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On Some q-Bessel Type Continuous Wavelet Transform

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ABSTRACT: In this paper we continue to exploit the modified variants of Bessel function in the framework of q-theory to construct wavelet operators. A generalized q-Bessel type function is introduced leading to an associated mother wavelet which in turns induced a continuous wavelet transform. Finally, Plancherel/Parceval type relations are proved. Such variant of wavelets permits to approximate solutions of PDEs by transforming them to recurrent sequences. Numerical examples are provided with graphical illustrations and error estimates to confirm these theoretical findings.

Key Words: Wavelets, Bessel function, q-Bessel function, modified Bessel functions, generalized q-Bessel functions, q-Bessel wavelets.

Contents

1. Introduction

The purpose of this paper is to propose decomposition and reconstruction formulas of a new variant of the wavelet transform. The framework is Bessel wavelet theory, where the usual translations used in wavelets are replaced by an integral operator based on Bessel functions. The other addendum is the setting of q-theory, where the Lebesgue measure is replaced by sums of Dirac masses set on a geometric sequence.

Wavelet theory has known a great success since its appearance in the mid-eighties of the last century when it has been introduced in the context of signal analysis and exploration of petroleum. Next, wavelets have been applied to analyse different signals such as seismic ones which are more sensitive than Fourier techniques. Therefore, the first appearance of the wavelet transform formula has been pointed out. Wavelet theory has become an active area of research is many fields, including electrical engineering (signal processing and data compression), mathematical analysis (harmonic analysis, operator theory), physics (fractals, quantum theory), ... etc. Wavelets have attracted much attention recently in signal processing problems especially non-stationary signals, and the detection of discontinuities or irregularities.

Like Fourier analysis, wavelets deal with the decomposition of functions in terms of a functional basis. Unlike Fourier analysis, wavelet analysis expands functions in terms of similar wavelets which are generated in the form of translations and dilations of a fixed function called the mother wavelet characterized by special properties. Wavelet theory provides for functional spaces good bases allowing their decomposition into spices associated to different horizons known as the levels of decomposition.

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This makes the finding of mother wavelet of great interest. One aim of the present paper is to construct indeed a mother wavelet in the framework of q-theory.

The theory or the concept of q-calculus is based on the concept of scaling, and permits among many properties to serve of the scale q to conduct an analysis on the subset $\mathbb{R}_q$. In statistical applications for example, especially in financial time series, the time scale is widely know. For example, in wavelet estimation of the systematic risk, the scaling process is sued twice and simultaneously in the distribution of the time intervals, and in the wavelet basis elements. In time intervals, the whole period is subdivided into sub-period according to some scale-based process for which each period has a length $L_j = q^{-j}(1-q)$. For example, for $q = 0.5$, we get the dyadic scale-based periods with corresponding lengths $l_j = 2^{j+1} - 2^j$. This process has been applied in many studies such as [2,7,33,34,37]. Besides, in [35], the concept of scaling combined with wavelets is used to measure the eventual dependence of financial time series issued from many stock markets. In [36], the authors proposed a multiscaled Neural Autoregressive Distributed Lag to conduct an empirical mode for the decomposition of time series into a scale-by-scale modes for forecasting. It is claimed that the scale-by-scale modes' features allow the model to capture the eventual nonlinear patterns hidden in the data.

The first step in wavelet analysis of functions is the so-called wavelet transform. There are in fact two types of transforms. The original one is known as the continuous wavelet transform (CWT) introduced originally in the works of Grossmann and Morlet in the beginning of the 1980's. The second is known as the discrete wavelet transform (DWT) which is a refinement copy of the CWT obtained by restricting the translation and dilation parameters on discrete grids.

Like Fourier analysis, wavelet analysis has been interconnected with many concepts in both theoretical and applied types such as windowed transforms, stationary transforms, Hankel transform or Fourier-Bessel. For example, Hankel transform called also Bessel-Fourier transform of a function $\varphi \in L^1(\mathbb{R}_+)$ is defined by

$$\widehat{\varphi}_\mu(y) = \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy)\varphi(x)dx, \quad x \in \mathbb{R}_+,\, \mu \geq -\frac{1}{2}.$$

It is invertible in the sense that, the analyzed function φ may be reconstructed as

$$\varphi(x) = \int_0^\infty (xy)^{\frac{1}{2}} J_\mu(xy)\widehat{\varphi}_\mu(y)dy, \quad y \in \mathbb{R}.$$

Here J_μ is the Bessel function of first kind and with order μ. It is also extended to the framework of fractional calculus to yield the fractional Hankel transform called also fractional Bessel-Fourier transform of parameter θ on $L^1(\mathbb{R}^+)$, defined by

$$\widehat{\varphi}^\theta_\mu(y) = \int_0^\infty K^\theta_\mu(x,y)\varphi(x)dx,$$

where K^θ_μ is the kernel

$$K^\theta_\mu(x,y) = \begin{cases} c^\theta_\mu e^{\frac{i}{2}(x^2+y^2)\cot\theta}(xy\cos\theta)^{\frac{1}{2}} J_\mu(xy\cos\theta), & \theta \neq n\pi, \\ (xy)^{\frac{1}{2}} J_\mu(xy), & \theta = \frac{\pi}{2}, \\ \delta(x-y), & \theta = n\pi, \forall n \in \mathbb{Z}. \end{cases}$$

Here, δ is the Dirac mass defined by

$$\delta(x-y) = \begin{cases} 1 & \text{si} \quad x = y, \\ 0 & \text{else,} \end{cases}$$

and

$$c^\theta_\mu = \frac{exp\left[i(1+\mu)(\frac{\pi}{2}-\theta)\right]}{\sin\theta}.$$

Here also, in wavelet theory, researchers have extended and thus interconnected wavelet transforms with similar concepts which in fact make a natural extension of Fourier analysis. We may refer to [24,25,26,27].

In the present work, our aim is to continue to develop an interconnection concept between wavelet analysis and Bessel one.

A first step in wavelet/Bessel transform interconnection is due to [16], where the authors generalized the theory of continuous wavelet transform to a class of generalized continuous wavelet transform associated with a class of singular differential operator. This class contains, in particular, the so called Bessel function (See also [28,38]).

It is naturally in fact to extend Fourier and wavelet analyses to the context of Bessel one. Indeed, Bessel functions form an important class of special functions applied almost everywhere in mathematical physics. They are known as cylindrical functions, or cylindrical harmonics, because of their strong link to the solutions of the Laplace equation in cylindrical coordinates. The history of these functions is traced back to Bernoulli, Euler and Poisson, and still attracts the interest of researchers in both theory and applications. Bessel functions are associated most commonly with the partial differential equations of the potential, wave motion, or diffusion, in cylindrical or spherical coordinates.

One of the interesting fields of extensions of these analyses among other ones such as Hankel and Dunkel transforms is the so-called q-theory which is an important sub-field in harmonic analysis and which provides some discrete and/or some refinement of continuous harmonic analysis in sub-spaces such as $\mathbb{R}_q$ composed of the discrete grid $\pm q^n$, $n \in \mathbb{Z}$, $q \in (0,1)$. Recall that for all $x \in \mathbb{R}^*$ there exists a unique $n \in \mathbb{Z}$ such that $q^{n+1} < |x| \leq q^n$ which guarantees some density of the set $\mathbb{R}_q$ in $\mathbb{R}$.

Koornwinder and Swarttouw studied the Jackson's third q-Bessel function claiming that it can be used to build a reliable harmonic analysis on $\mathbb{R}_q$, thus motivating several works on q-harmonic analysis associated to different q-differential operators. Motivations may also be related to the application of q-theory and/or q-wavelets in developing solutions of PDEs. Consider for example an elliptic equation

$$\Delta u + f(u,x) = 0,$$

where f is a suitable function, generally nonlinear in u. We may search for a numerical q-approximation by considering a grid points in $\mathbb{R}_q$ instead of finite difference/finite elements used usually. Indeed, the Laplace operator will be replaced by the q-analog

$$\Delta_q u(x) = \frac{u(q^{-1}x) - (1+q)u(x) + qu(qx)}{x^2}.$$

For $x = q^n$ in $\mathbb{R}_q^+$, we get

$$u_{n+1} - (1+q)u_n + qu_{n-1} = -(1-q)^2 q q^{2n} f_n,$$

where $u_n = u(q^n)$ and f_n is some discretization of $f(u,x)$. We thus obtain a recursive equation permitting to compute u_n recursively. More about applications of q-calculus in partial differential equations may be found in [4].

Many special functions have been shown to admit generalizations to a base q, and are usually reported as q-special functions. Interest in such q-functions is motivated by the recent and increasing relevance of q-analysis in exactly solvable models in statistical mechanics. Like ordinary special functions, q-analogues satisfy second order q-differential equations and various identities of recurrence relations. Basic analogues of Bessel function have been introduced by Jackson and Swarthouw as q-generalizations of the power series expansions.

In the present context, our aim is to develop new wavelet functions based on some special functions such as Bessel one. q (quantum)-version of the Bessel Wavelet Transform known also Weinstein Wavelet Transform associated to Bessel differential operator is introduced and Calderon-type reproducing formula is established. We aim precisely to apply the generalized q-Bessel function introduced in the context of q-theory and which makes a general variant of Bessel, Bessel modified and q-Bessel functions.

We serve of a new modified Bessel function in q-theory ([22]) to develop a 'new' q-wavelet analysis. We show in a first step an eigenvalue equation for such a function and next apply it to deduce oscillating and/or orthogonality properties of the mother wavelet constructed. The present work joins a series of works on q-analogues of the Bessel functions introduced early by Jackson [20] and which still make interesting subjects of investigations in mathematical analysis and quantum physics especially.

The present paper will be organized as follows. In section 2, a brief review focusing on the last developments of q-Bessel wavelets is developed. In section 3, we present our main results. We precisely introduce the generalized q-Bessel type function and some basic properties such as the eigenvalue equation and the orthogonality one. Next, we present a dilation-invariance relation. The generalized q-Bessel type function is then applied to define an associated mother wavelet which in turns leads to a continuous wavelet transform. Finally, Fourier-Plancherel as well as Parseval type relations are presented. Section 4 is devoted to the proofs of the main results.

2. q-Bessel wavelets brief review

In the literature, we may refer to different approaches to introduce Bessel wavelets. To avoid non-availability of references for readers, in the present work, we essentially refer to the most recent ones ([30] and [31]). Recall that Bessel wavelets are special wavelets introduced by replacing sine/cosine/kernels/wavelet in classical analysis by variants of Bessel function. Historically, special functions differ from elementary ones such as powers, roots, trigonometric, and their inverses mainly with the limitations that these latter classes have known. Many fundamental problems such that orbital motion, simultaneous oscillatory chains, spherical bodies gravitational potential were not well described using elementary functions. This makes it necessary to extend elementary functions' classes to more general ones that may describe well unresolved problems. The present section aims to recall some basic facts about Bessel wavelets. For $1 \leq p < \infty$ and $\mu > 0$, denote

$$L^p_\sigma(\mathbb{R}_+) := \left\{ f \;\text{ such as }\; \|f\|^p_{p,\sigma} = \int_0^\infty |f(x)|^p \, d\sigma(x) < \infty \right\},$$

where $d\sigma(x)$ is the measure defined by

$$d\sigma(x) = \frac{x^{2\mu}}{2^{\mu-\frac{1}{2}}\,\Gamma(\mu+\frac{1}{2})} \chi_{[0,\infty)}(x)\, dx.$$

Denote also

$$j_\mu(x) = 2^{\mu-\frac{1}{2}}\,\Gamma(\mu+\frac{1}{2})\, x^{\frac{1}{2}-\mu}\, J_{\mu-\frac{1}{2}}(x),$$

where $J_{\mu-\frac{1}{2}}(x)$ is the Bessel function of order $v = \mu - \frac{1}{2}$ given by

$$J_v(x) = (\frac{x}{2})^v \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+v+1)} (\frac{x}{2})^{2k}.$$

Formally, a Bessel wavelet is a function $\Psi \in L^2_\sigma(\mathbb{R}_+)$ satisfying the so-called admissibility condition

$$C_\psi = \int_0^\infty \frac{|\widehat{\psi}(\xi)|^2}{\xi} d\xi < \infty.$$

Denote next

$$D(x,y,z) = \int_0^\infty j_\mu(xt)\, j_\mu(yt)\, j_\mu(zt)\, d\sigma(t).$$

For a 1-variable function f, we define a translation operator

$$\tau_x f(y) = \widetilde{f}(x,y) = \int_0^\infty D(x,y,z)\, f(x)\, d\sigma(z), \;\; \forall\, 0 < x,y < \infty.$$

and for a 2-variables function f, we define a dilation operator

$$D_a f(x,y) = a^{-2\mu-1}\, f(\frac{x}{a}, \frac{y}{a}).$$

Recall that

$$\int_0^\infty j_\mu(zt)\, D(x,y,z)\, d\sigma(z) = j_\mu(xt)\, j_\mu(yt), \quad \forall\, 0 < x,y < \infty,\; 0 \leq t < \infty,$$

and

$$\int_0^\infty D(x,y,z)\,d\sigma(z)=1.$$

(See [30]). The Bessel wavelet copies $\Psi_{a,b}$ are defined from the Bessel mother wavelet $\Psi\in L^2_\sigma(\mathbb{R}_+)$ by

$$\Psi_{a,b}(x)=D_a\tau_b\Psi(x)=a^{-2\mu-1}\int_0^\infty D(\frac{b}{a},\frac{x}{a},z)\,\Psi(z)\,d\sigma(x),\ \ \forall a,b\geq 0. \tag{2.1}$$

As in the classical wavelet theory on $\mathbb{R}$, we define here-also the continuous Bessel Wavelet transform (CBWT) of a function $f\in L^2_\sigma(\mathbb{R}_+)$, at the scale a and the position b by

$$(B_\Psi f)(a,b)=a^{-2\mu-1}\int_0^\infty\int_0^\infty f(t)\,\overline{\Psi}(z)\,D(\frac{b}{a},\frac{t}{a},z)\,d\sigma(z)\,d\sigma(t). \tag{2.2}$$

In Bessel wavelet theory, it is already shown that such a transform is a continuous function according to the variable (a,b). Furthermore a reconstruction formula of Parceval/Plancherel type is known claiming that for any elements $f,g\in L^2_\sigma(\mathbb{R}_+)$, there holds that

$$a^{-2\mu-1}\int_0^\infty\int_0^\infty (B_\Psi f)(b,a)\,(\overline{B_\Psi g})(b,a)\,d\sigma(a)\,d\sigma(b)=C_\Psi\,\langle f,g\rangle, \tag{2.3}$$

At the beginning of the twentieth century Jackson introduced the theory of q-analysis by defining the notions of q-derivative and q-integral and giving q-analogues of some special functions such as Bessel's one. By virtue of their utilities, special functions and q-special functions continue to be fascinating subjects in both theoretical and applied researches till now. This is particularly the case for q-Bessel functions, which represent one of the most important examples of q-special functions. Backgrounds on q-theory and q-wavelets may be found in [10], [12], [14], [15] and the references therein.

We begin by introducing the context of q-theory. For $0<q<1$, denote

$$\mathbb{R}_q=\{\pm q^n,\ n\in\mathbb{Z}\},\ \widetilde{\mathbb{R}}_q=\mathbb{R}_q\cup\{0\},\ \mathbb{R}_q^+=\{q^n,\ n\in\mathbb{Z}\}\text{ and }\widetilde{\mathbb{R}}_q^+=\mathbb{R}_q^+\cup\{0\}.$$

On $\widetilde{\mathbb{R}}_q^+$, the q-Jackson integrals on $[a,b]$ ($a\leq b$ in $\widetilde{\mathbb{R}}_q^+$) is defined by

$$\int_a^b f(x)d_qx=(1-q)\sum_{n\in\mathbb{Z}}q^n(bf(bq^n)-af(aq^n)),$$

and for $[0,+\infty)$ it is defined by

$$\int_0^\infty f(x)d_qx=(1-q)\sum_{n\in\mathbb{Z}}f(q^n)\,q^n,$$

provided that the sums converge absolutely. On $[a,+\infty)$, we have to apply an analogue of Chasle's rule and thus get

$$\int_a^{+\infty}f(x)d_qx=\int_0^{+\infty}f(x)d_qx-\int_0^a f(x)d_qx.$$

(See [8], [15]). This leads next to define the functional space

$$\mathcal{L}_{q,p,v}(\widetilde{\mathbb{R}}_q)=\{f:\widetilde{\mathbb{R}}_q\to\mathbb{C},\ \text{even}\ ;\ \ \|f\|_{q,p,v}<\infty\},$$

where $\|.\|_{q,p,v}$ is the norm defined analogously by

$$\|f\|_{q,p,v}=\left[\int_0^\infty |f(x)|^p\,x^{2v+1}\,d_qx\right]^{\frac{1}{p}}.$$

where v is a fixed real parameter such that $2v > -1$. Denote next, $C_q^0(\widetilde{\mathbb{R}}_q^+)$ the space of functions defined on $\widetilde{\mathbb{R}}_q^+$, continuous at 0 and vanishing at $+\infty$, equipped with uniform norm

$$\|f\|_{q,\infty} = \sup_{x\in\widetilde{\mathbb{R}}_q^+} |f(x)| < \infty.$$

Finally, $C_q^b(\widetilde{\mathbb{R}}_q^+)$ designates the space of functions that are continuous at 0 and bounded on $\widetilde{\mathbb{R}}_q^+$. Recall here that $q^n \to \infty$ as $n \to -\infty$ in $\mathbb{Z}$. The q-derivative of a function $f \in \mathcal{L}_{q,p,v}(\widetilde{\mathbb{R}}_q^+)$ is defined by

$$D_q f(x) = \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)x}, & x \neq 0, \\ f'(0)\ , & else. \end{cases}$$

The q-derivative of a function is a linear operator. However for the product of functions we have a different form,

$$D_q\left(fg\right)(x) = f(qx)D_q g(x) + D_q f(x) g(x),$$

and whenever $g(x) \neq 0$ and $g(qx) \neq 0$, we have

$$D_q\left(\frac{f}{g}\right)(x) = \frac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(qx)g(x)}.$$

In q-theory, we may speak with an analogue of the integration by parts rule ([5]),

$$\int_a^b g(x)\, D_q f(x)\, d_q x = [f(b)g(b) - f(a)g(a)] - \int_a^b f(qx)\, D_q g(x)\, d_q x.$$

Besides, we may also have an analogue of the change variables theorem. Let $u(x) = \gamma x^s$, for γ, $s > 0$ fixed. We have

$$\int_0^\infty f(t)\, d_q t = \int_0^\infty f(u(x)) D_{q^{1/s}} u(x)\, d_{q^{1/s}} x.$$

Particularly,

$$\int_0^\infty f(t) t^{2v+1}\, d_q t = \frac{1}{a^{2v+2}} \int_0^\infty f(\frac{x}{a}) x^{2v+1}\, d_q x. \tag{2.4}$$

The q-Bessel operator is defined for $f \in \mathcal{L}_{q,p,v}(\widetilde{\mathbb{R}}_q^+)$ by

$$\Delta_{q,\,v} f(x) = \frac{f(q^{-1}x) - (1+q^{2v}) f(x) + q^{2v} f(qx)}{x^2}, \quad \forall x \neq 0. \tag{2.5}$$

The following relations are easy to show. The first is an analogue of Stokes rule and states that for $f, g \in \mathcal{L}_{q,2,v}(\widetilde{\mathbb{R}}_q^+)$ such that $\Delta_{q,v} f, \Delta_{q,v} g \in \mathcal{L}_{q,2,v}(\widetilde{\mathbb{R}}_q^+)$, we have

$$\int_0^\infty \Delta_{q,v} f(x)\, g(x)\, x^{2v+1}\, d_q x = \int_0^\infty f(x)\, \Delta_{q,v} g(x)\, x^{2v+1}\, d_q x. \tag{2.6}$$

The q-shifted factorials are defined by

$$(a,q)_0 = 1, \quad (a,q)_n = \prod_{k=0}^{n-1}(1 - aq^k), \quad (a,q)_\infty = \prod_{k=0}^{+\infty}(1 - aq^k).$$

The normalized q-Bessel function is introduced in [11] as

$$j_\alpha(x, q^2) = \sum_{n\geq 0} (-1)^n \frac{q^{n(n+1)}}{(q^{2\alpha+2}, q^2)_n\, (q^2, q^2)_n}\, x^{2n}, \tag{2.7}$$

The q-Bessel operator is related to the normalized q-Bessel function by the eigenvalue equation

$$\Delta_{q,\,\alpha} j_\alpha(\lambda x, q^2) = -\lambda^2 j_\alpha(x, q^2).$$

More precisely, $j_\alpha(x, q^2)$ is the unique solution of the Laplace eigenvalue problem for $\lambda \in \mathbb{C}$,

$$\begin{cases} \Delta_{q,\,\alpha} u(\lambda x) = -\lambda^2\, u(x), \\ u(0) = 1,\ u'(0) = 0. \end{cases}$$

As in the classical analysis, we have here also an orthogonality relation for the normalized q-Bessel function ([1]) as

$$\int_0^\infty j_\alpha(xt, q^2)\, j_\alpha(yt, q^2)\, t^{2\alpha+1}\, d_q t = \frac{1}{c_{q,\alpha}^2}\, \delta_{q,\alpha}(x, y),$$

where

$$c_{q,\alpha} = \frac{1}{1-q} \frac{(q^{2\alpha+2}, q^2)_\infty}{(q^2, q^2)_\infty}, \tag{2.8}$$

and

$$\delta_{q,\alpha}(x, y) = \begin{cases} \dfrac{1}{(1-q)\, x^{2(\alpha+1)}}, & if\ x = y, \\ 0 & else. \end{cases}$$

We now recall the q-Bessel Fourier transform $\mathcal{F}_{q,\alpha}$ already defined in ([15]) as

$$\mathcal{F}_{q,\alpha} f(x) = c_{q,\alpha} \int_0^\infty f(t)\, j_\alpha(xt, q^2)\, t^{2\alpha+1}\, d_q t, \tag{2.9}$$

where $c_{q,\alpha}$ is as above. Define next the q-Bessel translation operator by

$$T^\alpha_{q,x} f(y) = c_{q,\alpha} \int_0^\infty \mathcal{F}_{q,\alpha} f(t)\, j_\alpha(xt, q^2)\, j_\alpha(yt, q^2)\, t^{2\alpha+1}\, d_q t. \tag{2.10}$$

Such a translation operator satisfies for all $f \in \mathcal{L}_{q,2,\alpha}(\widetilde{\mathbb{R}}_q^+)$ a Fourier-Bessel invariance property ([1])

$$\mathcal{F}_{q,\alpha}(T^\alpha_{q,x} f)(\lambda) = j_\alpha(\lambda x, q^2)\, \mathcal{F}_{q,\alpha} f(\lambda), \ \ \forall \lambda, x \in \widetilde{\mathbb{R}}_q^+. \tag{2.11}$$

It satisfies also for $f \in \mathcal{L}_{q,2,\alpha}(\widetilde{\mathbb{R}}_q^+)$,

$$T^\alpha_{q,x} f(y) = T^\alpha_{q,y} f(x) \ \text{ and } \ T^\alpha_{q,x} f(0) = f(x), \tag{2.12}$$

and

$$\mathcal{T}^\alpha_{q,x} j_\alpha(ty, q^2) = j_\alpha(tx, q^2)\, j_\alpha(ty, q^2),\ \forall t, x, y \in \widetilde{\mathbb{R}}_q^+.$$

The q-Bessel wavelets have been introduced in [15]. Let $\Psi \in \mathcal{L}_{q,2,\alpha}(\widetilde{\mathbb{R}_q})$ be an even function satisfying an admissibility condition

$$\mathcal{A}_{\alpha,\Psi} = \int_0^\infty |\mathcal{F}_{q,\alpha}\Psi(a)|^2\, \frac{d_q a}{a} < \infty.$$

Then, Ψ is said to be a q-Bessel wavelet. An associated continuous q-Bessel wavelet transform is then introduced for any function $f \in \mathcal{L}_{q,2,\alpha}(\mathbb{R}_q^+)$ as

$$\mathcal{C}^\alpha_{q,\Psi}(f)(a, b) = c_{q,\alpha} \int_0^\infty f(x)\, \overline{\Psi^\alpha_{(a,b)}}(x)\, x^{2\alpha+1}\, d_q x,\ \forall a \in \mathbb{R}_q^+,\ \forall b \in \widetilde{\mathbb{R}}_q^+,$$

where

$$\Psi^\alpha_{(a,b)}(x) = \sqrt{a}\,\mathcal{T}^\alpha_{q,b}(\Psi_a);\ \forall a, b \in \mathbb{R}_q^+, \tag{2.13}$$

and

$$\Psi_a(x) = \frac{1}{a^{2\alpha+2}}\,\Psi(\frac{x}{a}). \tag{2.14}$$

In this context, variants of Parseval-Plancherel Theorems for the case of q-Bessel wavelets have been established claiming that for all $f, g \in \mathcal{L}_{q,2,\alpha}(\widetilde{\mathbb{R}}_q^+)$, there holds that

$$\int_0^\infty f(x)\,\overline{g}(x)\,x^{2\alpha+1}\,d_qx = \frac{1}{\mathcal{A}_{\alpha,\Psi}}\int_0^\infty\int_0^\infty C_{q,\Psi}^\alpha(f)(a,b)\,\overline{C_{q,\Psi}^\alpha}(g)(a,b)d_q(a,b), \tag{2.15}$$

with $d_q(a,b) = b^{2\alpha+1}\dfrac{d_qa\,d_qb}{a^2}$ which yields in turns that for all $f \in \mathcal{L}_{q,2,\alpha}(\widetilde{\mathbb{R}}_q^+)$, we have in the $\mathcal{L}_{q,2,\alpha}$-sense,

$$f(x) = \frac{c_{q,\alpha}}{\mathcal{A}_{\alpha,\Psi}}\int_0^\infty\int_0^\infty C_{q,\Psi}^\alpha(f)(a,b)\,\Psi_{(a,b)}^\alpha(x)\,b^{2\alpha+1}\frac{d_qa\,d_qb}{a^2},\ \forall x. \tag{2.16}$$

See [8] and [15] for proofs.

Lastly, in the most recent work on q-Bessel wavelets, a more general variant of such wavelets has been developed in [32]. The idea is based on a two-parameters q-theory leading to a two-parameters q-Bessel function and associated wavelet analysis. The parameter v in the classical theory is replaced by a couple of parameters $v = (\alpha, \beta) \in \mathbb{R}^2$, $\alpha + \beta > -1$, for which a modified functional space is associated with suitable integrating measure. For $1 \leq p < \infty$, we put

$$\mathcal{L}_{q,p,v}(\widetilde{\mathbb{R}}_q^+) = \left\{ f:\ \|f\|_{q,p,v} = \left[\int_0^\infty |f(x)|^p\,x^{2|v|+1}\,d_qx\right]^{\frac{1}{p}} < \infty \right\}.$$

where $|v| = \alpha + \beta$. (See [11] for details). The modified version of q-Bessel function is

$$\widetilde{j}_{q,v}(x, q^2) = x^{-2\beta}\,j_{\alpha-\beta}(q^{-\beta}x, q^2). \tag{2.17}$$

The generalized q-Bessel Fourier transform $\mathcal{F}_{q,v}$, is defined by

$$\mathcal{F}_{q,v}f(x) = c_{q,v}\int_0^\infty f(t)\,\widetilde{j}_{q,v}(tx, q^2)\,t^{2|v|+1}\,d_qt, \quad \forall f \in \mathcal{L}_{q,p,v}(\mathbb{R}_q^+). \tag{2.18}$$

where $c_{q,v}$ is a suitable constant analogue to (2.8). This has induced a generalized q-Bessel wavelet already as an even function $\Psi \in \mathcal{L}_{q,2,v}(\widetilde{\mathcal{R}}_q)$ satisfying an analogue admissibility condition as for the existing cases with suitable and necessary modifications. The associated translation operator has been already defined in [11] by

$$T_{q,x}^v f(y) = c_{q,v}\int_0^\infty \mathcal{F}_{q,v}f(t)\,\widetilde{j}_{q,v}(yt, q^2)\,\widetilde{j}_{q,v}(xt, q^2)\,t^{2|v|+1}\,d_qt, \tag{2.19}$$

where here also we get analogue translation operator as in (2.12).

In [32], the continuous generalized q-Bessel wavelet transform of a function $f \in \mathcal{L}_{q,2,v}(\widetilde{\mathbb{R}}_q^+)$ at the scale $a \in \mathbb{R}_q^+$ and the position $b \in \widetilde{\mathbb{R}}_q^+$ has been defined by

$$\mathcal{C}_{q,\Psi}^v(f)(a,b) = c_{q,v}\int_0^\infty f(x)\,\overline{\Psi_{(a,b),v}}(x)\,x^{2|v|+1}\,d_qx,\ \forall a \in \mathbb{R}_q^+,\ \forall b \in \widetilde{\mathbb{R}}_q^+,$$

where again $\Psi_{(a,b),v}$ and Ψ_a are the analogues of (2.13) and (2.14).

Properties of such transform has been shown and analogue of Calderon-type reproducing formula has been established as in the previous cases (2.15) and (2.16).

In the present work, we continue to develop variants of Bessel-type wavelet analysis in the framework of q-theory. The main differences with existing cases may be resumed in some points.

- The modified version of q-Bessel function introduced in [22] is the last one in the field. It includes all the existing versions.

- There is no previous developments of wavelet analysis associated to the modified version of q-Bessel function introduced in [22].
- We did not serve in hand of an eigenvalue equation for such version of the modified version of q-Bessel function due to [22]. Recall that such an equation is basic to guarantee the oscillating behaviour and/or orthogonality of wavelets constructed. In a first step here we established such an equation which showed a dependence on a scaling law form.
- Even for the last and more general two-parameters Bessel operator $\Delta_{(\alpha,\beta),q}$ introduced lastly in [11], surprisingly, the modified version of q-Bessel function due to [22] did not satisfy an analogue eigenvalue equation. It is not an eigenfunction for $\Delta_{(\alpha,\beta),q}$.
- This last point leads us to return again to the one-parameter Bessel operator $\Delta_{v,q} = \Delta_{(v,0),q}$, $v \in \mathbb{R}$ and to re-check for a suitable eigenvalue equation and next to a suitable wavelet analysis.

3. Main results

In this section, the purpose is to generalize Bessel and q-Bessel wavelets to a more general case already in q-theory framework by replacing the Bessel and/or q-Bessel function with the general one introduced in [22]. We propose to introduce new wavelet functions and new wavelet transforms and prove some associated famous relations such as Plancherel/Parcevall ones as well as reconstruction formula.

Definition 3.1. *[22] The generalized q-Bessel type function is defined by*

$$j_{v,q}(x) = \frac{(x/2)^v}{(q;q)_v} \sum_{k=0}^{\infty} \frac{q^{\frac{3k}{2}(k+v)}}{(q^{v+1};a)_k} \frac{(x^2/4)^k}{(q;q)_k}.$$

The function $j_{v,q}$ is a q-analogue of the ordinary modified Bessel function as it satisfies the limit relation

$$\lim_{q\to 1} j_{v,q}(x) = j_v(x),$$

where j_v is the ordinary modified Bessel function. If v is an integer it satisfies the equality

$$j_{v,q}(x) = j_{-v,q}(x).$$

The function $j_{v,q}$ is an even (odd) function if the parameter v is an even (odd) integer, since

$$j_{v,q}(-x) = (-1)^v j_{v,q}(x),$$

whenever $v \in \mathbb{N}$. (See [22]).

In the present paper, we denote

$$\widetilde{j}_{v,q}(x) = \frac{(q;q)_v}{(x/2)^v} j_{v,q}(x) = \sum_{k=0}^{\infty} \frac{q^{\frac{3k}{2}(k+v)}}{(q^{v+1};a)_k} \frac{(x^2/4)^k}{(q;q)_k}, \tag{3.1}$$

and

$$\widetilde{J}_v(x) = \widetilde{j}_{v,q^2}(x) = \sum_{k=0}^{\infty} \frac{q^{3k(k+v)}}{(q^{2v+2};q^2)_k} \frac{(x^2/4)^k}{(q^2;q^2)_k}. \tag{3.2}$$

Otherwise,

$$\widetilde{J}_v(x) = (q^2,q^2)_v \left(\frac{x}{2}\right)^{-v} j_{v,q^2}(x). \tag{3.3}$$

Our principal aim in the present paper is to develop wavelet operators relatively to the q-Bessel type function $\widetilde{J}_v$. We thus need to introduce an associated Fourier transform and a translation operator.

Definition 3.2. *The Fourier transform $\mathcal{F}_{q,v}$ associated to the generalized q-Bessel type function $\widetilde{J}_v$ is defined by*

$$\mathcal{F}_{q,v}f(x) = \int_0^\infty f(t)\,\widetilde{J}_v(xt)\,t^{2v+1}\,d_qt. \tag{3.4}$$

The q-Bessel translation operator is defined by

$$\tau^v_{q,x}f(y) = \int_0^\infty \mathcal{F}_{q,v}f(t)\,\widetilde{J}_v(xt)\,\widetilde{J}_v(yt)\,t^{2v+1}\,d_qt. \tag{3.5}$$

Now, we introduce the associated mother wavelet to the generalized q-Bessel type function $\widetilde{J}_v$.

Definition 3.3. *A generalized q-Bessel type wavelet is a function $\Psi \in \mathcal{L}_{q,2,v}(\widetilde{\mathbb{R}}^+_q)$ satisfying the following admissibility condition:*

$$\mathcal{A}_{v,\Psi} = \int_0^\infty |\mathcal{F}_{q,v}\Psi(a)|^2\,\frac{d_qa}{a} < \infty.$$

The continuous q-Bessel wavelet transform of a function $f \in \mathcal{L}_{q,2,v}(\widetilde{\mathbb{R}}^+_q)$ is defined by

$$C^v_{q,\Psi}(f)(a,b) = \int_0^\infty f(x)\,\overline{\Psi^v_{(a,b)}}(x)\,x^{2v+1}\,d_qx,\ \forall a \in \mathbb{R}^+_q,\ \forall b \in \widetilde{\mathbb{R}}^+_q,$$

where

$$\Psi^v_{(a,b)}(x) = \sqrt{a}\tau^v_{q,b}(\Psi_a);\ \forall a,b \in \mathbb{R}^+_q,$$

and

$$\Psi_a(x) = \frac{1}{a^{2v+2}}\,\Psi(\frac{x}{a}).$$

The following result shows some properties of the generalized q-Bessel continuous wavelet transform.

Theorem 3.4. *Let Ψ be a generalized q-Bessel type wavelet in $\mathcal{L}_{q,2,v}(\widetilde{\mathbb{R}}^+_q)$. Then for all $f \in \mathcal{L}_{q,2,v}(\widetilde{\mathbb{R}}^+_q)$ and all $a \in \mathbb{R}^+_q$ the function $C^v_{q,\Psi}(f)(a,.)$ is continuous on $\widetilde{\mathbb{R}}^+_q$ and*

$$\lim_{b\to\infty} C^v_{q,\Psi}(f)(a,b) = 0.$$

Furthermore, we have

$$|C^v_{q,\Psi}(f)(a,b)| \le C(q,v,a)\,\|\Psi\|_{q,2,v}\,\|f\|_{q,2,v},$$

where $C(q,v,a) > 0$ is a constant depending eventually on q, v and a.

Recall again that whenever $b = q^n$, $n \in \mathbb{Z}$, we get $b \to \infty$ as $n \to -\infty$. The following result shows Plancherel and Parceval formulas for the generalized q-Bessel wavelet transform.

Theorem 3.5. *Let Ψ be a generalized q-Bessel wavelet in $\mathcal{L}_{q,2,v}(\widetilde{\mathbb{R}}^+_q)$. Then we have*

1. $\forall f \in \mathcal{L}_{q,2,v}(\mathbb{R}^+_q)$,
$$\|f\|^2_{q,2,v} = \frac{1}{\mathcal{A}_{v,\Psi}}\int_0^\infty\int_0^\infty |C^v_{q,\Psi}(f)(a,b)|^2\,b^{2v+1}\frac{d_qa\,d_qb}{a^2}.$$

2. $\forall f,g \in \mathcal{L}_{q,2,v}(\widetilde{\mathbb{R}}^+_q)$,
$$\int_0^\infty f(x)\overline{g}(x)\,x^{2v+1}\,d_qx = \frac{1}{\mathcal{A}_{v,\Psi}}\int_0^\infty\int_0^\infty C^v_{q,\Psi}(f)(a,b)\,\overline{C^v_{q,\Psi}}(g)(a,b)\,d_q(a,b),$$
where $d_q(a,b) = b^{2v+1}\dfrac{d_qa\,d_qb}{a^2}$.

Theorem 3.6. *Let Ψ be a generalized q-Bessel type wavelet in $\mathcal{L}_{q,2,v}(\widetilde{\mathbb{R}}^+_q)$, then for all $f \in \mathcal{L}_{q,2,v}(\mathbb{R}^+_q)$ we have*

$$f(x) = \frac{1}{\mathcal{A}_{v,\Psi}}\int_0^\infty\int_0^\infty C^v_{q,\Psi}(f)(a,b)\,\Psi_{(a,b),\alpha}(x)\,b^{2v+1}\frac{d_qb\,d_qa}{a^2} \tag{3.6}$$

in the $\mathcal{L}_{q,2,v}$-sense.

4. Proof of Main Results

In this part we develop the proofs of our main results. To do this we need a series of preliminary and technical lemmas.

Lemma 4.1. *The generalized q-Bessel type function $\widetilde{J}_v$ satisfies the eigenvalue equation for all $\alpha_q \in \mathbb{R}_q^+$,*

$$(\Delta_{q,v}u)(\alpha_q x) = \beta_q\, u(\alpha_q \gamma_q x), \quad \textit{where } \beta_q = \frac{q^{3v+1}}{4} \textit{ and } \gamma_q = q^2.$$

Proof. We set in a general form

$$\Delta_{q,v}\widetilde{J}_v(\alpha_q x) = \beta_q \widetilde{J}_v(\delta_q x).$$

We shall search for suitable constants α_q, β_q and δ_q satisfying the Lemma. So, denote for $k \in \mathbb{N}$,

$$a_k(q,v) = \frac{q^{3k(k+v)}}{(q^{2v+2};q^2)_k(q^2;q^2)_k}.$$

This way, $\widetilde{J}_v$ becomes

$$\widetilde{J}_v(x) = \sum_{k=0}^{\infty} a_k(q,v)\left(\frac{x^2}{4}\right)^k.$$

Substituting in the Laplacian equation of the Lemma, we get

$$a_k(q,v)\alpha_q^{2(k-1)}\left[\frac{1}{q^{2k}} - (1+q^{2v}) + q^{2k+2v}\right] = 4\beta_q \delta_q^{2(k-1)} a_{k-1}(q,v), \ \forall k \geq 1.$$

Otherwise,

$$a_k(q,v)\alpha_q^{2(k-1)}\left[\frac{(1-q^{2k})(1-q^{2k+2v})}{q^{2k}}\right] = 4\beta_q \delta_q^{2(k-1)} a_{k-1}(q,v), \ \forall k \geq 1.$$

For $k=1$, we get

$$a_1(q,v)\left[\frac{(1-q^2)(1-q^{2+2v})}{q^2}\right] = 4\beta_q a_0(q,v).$$

Observing that

$$a_0(q,v) = 1 \qquad \text{and} \qquad a_1(q,v) = \frac{q^{3(v+1)}}{(1-q^2)(1-q^{2+2v})},$$

we get

$$\beta_q = \frac{q^{3v+1}}{4}.$$

For $k=2$, we get

$$a_2(q,v)\alpha_q^2\left[\frac{(1-q^4)(1-q^{4+2v})}{q^4}\right] = 4\beta_q \delta_q^2 a_1(q,v).$$

Now, analogously, we have

$$a_2(q,v) = \frac{q^{6(2+v)}}{(1-q^{2v+2})(1-q^{2v+4})(1-q^2)(1-q^4)}.$$

We get

$$q^2\alpha_q = \delta_q.$$

As a result of Lemma 4.1, we get the following orthogonality relation.

Lemma 4.2. *The generalized q-Bessel type function j_v satisfies the following orthogonality relation,*

$$\int_0^\infty \widetilde{J}_v(xt)\,\widetilde{J}_v(yt)\,t^{2v+1}\,d_qt = \delta_{q,v}(x,y)\,,\ \ \forall x,y,$$

where $\delta_{q,v}$ is the (q,v)-delta operator defined by

$$\delta_{q,v}(x,y) = \begin{cases} \dfrac{1}{(1-q)\,x^{2(v+1)}}, & if\ x=y, \\ 0 & else. \end{cases}$$

The proof may be inspired from the general cases developed already in [21]. See also [32]. We may show easily the following result for such a translation operator.

Lemma 4.3. *For all $f \in \mathcal{L}_{q,2,v}(\widetilde{\mathbb{R}}_q^+)$ and all $t \in \mathbb{R}_q^+$, we have*

$$f(t) = \int_0^\infty f(x)\,\delta_{q,v}(x,t)\,x^{2(v+1)}\,d_qx.$$

Proof. For $t \in \mathbb{R}_q^+$, let k be the unique integer satisfying $t = q^k$. We have

$$\begin{aligned}\int_0^\infty f(x)\delta_{q,v}(x,t)x^{2v+1}d_qx &= (1-q)\sum_{n=0}^\infty f(q^n)\delta_{q,v}(q^n,s)q^{n(2v+2)} \\ &= (1-q)f(q^k)\delta_{q,v}(q^k,t)q^{k(2v+2)} \\ &= f(q^k)\end{aligned}$$

Now, to move to the proof of Theorem 3.4 we need the following preliminary results.

Lemma 4.4. *For all $f \in \mathcal{L}_{q,2,v}(\widetilde{\mathbb{R}}_q^+)$, the following Fourier dilation-invariance property holds,*

$$\mathcal{F}_{q,v}(\tau_{q,x}^v f)(\lambda) = \widetilde{J}_v(\lambda x)\,\mathcal{F}_{q,v}f(\lambda),\ \ \forall \lambda, x \in \widetilde{\mathbb{R}}_q^+. \tag{4.1}$$

Furthermore,

$$\tau_{q,x}^v f(y) = \tau_{q,y}^v f(x)\ \ and\ \ \tau_{q,x}^v f(0) = f(x),\ \ \forall f \in \mathcal{L}_{q,2,v}(\widetilde{\mathbb{R}}_q^+), \tag{4.2}$$

and

$$\tau_{q,x}^v \widetilde{J}_v(ty) = \widetilde{J}_v(tx)\,\widetilde{J}_v(ty),\ \forall t,x,y \in \widetilde{\mathbb{R}}_q^+. \tag{4.3}$$

Proof. Remark firstly that

$$\tau_{q,x}^v f(y) = \mathcal{F}_{q,v}\Big(\mathcal{F}_{q,v}(f)(.)\widetilde{J}_v(x.)\Big)(y).$$

Consequently, Lemma 4.3 yields that

$$\mathcal{F}_{q,v}\big(\tau_{q,x}^v f\big)(\lambda) = \mathcal{F}_{q,v}\big(\mathcal{F}_{q,v}\big(\mathcal{F}_{q,v}(f)(.)\widetilde{J}_v(x.)\big)\big)(\lambda) = \mathcal{F}_{q,v}(f)(\lambda)\widetilde{J}_v(\lambda x).$$

So as equation (4.1). Equation (4.2) is obvious. To show (4.3) denote as in [30] and [32]

$$D(x,y,z) = \int_0^\infty \widetilde{J}_v(xt)\,\widetilde{J}_v(yt)\,\widetilde{J}_v(zt)\,t^{2v+1}d_qt.$$

It is straightforward that

$$\int_0^\infty \widetilde{J}_v(zt)\,D(x,y,z)\,z^{2v+1}d_qz = \widetilde{J}_v(xt)\widetilde{J}_v(yt),\quad \forall\ x,y,t.$$

Using Fubini's rule, we get

$$\tau_{q,x}^v \widetilde{J}_v(ty) = \int_0^\infty \widetilde{J}_v(zt)\,D(x,y,z)\,z^{2v+1}d_qz = \widetilde{J}_v(xt)\widetilde{J}_v(yt).$$

So as equation (4.3).

Lemma 4.5. *For all $f \in L_{q,2,v}(\widetilde{\mathbb{R}}_q^+)$, we have*

$$\|\mathcal{F}_{q,v}f\|_{q,2,v} = \|f\|_{q,2,v}.$$

Proof. Denote

$$\mathcal{K}_v(x,t,s) = \widetilde{J}_v(xt)\widetilde{J}_v(xs).$$

We have

$$\|\mathcal{F}_{q,v}f\|_{q,2,v}^2 = \int_0^\infty \int_0^\infty \int_0^\infty f(t)\overline{f(s)}\mathcal{K}_v(x,t,s)(tsx)^{2v+1}d_qtd_qsd_qx.$$

Using Fubini's rule we get

$$\|\mathcal{F}_{q,v}f\|_{q,2,v}^2 = \int_0^\infty \int_0^\infty f(t)\overline{f(s)} \int_0^\infty \mathcal{K}_v(x,t,s)x^{2v+1}d_qx(ts)^{2v+1}d_qtd_qs.$$

Using Lemma 4.2 we obtain

$$\|\mathcal{F}_{q,v}f\|_{q,2,v}^2 = \int_0^\infty \int_0^\infty f(t)\overline{f(s)}\delta_{q,v}(t,s)t^{2v+1}s^{2v+1}d_qtd_qs,$$

which by Fubini's rule again yields that

$$\|\mathcal{F}_{q,v}f\|_{q,2,v}^2 = \int_0^\infty f(t)t^{2v+1} \int_0^\infty \overline{f(s)}\delta_{q,v}(t,s)s^{2v+1}d_qsd_qt.$$

Now, Lemma 4.3 yields that

$$\|\mathcal{F}_{q,v}f\|_{q,2,v}^2 = \int_0^\infty |f(t)|^2t^{2v+1}d_qt = \|f\|_{q,2,v}^2.$$

Lemma 4.6. *For all $\Psi \in L_{q,2,v}(\widetilde{\mathbb{R}}_q^+)$, the following assertions are true.*

1. $$\|\tau_{q,x}^v\Psi\|_{q,2,v} \leq \frac{1}{(q,q^2)_\infty^2}\|\Psi\|_{q,2,v}.$$

2. $$\|\Psi_a\|_{q,2,v} = \frac{1}{a^{2v+2}}\|\Psi\|_{q,2,v}.$$

Proof. (1) Denote

$$\widetilde{\mathcal{K}}_v(x,y,t,s) = \mathcal{K}_v(x,t,s)\mathcal{K}_v(y,t,s),$$

where $\mathcal{K}_v(x,t,s)$ is defined previously in the proof of Lemma 4.5. Let also

$$\mathcal{Q}_v\Psi(t,s) = \mathcal{F}_{q,v}\Psi(t)\overline{\mathcal{F}_{q,v}\Psi(s)}.$$

We have

$$\begin{aligned}
&\|\tau_{q,x}^v\Psi\|_{q,2,v}^2 \\
= &\int_0^\infty \int_0^\infty \int_0^\infty \mathcal{Q}_v\Psi(t,s)\widetilde{\mathcal{K}}_v(x,y,t,s)(tsy)^{2v+1}d_qtd_qsd_qy \\
= &\int_0^\infty \int_0^\infty \mathcal{Q}_v\Psi(t,s)\int_0^\infty \widetilde{\mathcal{K}}_v(x,y,t,s)y^{2v+1}d_qy(ts)^{2v+1}d_qtd_qs \\
= &\int_0^\infty \int_0^\infty \mathcal{Q}_v\Psi(t,s)\delta_{q,v}(t,s)\mathcal{K}_v(x,t,s)(ts)^{2v+1}d_qtd_qs \\
= &\int_0^\infty \mathcal{F}_{q,v}\Psi(t)\widetilde{J}_v(xt)t^{2v+1}\int_0^\infty \overline{\mathcal{F}_{q,v}\Psi(s)}\delta_{q,v}(t,s)\overline{\widetilde{J}_v(xs)}s^{2v+1}d_qsd_qt \\
= &\int_0^\infty |\mathcal{F}_{q,v}\Psi(t)|^2|\widetilde{J}_v(xt)|^2t^{2v+1}d_qt.
\end{aligned}$$

The second and the fourth equalities follow from Fubini's rule. The third and the fifth follow from Lemma 4.2 and Lemma 4.3 respectively. Next, observing that

$$|\widetilde{J}_v(xt)| \leq \frac{1}{(q,q^2)_\infty^2},$$

we get

$$\begin{aligned}\int_0^\infty |\mathcal{F}_{q,v}\Psi(t)|^2|\widetilde{J}_v(xt)|^2t^{2v+1}d_qt &\leq \frac{1}{(q,q^2)_\infty^4}\int_0^\infty |\mathcal{F}_{q,v}\Psi(t)|^2t^{2v+1}d_qt\\ &= \frac{1}{(q,q^2)_\infty^4}\|\mathcal{F}_{q,v}\Psi\|_{q,2,v}^2.\end{aligned}$$

(2) Recall that

$$\begin{aligned}\|\Psi_a\|_{q,2,v}^2 &= \int_0^\infty |\Psi_a(x)|^2x^{2v+1}d_q(x)\\ &= \frac{1}{a^{4v+4}}\int_0^\infty |\Psi(\frac{x}{a})|^2x^{2v+1}d_q(x).\end{aligned}$$

Using (2.4), this yields that

$$\begin{aligned}\|\Psi_a\|_{q,2,v} &= \frac{1}{a^{2v+2}}\int_0^\infty |\Psi(u)|^2u^{2v+1}d_q(u)\\ &= \frac{1}{a^{2v+2}}\|\Psi\|_{q,2,v}^2.\end{aligned}$$

Proof of Theorem 3.4 For $a \in \mathbb{R}_q^+$ and $b \in \widetilde{\mathbb{R}}_q^+$, we have

$$C_{q,\Psi}^v(f)(a,b) = \int_0^\infty f(x)\overline{\Psi_{(a,b),v}}(x)x^{2v+1}d_qx.$$

Observing that

$$\Psi_{(a,b),v}(x) = \sqrt{a}\tau_{q,b}^v(\Psi_a),$$

we get

$$C_{q,\Psi}^v(f)(a,b) == \sqrt{a}\int_0^\infty f(x)\overline{T_{q,b}^v\Psi_a}(x)x^{2v+1}d_qx.$$

Next, Hölder's inequality yields that

$$\left|C_{q,\Psi}^v(f)(a,b)\right| \leq \sqrt{a}\|f\|_{q,2,v}\|\tau_{q,b}^v\Psi_a\|_{q,2,v}.$$

Which by Lemma 4.6 implies that

$$\left|C_{q,\Psi}^v(f)(a,b)\right| \leq \frac{1}{(q,q^2)_\infty^2\,a^{v+\frac{1}{2}}}\,\|\Psi\|_{q,2,v}\,\|f\|_{q,2,v}.$$

Proof of Theorem 3.5 (1) We have

$$\begin{aligned}&q^{4v+2}\int_0^\infty\int_0^\infty |C_{q,\Psi}^v(f)(a,b)|^2\,b^{2v+1}\frac{d_qa\,d_qb}{a^2}\\ &= q^{4v+2}\int_0^\infty\left(\int_0^\infty |\mathcal{F}_{q,v}(f)(x)|^2|\mathcal{F}_{q,v}(\overline{\Psi_a})|^2(x)x^{2v+1}d_qx\right)\frac{d_qa}{a}\\ &= \int_0^\infty |\mathcal{F}_{q,v}(f)(x)|^2\left(|\mathcal{F}_{q,v}(\Psi)(ax)|^2\frac{d_qa}{a}\right)x^{2v+1}d_qx\\ &= C_{v,\Psi}\int_0^\infty |\mathcal{F}_{q,v}(f)(x)|^2x^{2v+1}d_qx\\ &= C_{v,\Psi}\|f\|_{q,2,v}^2.\end{aligned}$$

Hence, the first assertions is proved.
(2) We apply the first assertion above with $f+g$ instead of f and the linearity of the wavelet transform operator.
Proof of Theorem 3.6 Consider for $x\in\mathbb{R}_q^+$ the function $g_x=\delta_{q,v}(x,.)$. It holds that

$$C_{q,\Psi}^v(g_x)(a,b)=\Psi_{(a,b),v}(x).$$

Therefore, Theorem 3.5 yields that

$$\int_0^\infty f(t)\overline{g}_x(t)\,t^{2v+1}\,d_qt=\frac{1}{\mathcal{A}_{v,\Psi}}\int_0^\infty\int_0^\infty C_{q,\Psi}^v(f)(a,b)\overline{\Psi_{(a,b),v}}(x)\;b^{2v+1}\frac{d_qa\,d_qb}{a^2}.$$

On the other hand, Lemma 4.3 implies that

$$f(x)=\int_0^\infty f(x)\overline{g}_x(t)\,t^{2v+1}\,d_qt;\quad\forall f\in L_{q,2,v}(\widetilde{\mathbb{R}}_q^+).$$

As a result,

$$f(x)=\frac{1}{\mathcal{A}_{v,\Psi}}\int_0^\infty\int_0^\infty C_{q,\Psi}^v(f)(a,b)\overline{\Psi_{(a,b),v}}(x)\;b^{2v+1}\frac{d_qa\,d_qb}{a^2}.$$

5. Some illustrative numerical examples

In this section we propose to conduct a numerical examples to illustrate the use of the q-wavelets developed previously, and the use of q-theory in the numerical approximation of PDEs.

5.1. Example 1

We consider for this aim a simple example dealing with the non-homogeneous stationary cubic Schrodinger equation

$$u''(x)+|u(x)|^2u(x)=\Phi(x),\;x\in\mathbb{R},\tag{5.1}$$

where

$$\Phi(x)=-i2a\sqrt{2a}\sinh(\sqrt{a}x)sech^2(\sqrt{a}x)e^{i\sqrt{a}x},$$

with a constant $a>0$ fixed. Remark that the exact solution of (5.1) is given by

$$u_{exact}(x)=\sqrt{2a}sech(\sqrt{a}x)e^{i\sqrt{a}x},$$

which my be seen as a stationary form of the soliton-type disturbance travels with speed c and with a-governed amplitude

$$u(x,t)=\sqrt{2a}\,sech(\sqrt{a}(x-ct))e^{i(\frac{c}{2}x-\theta t)},$$

with $\theta=a-\frac{c^2}{4}$, and which is a solution of the classical cubic Schrodinger equation

$$iu_t(x,t)+u_{xx}(x,t)+|u(x,t)|^2u(x,t)=0,\;(x,t)\in\mathbb{R}\times\mathbb{R}_+.$$

u_t and u_{xx} are the first and the second partial derivatives of the function u in time and space respectively. The q-analog of equation (5.1) is written as

$$\Delta_qu(x)+|u(x)|^2u(x)=\frac{u(q^{-1}x)-(1+q)u(x)+qu(qx)}{x^2}+|u(x)|^2u(x)=\Phi(x),\;x\in\mathbb{R}_q^+,\tag{5.2}$$

taken with the initial condition $u(1)=1$ and $D_qu(1)=-1$, where the differential operator D_q is the analog of the first order derivative defined on $\mathbb{R}_q$ by

$$D_qu(x)=\frac{u(x)-u(qx)}{(1-q)x}.$$

q	$\|u-u_{exact}\|_{2,N}$
5/7	$1.12\,10^{-4}$
2/3	$2.28\,10^{-5}$

Table 1: L_2-discrete error estimate due to (5.4) for $N=10$.

By denoting $x=q^n$ and $u_n=u(q^n)$, $\Phi_n=q^{2n}\Phi(q^n)$, $n\geq 0$, the equation (5.2) may be written as

$$qu_{n+1}-(1+q)u_n+u_{n-1}+q^{2n}|u_n|^2u_n=\Phi_n,\ n\geq 0,\text{ and } u_0=1,\ u_1=-q. \tag{5.3}$$

We will evaluate the closeness of the numerical solution to the exact one via the L_2 discrete norm

$$\|u-u_{exact}\|_{2,N}=\Big(\sum_{n=0}^{N}|u_n-u_{exact}(q^n)|^2\Big)^{1/2}, \tag{5.4}$$

where $u=(u_n)_{n\geq 0}$ is the numerical solution to (5.3), and N is the number of points used for the discrete grid.

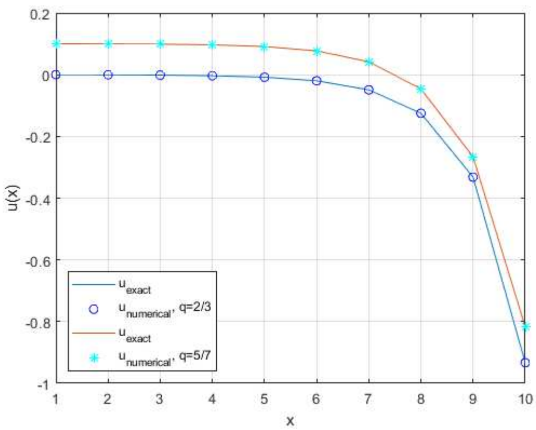

Figure 1: Exact and numerical solutions due to (5.3) for $q\in\{\frac{2}{3},\frac{5}{7}\}$.

Table 1 illustrates the error estimates between the numerical solution and the exact one for different values of the quantum parameter q. The space grid is fixed to a number $N=10$ points. It is noticeable from Table 1 that the numerical quantum discrete scheme converges with good error estimates.

5.2. Example 2

We consider in this example a simple model

$$u''(x)+\pi^2u(x)=\pi^2x,\ x\in[0,1], \tag{5.5}$$

with the boundary values $u(1)=0$, $u'(1)=1-\pi$. We get here an exact solution given by

$$u_{exact}(x)=\cos\pi x+\sin\pi x+x.$$

The q-analog of equation (5.5) is written as

$$\Delta_q u(x)+u(x)=x,\ x\in\mathbb{R}_q^+, \tag{5.6}$$

taken with the initial condition $u(1)=0$ and $D_qu(1)=1-\pi$. By using the wavelet approach, we consider a system of q-adic q-Bessel wavelets (ψ_n) to decompose the function u at an approximation level $J\in\mathbb{N}$ as

$$u_J(x)=\sum_{k=0}^{J}u_{J,k}\psi_{J,k}(x), \tag{5.7}$$

q	$\|u - u_{exact}\|_{2,J}$
5/7	$2.72\,10^{-6}$
2/3	$2.32\,10^{-8}$

Table 2: L_2-discrete error estimate due to (5.9) for $J = 4$.

and substitute it in the equation (5.6). Using Lemma 4.1, we get a recursive equation on the coefficients $u_{J,k}$ as

$$\beta_q u_{J+2,k} + u_{J,k} = f_{J,k},\ \forall k, \tag{5.8}$$

where $f_{J,k}$ are the coefficients of the function $f(x) = \pi^2 x$, $x \in [0,1]$.
Here, the closeness of the numerical solution to the exact one will be evaluated via the L_2 discrete norm

$$\|u - u_{exact}\|_{2,J} = \Big(\sum_k |u_{J,k} - u_{exact,J,k}|^2\Big)^{1/2}. \tag{5.9}$$

Figure 2 illustrates the closeness of the numerical solution to the exact one, while Table 2 shows the error estimates between the numerical solution and the exact one. As for the previous example, we notice

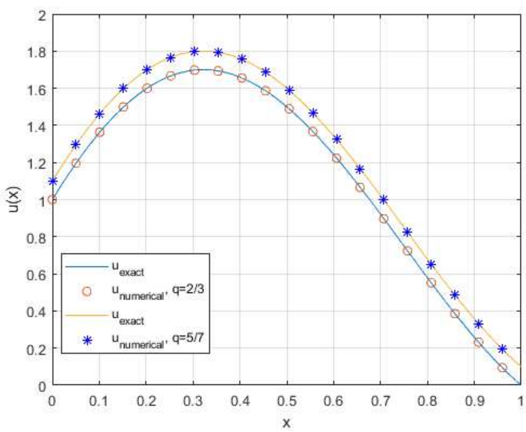

Figure 2: Exact and numerical solutions due to (5.3) for $q \in \{\frac{2}{3}, \frac{5}{7}\}$ and $\alpha = 0.25$.

from Table 2 that the numerical quantum discrete scheme converges with good error estimates.

6. Conclusion

In this paper new wavelet operators have been constructed by exploiting a generalized and/or modified variant of q-Bessel function. q-Continuous wavelet transform and Plancherel/Parceval type relations have been proved. It is shown in the introduction that the variant may be applied to solve ODEs and PDEs by transforming them to recurrent sequences. This fact is confirmed by the development of numerical examples where the efficiency of the theoretical results is shown already with graphical illustrations and error estimates. The present results may be also extended to other future directions especially in statistical applications. Including the q-scale into the estimation and the modeling of time series, statistical series, and financial data may induce good results. A starting step may be for instance done by mixing the ideas in [2,7,33,34,35,36,37]. In some of these references it is already shown that the use of new time scales is performant than the classical uniform method.

References

1. A. Abouelaz, R. Daher and L. El Mehdi, Harmonic Analysis associated with the generalized q-Bessel operator, International Journal of Analysis and Applicayions ISSN 2291-8639, 10 (2016), pp. 17-23.

2. B. Aktan, A. Ben Mabrouk, M. Ozturk, N. Rhaiem, Wavelet-Based systematic risk estimation an application on istanbul stock exchange. International Research Journal of Finance and Economics 23(3) (2009), pp. 33-45.

3. S. Albeverio, P. E. T. Jorgensen, and A. M. Paolucci, Multiresolution wavelet analysis of integer scale Bessel functions J. Math. Phys. 48, 073516 (2007); https://doi.org/10.1063/1.2750291.

4. M. H. Annaby and Z. S. Mansour, q-Fractional Calculus and Equations. Lecture Notes in Mathematics 2056, Editors: J.-M. Morel and B. Teissier, Springer 2012.

5. A. Aral, V. Gupta and R. P. Agarwal, Applications of q-calculs in operator theory, Springer, New York, 2013.

6. S. Arfaoui, I. Rezgui and A. Ben Mabrouk, Wavelet Analysis on the Sphere: Spheroidal Wavelets. Walter de Gruyter, March 20, 2017.

7. A. Ben Mabrouk, Wavelet-based systematic risk estim-ation: application on GCC stock markets: the Saudi Arabia case. Quantitative Finance and Economics 7(1) (2020), pp. 117-118.

8. S. Bouaziz, The q-Bessel wavelet packets, Advances in Analysis, 1(1) (2016), pp. 27-39.

9. F. Bouzeffour, H. Ben Mansour, and M. Garayev, On the Zeros of the Big q-Bessel Functions and Applications. Mathematics 2020, 8(2), 237; https://doi.org/10.3390/math8020237

10. L. Dhaouadi, On the q-Bessel Fourier transform, Bulletin of mathematical analysis and applications, 5(2) (2013), pp. 42-60.

11. L. Dhaouadi and M. Hleili, Generalized q-Bessel operator, Bulletin of mathematical analysis and applications, 7(1) (2015), pp. 20-37.

12. L. Dhaouadi, A. Fitouhi and J. El Kamel, Inequalities in q-Fourier analysis, J. of Inequalities in Pure and Applied Mathematics, 7(5) (2006), Article 171, 14 pages.

13. M. M. Dixit, C. P. Pandey and D. Das, The Continuous Generalized Wavelet Transform Associated with q-Bessel Operator. Bol. Soc. Paran. Mat. (3s.) v. 2023 (41), pp. 1-10. doi:10.5269/bspm.52810.

14. A. Fitouhi and N. Bettaibi, Wavelet transforms in quantum calculus, J. Nonlinear Mathematical Physics, 13(3) (2006), pp. 492-506.

15. A. Fitouhi, N. Bettaibi and W. Binous, Inversion Formulas For The q-Riemann-Liouville and q-Weyl Transforms Using Wavelets, Fractional calculus and Applied Analysis, 10(4) (2007), pp. 327-342.

16. A. Fitouhi, K. Trimeche and J. L. Lions, Transmutation operators and generalized continuous wavelets, Preprint, Faculty of Science of Tunis (1995).

17. G. Gasper and M. Rahman, Basic Hypergeometric serie Second edition, Combridge university Press, 2004.

18. D. T. Haimo, Integral equations associated with Hankel convolution, Trans .Amer. Math. Soc 116 (1965), pp. 330-375.

19. I. I. Hirschman, Variation diminishing Hankel transform, Journal d'Analyse Mathématique 8(1) (1860), pp. 307-336.

20. F. H. Jackson, The application of basic numbers to Bessel's and Legendre's functions, Proc. London math.Soc. 2(2) (1905), pp. 192-220.

21. T. H. Koornwinder and R. F. Swarttow, On q-Analogues of the Hankel and Fourier transform, Trans. A.M.S., 333 (1992), pp. 445–461.

22. M. Mansour and M. M. Al-Shomrani, New q-analogue of modified Bessel function and the quantum algebra $E_q(2)$. J. Computational Analysis and Applications, 15(4) (2013), pp. 655–664.

23. M. Martinet, Analyse multirésolution en ondelettes du rayonnement acoustique des structures planes, Thése de Doctorat en Acoustique, INSA de Lyon, 2001.

24. H. Mejjaoli and N. Sraieb, The q-Dunkl wavelet theory and localization operators. Int. J. Open Problems Compt. Math., 13(3) (2020), pp. 106-128.

25. H. Mejjaoli and N. Sraieb, Gabor transform in quantum calculus and applications. Fractional Calculus and Applied Analysis, 12(3) (2009), pp. 319-336.

26. H. Mejjaoli and N. Sraieb, Generalized Sobolev spaces in quantum calculus and applications. Journal of Inequalities and Special Functions, 3(2) (2012), pp. 1-19.

27. H. Mejjaoli and N. Sraieb, Wavelet-multipliers analysis in the framework of the q-Dunkl theory. Int. J. Open Problems Complex Analysis, 11(3) (2019), pp. 1-27.

28. H. Mejjaoli and N. Sraieb, Localization operators associated with the q-Bessel wavelet transform and applications. International Journal of Wavelets, Multiresolution and Information Processing, 19(4) (2021), 2050094.

29. A. Prasad, A. Mahato, V. K. Singh and M. M. Dixit, The continuous fractional Bessel wavelet transformation, Boundary value Problems 40 (2013), pp. 1-16.

30. R. S. Pathak, M. M. Dixit, Continuous and discrete Bessel wavelet transforms, Journal of Computational and Applied Mathematics, 160 (2003), pp. 240-250.

31. R. S. Pathak, S. K. Upadhyay and R. S. Pandey, The Bessel wavelet convolution product, Rend.Sem. Mat. Univ. Politec. Torino, 96(3) (2011), pp. 267-279.

32. I. Rezgui and A. Ben Mabrouk, Some Generalized q-Bessel type Wavelets and associated transforms, Anal. Theory Appl, 34(1) (2017), pp. 1-15.

33. N. Rhaiem, A. Ben Mabrouk, and S. Ben Ammou, Estimation of Capital Asset Pricing Model at Different Time Scales: application to French stock market. The International Journal of Applied Economics and Finance 80 (2007), pp. 79-87.

34. N. Rhaiem, A. Ben Mabrouk, and S. Ben Ammou, Wavelet estimation of systematic risk at different time scales: application to French stock market. The International Journal of Applied Economics and Finance 1(2) (2007), pp. 113-119.

35. F. Saadaoui, Using wavelets in the measurement of multiscale dependence between Saudi and selected foreign stock markets. Intl. Trans. in Op. Res. (2023), https://doi.org/10.1111/itor.13019.

36. F. Saâdaoui and O. Ben Messaoud, Multiscaled Neural Autoregressive Distributed Lag: A New Empirical Mode Decomposition Model for Nonlinear Time Series Forecasting. International Journal of Neural Systems 30(08) (2020), 2050039. https://doi.org/10.1142/S0129065720500392.

37. M. Sarraj, and A. Ben Mabrouk, The Systematic Risk at the Crisis - A Multifractal Non-Uniform Wavelet Systematic Risk Estimation. Fractal Fract. 2021, 5, 135. https://doi.org/ 10.3390/fractalfract5040135.

38. K. Trimèche, Generalized Harmonic Analysis and wavelet Packets. Gordon and Breach. Science Publishers, Amesterdam etc. 2001.

39. S. K. Upadhyaya, On continuous Bessel wavelet transformation associated with the Hankel-Hausdorff operator. Integral Transforms and Special Functions, 23(5) (2012), pp. 315-323.

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