mathbb{R}$, S being the Cauchy surface, for more details see [3,12,13,14,15,16].

We now prove the following theorem which is closely related to the Proposition 2.5 of [1], but we have reshaped it and proved in a different way.

Theorem 3.3. *Existence of Cauchy surface in a spacetime (M, g) ensures the compactness of causally convex hulls of compact sets there.*

Proof. Let (M, g) be a spacetime having a Cauchy surface S, so M does not contain any closed causal curves as S is achronal. Let $p, q \in M$ and $\{\Gamma_i\}$ be a sequence of past directed causal curves starting from p to q. Here two cases may arise. Firstly, if p and q both are in future domain of dependence $D^+(S)$ of S then every Γ_i can be enlarged to intersect S. Now S being a Cauchy surface, ensures that the collection of all the causal curves having past endpoint in S and future endpoint at p is compact. Therefore, the extended new sequence of the previous sequence $\{\Gamma_i\}$ has a limit curve Γ. As every Γ_i contains q, so does Γ and the portion of Γ between p and q is a limit curve of $\{\Gamma_i\}$. Secondly, if $p \in D^+(S)$ and $q \in I^-(S)$, then each of Γ_i must intersects S. Similarly as previous case, the curves $\Gamma_i \cap D^+(S)$ have again a limit curve Γ, having past endpoint s, say, which belong to S. Choosing a subsequence $\{\Gamma_i^s\}$ of $\{\Gamma_i\}$ such that $\{\Gamma_i^s\} \cap S$ approaches to s, then the curves $\Gamma_i \cap D^-(S)$ have a limit curve Γ'. So the curves Γ and Γ' have past endpoint and future endpoint respectively. So $\Gamma \cup \Gamma'$ is a limit curve $\{\Gamma_i\}$. Thus for every p and q, the set of all future directed causal curves from p to q, is compact. So (M, g) is globally hyperbolic spacetime. Thus (M, g) is causal spacetime having compact causal diamonds. Now let K and $\hat{K}$ be two compact sets in (M, g) such that $K \subset int\hat{K}$. So for each $p \in K$ we can have $q, r \in \hat{K}$ such that $p \in I^+(p) \cap I^-(r)$. Thus we can find a finite covering of K, which can explicitely be of the form $I^+(q_i) \cap I^-(r_i)$ for $i = 1, 2, \ldots n$. Then $J^+(K) \cap J^-(K) \subset \{J^+(q_i) \cap J^-(r_j)\}$, which is again compact. As $J^+(K) \cap J^-(K)$ is a closed subset of a compact set in (M, g), so $J^+(K) \cap J^-(K)$ is compact for arbitrary compact subset K. ☐

Remark 3.2. The converse of this theorem is not true in general. However if a spacetime is causal and every causally convex hulls respects the compactness, then the spacetime turns to become globally hyperbolic, as a result it does contain a Cauchy surface.

Theorem 3.4. *If S be a Cauchy surface in a spacetime (M, g), then the domain of dependence of S will be the whole spacetime, [18,19].*

Corollary 3.5. *If a spacetime (M, g) be causal and causally convex hulls of compact sets be compact then there exists a set S in M such that $D(S) = M$.*

4. Convex hull operator

Let (M, g) be a spacetime. Let us defined convex hull operator in the spacetime which converts every subset S of M into a causally convex hull as $\phi : \mathcal{P}(M) \to \mathcal{P}(M)$ by $\phi(S) = J^+(S) \cap J^-(S)$.

Theorem 4.1. *If S be a Cauchy surface in a spacetime (M, g), then $\phi(S)$ covers the whole spacetime.*

Proof. By the definition of convex hull operator we have,

$$\phi(S) = J^+(S) \cap J^-(S) = M \cap M = M,$$

as S is a Cauchy surface. □

Theorem 4.2. *Let S_1 and S_2 be any two subsets of a spacetime (M, g). Then*
(i) $\phi(S_1 \cup S_2) = \phi(S_1) \cup \phi(S_2)$,
(ii) $\phi(S_1 \cap S_2) = \phi(S_1) \cap \phi(S_2)$.

Proof. (i) Here, we show that $J^+(S_1 \cup S_2) \cap J^-(S_1 \cup S_2) = (J^+(S_1) \cap J^-(S_1)) \cup (J^+(S_2) \cap J^-(S_2))$.
Let, $p \in J^+(S_1 \cup S_2) \cap J^-(S_1 \cup S_2)$ be an event in the spacetime (M, g).
$\Rightarrow p \in J^+(S_1 \cup S_2)\,\text{and}\, p \in J^-(S_1 \cup S_2)$
$\Rightarrow$ (p belongs to the causal future of S_1 or p belongs to the causal future of S_2) and (p belongs to the causal past of S_1 or p belongs to the causal past of S_2)
$\Rightarrow$ (p belongs to the causal future of S_1 and p belongs to the causal past of S_1) or (p belongs to the causal future of S_2 and p belongs to the causal past of S_2)
$\Rightarrow p \in J^+(S_1) \cap J^-(S_1)$ or $p \in J^+(S_2) \cap J^-(S_2)$
$\Rightarrow p \in (J^+(S_1) \cap J^-(S_1)) \cup (J^+(S_2) \cap J^-(S_2))$.
So, $J^+(S_1 \cup S_2) \cap J^-(S_1 \cup S_2) \subset (J^+(S_1) \cap J^-(S_1)) \cup (J^+(S_2) \cap J^-(S_2))$. Similarly, the reverse direction of the inclusion relation can be shown.

(ii) This result can be proved analogously.

□

• **Epigraph of a time function:** Let (M, g) be a globally hyperbolic spacetime and Σ be a Cauchy surface in (M, g). Let $f : \Sigma \to \mathbb{R}$ be a time function then the epigraph of a time function is defined as $epif = \{(u, t) \in \Sigma \times \mathbb{R} : f(u) \leq t\}$.

Theorem 4.3. *The convex hull operator ϕ shifts the epigraph of a time function f into its causal past.*

Proof. As the time function has Cauchy surface as its domain, therefore $J^+(epif) = M$.

$$\begin{aligned} \phi(epif) &= J^+(epif) \cap J^-(epif) \\ &= M \cap J^-(epif) \\ &= J^-(epif) \end{aligned}$$

□

5. Conclusion

Here we have shown one of the very useful consequence of the existence of Cauchy surface in a spacetime. The investigation of the existence of Cauchy surface in a spacetime is of special importance since its discovery, as there are so many practical aspects of it. Also at the same time, the compactness of causally convex hulls has a beautiful role to play to make a spacetime globally hyperbolic, the most realistic, useful and common kind of spacetime used in different branches of Mathematics and Modern Physics. We have demonstrated that the existence of Cauchy surface affirms the compactness of all the causally convex hulls of compact sets. The spacetimes other than globally hyperbolic spacetimes can also be defined and presented through the idea of causally convex hulls which would be very useful tool to study the causality of spacetimes. Lastly, introducing the convex hull operator, we have studied couple of results regarding set union-intersection and epigraph of a time function which may lead to engrain some other important and useful mathematical ideas into causal theory of spacetime.

Conflict of interest. The authors have no conflict of interest.

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