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Generalized Quasi-Conformal Curvature Tensor and the Spacetime of General Relativity

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ABSTRACT: In this paper, a study of generalized quasi-conformal curvature tensor has been made on the four dimensional spacetime of general relativity. Some results related to the application of such spacetime in the general relativity are obtained. Perfect fluid and dust fluid cosmological models have also been studied.

Key Words: Perfect fluid spacetime, Einstein, generalized quasi-conformal curvature tensor, energy momentum tensor.

Contents

1 Introduction 1

2 Spacetime with Vanishing Generalized Quasi-Conformal Curvature Tensor 2

3 Perfect Fluid Spacetime with Vanishing Generalized Quasi-Conformal Curvature Tensor 5

4 Dust Fluid Spacetime with Vanishing Generalized Quasi-Conformal Curvature Tensor 7

1. Introduction

In this paper we study some results on general relativity by the co-ordinate free method of differential geometry. In this method we consider a four dimensional semi-Riemannian manifold (M^4, g) with a Lorentzian metric g with signature $(-,+,+,+)$. The geometry of the Lorentzian manifold begins with the study of the casual character of vectors of the manifold. It is due to this causality that the Lorentzian manifold became a convenient choice for the study of general relativity.

In [12] Einstein equation on the energy-momentum tensor is of vanishing divergence and the requirement is satisfied if the energy-momentum tensor is covariant-constant. In paper [7] M.C.Chaki and Sarbari Roy showed that a general relativistic spacetime with covariant-constant energy-momentum tensor is Ricci symmetric, that is $\nabla S = 0$, where S is the Ricci tensor of the spacetime.

The generalized quasi conformal curvature tensor of 4- dimensional space time (M^n, g) is given by (See [6])

$$\begin{aligned}\omega(Y,U)V =& R(Y,U)V + a_1\left(S(U,V)Y - S(Y,V)U\right) + b_1(g(U,V)QY \\ &- g(Y,V)QU) - \frac{c_1}{4} r\left(\frac{1}{3} + a_1 + b_1\right)\left(g(U,V)Y - g(Y,V)U\right),\end{aligned} \tag{1.1}$$

where a_1 and b_1 are real constants, we note that for four dimensions, it has the flavour of Riemannian curvature tensor R if $(a_1, b_1, c_1) = (0,0,0)$; conformal curvature tensor C if $(a_1, b_1, c_1) = (\frac{-1}{2}, \frac{-1}{2}, 1)$; conharmonic curvature tensor L if $(a_1, b_1, c_1) = (\frac{-1}{2}, \frac{-1}{2}, 0)$;concircular curvature tensor E if$(a_1, b_1, c_1) = (0,0,1)$; projective curvature tensor P if $(a_1, b_1, c_1) = (\frac{-1}{3}, 0, 0)$;$W_1$-curvature tensor if $(a_1, b_1, c_1) = (\frac{1}{3}, 0, 0)$;$W_2$-curvature tensor if $(a_1, b_1, c_1) = (0, \frac{-1}{3}, 0)$;$M$-projective curvature tensor if $(a_1, b_1, c_1) = (\frac{-1}{6}, \frac{-1}{6}, 0)$ and S,r are the Ricci tensor,scalar curvature respectively.

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2. Spacetime with Vanishing Generalized Quasi-Conformal Curvature Tensor

Let V_4 be the spacetime of general relativity, then from (1.1) we have

$$\begin{aligned}\omega(Y,U,V,X) =& R(Y,U,V,X) + a_1\left(S(U,V)g(Y,X) - S(Y,V)g(U,X)\right)\\ &+ b_1\left(g(U,V)g(QY,X) - g(Y,V)g(QU,X)\right)\\ &- \frac{c_1}{4}r\left(\frac{1}{3} + a_1 + b_1\right)\left[g(U,V)g(Y,X) - g(Y,V)g(U,X)\right],\end{aligned} \tag{2.1}$$

where $\omega(Y,U,V,X) = g(\omega(Y,U)V,X)$ and $R(Y,U,V,X) = g(R(Y,U)V,X)$.

If $\omega(Y,U,V,X) = 0$, then (2.1) becomes

$$\begin{aligned}&R(Y,U,V,X) + a_1\left(S(U,V)g(Y,X) - S(Y,V)g(U,X)\right)\\ &+ b_1\left(g(U,V)g(QY,X) - g(Y,V)g(QU,X)\right)\\ &- \frac{c_1}{4}r\left(\frac{1}{3} + a_1 + b_1\right)\left[g(U,V)g(Y,X) - g(Y,V)g(U,X)\right] = 0.\end{aligned} \tag{2.2}$$

Let Q be the Ricci operator given by $g(QY,X) = S(Y,X)$, then (2.2) becomes

$$\begin{aligned}&R(Y,U,V,X) + a_1\left(S(U,V)g(Y,X) - S(Y,V)g(U,X)\right)\\ &+ b_1\left(g(U,V)S(Y,X) - g(Y,V)S(U,X)\right)\\ &- \frac{c_1}{4}r\left(\frac{1}{3} + a_1 + b_1\right)\left[g(U,V)g(Y,X) - g(Y,V)g(U,X)\right] = 0.\end{aligned} \tag{2.3}$$

Taking a frame field over X and Y, equation (2.3) becomes

$$S(U,V) = -r\left(\frac{b_1 - \frac{3}{4}.c_1.(\frac{1}{3} + a_1 + b_1)}{1 + 3a_1 - b_1}\right)g(U,V). \tag{2.4}$$

Hence we state the following theorem.

Theorem 2.1. *A generalized quasi-conformally flat spacetime is an Einstein's spacetime, provided*

$$\left(\frac{b_1 - \frac{3}{4}.c_1.(\frac{1}{3} + a_1 + b_1)}{1 + 3a_1 - b_1}\right) \neq 0.$$

Substituting (2.4) in (2.3), we have

$$\begin{aligned}R(Y,U,V,X) =& -r\left((a_1 + b_1)\frac{b_1 - \frac{3}{4}.c_1.\left(\frac{1}{3} + a_1 + b_1\right)}{1 + 3a_1 - b_1} + \frac{c_1}{4}.\left(\frac{1}{3} + a_1 + b_1\right)\right)\\ &\left[g(Y,V)g(U,X) - g(U,V)g(Y,X)\right].\end{aligned} \tag{2.5}$$

Hence we can state the following theorem:

Theorem 2.2. *A generalized quasi-conformally flat spacetime is a spacetime of constant curvature, provided*

$$\left((a_1 + b_1)\frac{b_1 - \frac{3}{4}.c_1.\left(\frac{1}{3} + a_1 + b_1\right)}{1 + 3a_1 - b_1} + \frac{c_1}{4}.\left(\frac{1}{3} + a_1 + b_1\right)\right) \neq 0.$$

Now, we consider a spacetime satisfying the Einstein's field equation with cosmological constant is given by

$$S(X,Y) - \frac{r}{2}g(X,Y) + \lambda g(X,Y) = \kappa T(X,Y), \tag{2.6}$$

where S and r denote the Ricci tensor and scalar curvature, λ is the cosmological constant, κ is the gravitational constant and $T(X, Y)$ is the energy momentum tensor.

In view of (2.4) and (2.6), we have

$$T(X,Y) = -\frac{1}{\kappa}\left(r\left(\frac{b_1 - \frac{3}{4}.C_1.\left(\frac{1}{3}+a_1+b_1\right)}{1+3a_1-b_1}+\frac{1}{2}\right)-\lambda\right)g(X,Y). \tag{2.7}$$

Taking covariant derivative of (2.7), we get

$$(\nabla_Z T)(X,Y) = -\frac{1}{\kappa}dr(Z)\left(\frac{b_1 - \frac{3}{4}.C_1.\left(\frac{1}{3}+a_1+b_1\right)}{1+3a_1-b_1}+\frac{1}{2}\right)g(X,Y). \tag{2.8}$$

Since generalized quasi-conformally flat spacetime is Einstein, therefore the scalar curvature r is constant. Hence

$$dr(Z) = 0, \tag{2.9}$$

for all Z.
In view of (2.8) and (2.9), we have

$$(\nabla_Z T)(X,Y) = 0. \tag{2.10}$$

Hence we state the following theorem:

Theorem 2.3. *In a generalized quasi-conformally flat spacetime satisfying Einstein's field equation with cosmological constant, the energy momemtum tensor is covrariant constant.*

The curvature collineation studied by Katzin et al. [10], in context of the related particle and field conservation laws that may be admitted in the standard form of general relativity.

The geometrical symmetries of a spacetime is

$$L_\xi B - 2\varphi B = 0, \tag{2.11}$$

where B represents a geometrical/physical quantity, L_ξ denotes the Lie derivative with respect to the vector field ξ and φ is a scalar.

One of the most simple and widely used example is the metric inheritance symmetry for $B = g(X, Y)$ in (2.11). And ξ is the killing vector field if $\varphi = 0$.

$$(L_\xi g)(X,Y) = 2\varphi g(X,Y). \tag{2.12}$$

A spacetime M is said to admit a symmetry called a curvature collineation [8,9] provided there exist a vector field ξ such that

$$(L_\xi R)(X,Y)Z = 0, \tag{2.13}$$

where R is the Riemannian curvature tensor.

Now we shall investigate the role of such symmetry inheritance for the spacetime admitting generalized quasi-conformal curvature tensor with a killing vector field ξ as a curvature collineation. Then we have

$$(L_\xi g)(X,Y) = 0. \tag{2.14}$$

If M admits a curvature collineation and then (2.13) becomes

$$(L_\xi S)(X,Y) = 0. \tag{2.15}$$

where S is the Ricci tensor of the manifold.

Taking Lie derivative of (1.1) and using (2.13), (2.14), (2.15), we have

$$(L_\xi \omega)(Y, U)V = 0. \tag{2.16}$$

Hence we can state the following theorem:

Theorem 2.4. *If a spacetime M admitting the generalized quasi-conformal curvature tensor with ξ as a Killing vector field is curvature collineation, then the Lie derivative of the generalized quasi-conformal curvature tensor vanishes along the vector field ξ.*

Next, the symmetry of the energy momemtum tensor T is the matter collineation defined by

$$(L_\xi T)(X, Y) = 0, \tag{2.17}$$

where ξ is the vector field generating the symmetry and L_ξ is the lie derivative operator along the vector field ξ.

If ξ is Killing vector field on the spacetime with vanishing generalized quasi-conformal curvature tensor, then

$$(L_\xi g)(X, Y) = 0, \tag{2.18}$$

where L_ξ denotes the Lie derivative with respect to ξ.

Taking Lie derivative on equation (2.7), we get

$$(L_\xi T)(X, Y) = -\frac{1}{\kappa}\left(r\left(\frac{b_1 - \frac{3}{4}.C_1.\left(\frac{1}{3} + a_1 + b_1\right)}{1 + 3a_1 - b_1} + \frac{1}{2}\right) - \lambda\right)(L_\xi g)(X, Y). \tag{2.19}$$

From (2.18) and (2.19), we have

$$(L_\xi T)(X, Y) = 0,$$

which implies that the spacetime admits matter collineation.

Conversely, if $(L_\xi T)(X, Y) = 0$, then (2.19) becomes

$$(L_\xi g)(X, Y) = 0.$$

Hence we can state the following theorem:

Theorem 2.5. *If a spacetime obeying Einstein's field equation has vanishing generalized quasi-conformal curvature tensor, then the spacetime admits matter collineation with respect to a vector field ξ if and only if ξ is a Killing vector field.*

If ξ is a conformal Killing vector field, then

$$(L_\xi g)(X, Y) = 2\alpha g(X, Y), \tag{2.20}$$

where α is a scalar.

Using (2.20) in (2.19), we get

$$\kappa \frac{(L_\xi T)(X, Y)}{-\left(r\left(\frac{b_1 + \frac{3}{4}.C_1.\left(\frac{1}{3} + a_1 + b_1\right)}{1 + 3a_1 - b_1} + \frac{1}{2}\right) - \lambda\right)} = 2\alpha g(X, Y). \tag{2.21}$$

In view of (2.7) and (2.21), we have

$$(L_\xi T)(X,Y) = 2\alpha T(X,Y). \tag{2.22}$$

Therefore from (2.22) we can say that the energy-momentum tensor has Lie inheritance property along ξ.

Conversely, if (2.22) holds, then it follows that (2.20) holds, that is, ξ is a conformal Killing vector field. Thus we can state the following theorem.

Theorem 2.6. *If a spacetime obeying Einstein's field equation has vanishing generalized quasi-conformal curvature tensor, then the vector field ξ on the spacetime is a conformal Killing vector field if and only if the energy momemtum tensor has the Lie inheritance property along ξ.*

3. Perfect Fluid Spacetime with Vanishing Generalized Quasi-Conformal Curvature Tensor

We consider a perfect fluid spacetime with vanishing generalized quasi-conformal curvature tensor obeying Einstein's field equation without cosmological constant.

The energy momentum tensor T of a perfect fluid is given by (See [12])

$$T(X,Y) = (\delta+\rho)\gamma(X)\gamma(Y) + \rho g(X,Y), \tag{3.1}$$

where δ is the energy density, ρ the isotropic pressure and γ is a non-zero 1-form such that $g(X,U) = \gamma(X)$, $\forall\ X$. U being the velocity vector field of the flow,that is, $g(U,U) = -1$.

Einstein's field equation without cosmological constant is given by

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y), \tag{3.2}$$

where r is the scalar curvature of the manifold and $\kappa \neq 0$.

Substituting (2.4) and (3.1) in (3.2), we have

$$-\left(r\left(\frac{b_1 - \frac{3}{4}.c_1.(\frac{1}{3}+a_1+b_1)}{1+3a_1-b_1}+\frac{1}{2}\right)+\kappa\rho\right)g(X,Y) = \kappa(\delta+\rho)\gamma(X)\gamma(Y). \tag{3.3}$$

Taking a frame field and after contraction over X and Y, we get

$$r = \kappa\left(\frac{(\delta-3\rho)}{4\left(\frac{b_1-\frac{3}{4}.c_1.(\frac{1}{3}+a_1+b_1)}{1+3a_1-b_1}+\frac{1}{2}\right)}\right). \tag{3.4}$$

Substituting (3.4) in (2.4), we have

$$\begin{aligned} S(X,Y) =& \frac{\kappa(\delta-3\rho)}{4}\left(\frac{\frac{1}{2}(1+3a_1-b_1)}{b_1-\frac{3}{4}.c_1.(\frac{1}{3}+a_1+b_1)+\frac{1}{2}(1+3a_1-b_1)}-1\right) \\ & g(X,Y). \end{aligned} \tag{3.5}$$

Let Q be the Ricci operator given by $g(QX,Y) = S(X,Y)$ and $S(QX,Y) = S^2(X,Y)$. Then we have that $\gamma(QX) = g(QX,U) = S(X,U)$.

Hence equation (3.5) becomes

$$S(QX,Y) = \frac{\kappa^2(\delta-3\rho)^2}{16}\left(\frac{\frac{1}{2}(1+3a_1-b_1)}{b_1-\frac{3}{4}.c_1.(\frac{1}{3}+a_1+b_1)+\frac{1}{2}(1+3a_1-b_1)}-1\right)g(X,Y).$$

Taking a frame field and after contraction over X and Y in (3.6), we get

$$\parallel Q \parallel^2 = \frac{\kappa^2(\delta - 3\rho)^2}{4}\left(\frac{\frac{1}{2}(1+3a_1-b_1)}{b_1 - \frac{3}{4}.c_1.(\frac{1}{3}+a_1+b_1)+\frac{1}{2}(1+3a_1-b_1)} - 1\right). \tag{3.6}$$

Hence we can state the following theorem.

Theorem 3.1. *If a generalized quasi-conformally flat perfect fluid spacetime obeys Einstein's field equation without cosmological constant, then the square of the length of the Ricci operator of the spacetime is*

$$\frac{\kappa^2(\delta - 3\rho)^2}{4}\left(\frac{\frac{1}{2}(1+3a_1-b_1)}{b_1 - \frac{3}{4}.c_1.(\frac{1}{3}+a_1+b_1)+\frac{1}{2}(1+3a_1-b_1)} - 1\right).$$

Now putting $X = Y = U$ in (3.3), we get

$$r = \kappa\left(\frac{\delta}{\left(\frac{b_1 - \frac{3}{4}.c_1.(\frac{1}{3}+a_1+b_1)}{1+3a_1-b_1} + \frac{1}{2}\right)}\right). \tag{3.7}$$

In view of (3.4) and (3.7), we get

$$\delta + \rho = 0. \tag{3.8}$$

Substituting (3.8) in (3.1), we get

$$T(X,Y) = \rho g(X,Y). \tag{3.9}$$

Since the scalar curvature r of generalized quasi-conformally flat spacetime is constant. Hence from (3.7) we have δ = constant, then from (3.8) we obtain ρ = constant.

Now $\delta + \rho = 0$ means the fluid behaves as a cosmological constant (See [24]). This is also termed as phantom barrier (See [11]). Now in cosmology we know such a choice $\delta = -\rho$ leads to rapid expansion of the spacetime which is now termed as inflation (See [2]). Thus we can state the following theorem.

Theorem 3.2. *If a perfect fluid spacetime with vanishing generalized quasi-conformal curvature tensor obeying Einstein's equation without cosmological constant, then the spacetime has constant energy density and isotropic pressure and the spacetime represents inflation and also the fluid behaves as a cosmological constant.*

In [13] the Ricci tensor S of type $(0,2)$ of the spacetime satisfies condition

$$S(X,X) > 0, \tag{3.10}$$

for every timelike vector field X, then (3.10) is called the timelike convergence condition.

In view of (3.1) and (3.2), we have

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa\left((\delta+\rho)\gamma(X)\gamma(Y) + \rho(X,Y)\right), \tag{3.11}$$

Putting $X = Y = V$ in (3.11) and using (3.4), we get

$$S(V,V) = \kappa G_1.(\delta + \frac{3\rho}{G_2}), \tag{3.12}$$

where

$$G_1 = \left(\frac{2\left(\frac{4b_1 - 3.c_1.(\frac{1}{3}+a_1+b_1)}{1+3a_1-b_1} + 2\right) - 1}{\left(\frac{4b_1 - 3.c_1.(\frac{1}{3}+a_1+b_1)}{1+3a_1-b_1} + 2\right)}\right),$$

$$G_2 = 2\left(\frac{4b_1 + 3.c_1.\left(\frac{1}{3}+a_1+b_1\right)}{1+3a_1-b_1} + 2\right) - 1.$$

Since the spacetime under consideration satisfies the timelike convergence condition and $\kappa > 0$ then we have

$$(\delta + \frac{3\rho}{G_2}) > 0. \tag{3.13}$$

Thus we can state the following theorem

Theorem 3.3. *If a generalized quasi-conformally flat perfect fluid spacetime satisfying Einstein's equation without cosmological constant obeys the timelike convergence condition, then such a spacetime also satisfies cosmic strong energy condition.*

Let us suppose that the scalar curvature r of the spacetime be positive. Then from (3.4) we have that

$$\delta > 3\rho. \tag{3.14}$$

From (3.13) and (3.14) we have $\delta > 0$. This means that spacetime under consideration consists of pure matter. Thus we can state the following theorem.

Theorem 3.4. *If a generalized quasi-conformally flat perfect fluid spacetime satisfying Einstein's equation without cosmological constant obeys the timelike convergence condition, then such a spacetime contains pure matter, provided the scalar curvature r is positive.*

Taking frame field after contraction over X and Y, equation (3.2) becomes

$$r = -\kappa t \tag{3.15}$$

where $t = trace\mathrm{T}$. Equation (3.2) becomes

$$S(X,Y) = \kappa \left(T(X,Y) - \frac{t}{2} g(X,Y) \right). \tag{3.16}$$

Einstein's field equation without cosmological constant for a purely electromagnetic distribution takes the form (See [1])

$$S(X,Y) = \kappa T(X,Y). \tag{3.17}$$

In view of (3.16) and (3.17), we get $t = 0$. Thus from (3.15) we have $r = 0$. Hence equation (2.5) becomes $R(Y,U,V,X) = 0$ which means that the spacetime is flat. Hence we can state the following theorem.

Theorem 3.5. *A generalized quasi-conformally flat spacetime satisfying Einstein's equation without cosmological constant for a purely electromagnetic distribution is an Euclidean space.*

4. Dust Fluid Spacetime with Vanishing Generalized Quasi-Conformal Curvature Tensor

In a dust or pressureless fluid spacetime, the energy momentum tensor is of the form (See [23]):

$$T(X,Y) = \delta\gamma(X)\gamma(Y), \tag{4.1}$$

where δ is the energy density of the dust-like matter and γ is a non-zero 1-form such that $g(X,U) = \gamma(X)$, $\forall\ X$, U be the velocity vector field of the flow, i.e. $g(U,U) = -1$.

Using (2.7) in (4.1), we have

$$\left(\lambda - r \left(\frac{b_1 - \frac{3}{4}.C_1.\left(\frac{1}{3} + a_1 + b_1\right)}{1 + 3a_1 - b_1} + \frac{1}{2} \right) \right) g(X,Y) = \kappa\delta\gamma(X)\gamma(Y). \tag{4.2}$$

Taking frame field over X and Y, equation (4.2) becomes

$$\lambda = r \left(\frac{b_1 - \frac{3}{4}.c_1.(\frac{1}{3} + a_1 + b_1)}{1 + 3a_1 - b_1} - \frac{1}{2} \right) - \frac{\kappa\delta}{4}. \tag{4.3}$$

Replacing $X = Y = U$ in (4.2), we get

$$\lambda = r\left(\frac{b_1 - \frac{3}{4}.c_1.(\frac{1}{3} + a_1 + b_1)}{1 + 3a_1 - b_1} - \frac{1}{2}\right) - \kappa\delta. \tag{4.4}$$

In the view of (4.3) and (4.4), we get

$$\delta = 0. \tag{4.5}$$

Substituting (4.5) in (4.1), we get

$$T(X, Y) = 0. \tag{4.6}$$

Hence we can state the following theorem.

Theorem 4.1. *A generalized quasi-conformally flat dust fluid spacetime satisfying Einstein's field equation with cosmological constant is vacuum.*

The spacetime of general relativity can be studied for the concepts of the papers [3-5, 14-22].

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References

1. Ahsan, Z. and Siddiqui, S. A., *Concircular curvature tensor and fluid space-times*, Int. J. Theor. Phys., 48, 3202–3212, (2009).
2. Amendola, L. and Tsujikawa, S., *Dark Energy-Theory and Observations*, Cambridge University Press, Cambridge, (2010).
3. Angadi, P. G., Shivaprasanna, G. S., Somashekhara, G. and Siva Kota Reddy, P., *Ricci-Yamabe Solitons on Submanifolds of Some Indefinite Almost Contact Manifolds*, Adv. Math., Sci. J., 9(11), 10067-10080, (2020).
4. Angadi, P. G., Shivaprasanna, G. S., Somashekhara, G. and Siva Kota Reddy, P., *Ricci Solitons on (LCS)-Manifolds under D-Homothetic Deformation*, Italian Journal of Pure & Applied Mathematics, 46, 672-683, (2021).
5. Angadi, P. G., Siva Kota Reddy, P., Shivaprasanna, G. S. and Somashekhara, G., *On Weakly Symmetric Generalized (k, μ)-Space Forms*, Proceedings of the Jangjeon Math. Soc., 25(2), 133-144, (2022).
6. Baishya, K. K., *GRW-space-time and certain type of energy-momentum tensor*, Journal of Geometry and Physics, 157, Article Id. 103855, (2020).
7. Chaki, M. C. and Ray, S, *Space-times with covariant-constant energy-momentum tensor*, International Journal of Theoretical Physics, 35(5), 1027-1032, (1996).
8. Duggal, K. L., *Curvature collineations and conservation laws of general relativity*, Presented at the Canadian Conference on General Relativity and Relativistic Astrophysics, Halifax, Canada, (1985).
9. Duggal, K. L., *Curvature inheritance symmetry in Riemannian spaces with applications to fluid spacetimes*, J. Math. Phys., 33, 2989-2997, (1992).
10. Katzin, G. H., Levine, J. and Davis, W. R., *Curvature collineations: a fundamental property of the space-times of general relativity defined by the vanishing Lie derivative of the Riemann curvat ure tensor*, J. Math. Phys., 10, 617-629, (1969).
11. Mazumder, N., Biswas, R. and Chakraborty, S., *Cosmological evolution across phantom crossing and the nature of the horizons*, Astrophys Space Sci, 334, 183-186, (2011).
12. O'Neill B., *Semi-Riemannian geometry with applications to relativity*, Academic Press, New York-London, (1983).
13. Sach, R. K. and Hu, W., *General Relativity for Mathematician*, Springer Verlag, New York, (1977).
14. Somashekhara, G., Pavani, N. and Siva Kota Reddy, P., *Invariant Sub-manifolds of LP-Sasakian Manifolds with Semi-Symmetric Connection*, Bull. Math. Anal. Appl., 12(2), 35-44, (2020).
15. Somashekhara, G., Girish Babu, S. and Siva Kota Reddy, P., *Indefinite Sasakian Manifold with Quarter-Symmetric Metric Connection*, Proceedings of the Jangjeon Math. Soc., 24(1), 91-98, (2021).
16. Somashekhara, G., Girish Babu, S. and Siva Kota Reddy, P., *Conformal Ricci Soliton in an Indefinite Trans-Sasakian manifold*, Vladikavkaz Math. J, 23(3), 43-49, (2021).

17. Somashekhara, G., Girish Babu, S. and Siva Kota Reddy, P., *Ricci Solitons and Generalized Weak Symmetries under D-Homothetically Deformed LP-Sasakian Manifolds*, Italian Journal of Pure & Applied Mathematics, 46, 684–695, (2021).

18. Somashekhara, G., Girish Babu, S. and Siva Kota Reddy, P., *Conformal η-Ricci Solitons in Lorentzian Para-Sasakian Manifold Admitting Semi-Symmetric Metric Connection*, Italian Journal of Pure & Applied Mathematics, 46, 1008-1019, (2021).

19. Somashekhara, G., Siva Kota Reddy, P., Shivashankara, K. and Pavani, N., *Slant Sub-manifolds of Generalized Sasakian-Space-Forms*, Proceedings of the Jangjeon Math. Soc., 25(1), 83–88, (2022).

20. Somashekhara, G., Girish Babu, S., Siva Kota Reddy, P., and Shivashankara, K., *On LP-Sasakian Manifolds admitting Generalized Symmetric Metric Connection*, *Proceedings of the Jangjeon Math. Soc.*, 25(3), 287-296, (2022).

21. Somashekhara, G., Girish Babu, S. and Siva Kota Reddy, P., *η-Ricci soliton in an indefinite trans-Sasakian manifold admitting semi-symmetric metric connection*, Bol. Soc. Parana. Mat. (3), 41, 1-9, (2023).

22. Somashekhara, G., Rajendra, R., Shivaprasanna, G. S. and Siva Kota Reddy, P., *Pseudo Parallel and Generalized Ricci Pseudo Parallel Invariant Submanifolds of a Generalized Sasakian Space Form*, Proceedings of the Jangjeon Math. Soc., 26(1), 69-78, (2023).

23. Srivastava, S. K., *General Relativity and Cosmology*, PHI Learning, New Delhi, (2008).

24. Stephani, H., Kramer, D., MacCallum, M., Hoenselaers, C. and Herlt, E., *Exact Solutions of Einstein's Field Equations*, Second Edition. 732 p., Cambridge University Press, Cambridge, (2003).

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