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Sentinels of nth-Order Insensitivity for Identification Problems With High-Order Incomplete Data

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ABSTRACT: This study generalizes the definition of the sentinel function, which was introduced by J.L. Lions to study identification problems, to a more insensitive kind of sentinel, which is applied to identify pollution terms of the polynomial form w. r. t. the real λ. The main idea is to reconstruct the sentinel function to be nth-order independent of the incomplete data. Contrary to the original definition, information about the pollution term with an error of order $n+1$ is given by the sentinel of nth-order insensitivity.

Key Words: Sentinels, identification problem, incomplete data, null controllability.

Contents

1. Introduction

Identifying missing terms of PDEs from given observations on the state was a very attractive part of the studies on inverse problems. In 1988, J. L. Lions have introduced a function called "sentinel" to treat some kind of identification problems with incomplete data, where he distinguishes between important one (pollution term) and the unimportant terms. The sentinel provides information about the pollution terms via a given adjoint state [6] and [7]. The study of several identification problems by the use of sentinel method has received much attention in the last two decades (see for instance [1], [2], [4,5,6], and [11]).

Many theoretical and numerical results exist as well as applications of real physical problems motivated by researchers and industrialists, we can cite as an example the work [1,3,4] and [5]. Moreover, the method has applied to study several problems in ecology and meteorology. For example, we refer to [9] and [10] for the of problems of identification of pollution terms in distributed systems, the detection of pollution in an aquifer, the determination of missing parameters in a lake and the search for pollution in a river, also, sentinels are adapted the determination of pollution in environment.

All those previous works have only focused on the missing terms of the form $f_0(x) + \lambda f(x)$ unlike the present paper is devoted to defining a more insensitive sentinel functional, this function is expected to provide information about the pollution term of the higher order form $\sum_{k=1}^{n} \lambda^k f_k$ in contrast to the classic definition of sentinel [6], where only pollution terms of the following form λf have been considered.

Technically, we prove that the construction of an nth-order insensitive sentinel, which is linked to a control function with a specific property, is equivalent to a null controllability problem for the adjoint state.

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2. Preliminaries

Let Ω be a bounded domain of $\mathbb{R}^p$ ($p = 1, 2$ or 3) with a boundary Γ of class C^2, $Q = \Omega \times (0, T)$, and $\Sigma = \Gamma \times (0, T)$. As a typical model, we will consider the following semilinear heat equation with incomplete data

$$\left| \begin{array}{lr} \frac{\partial y}{\partial t} - \Delta y + g(y) = f_0(x) + \sum_{k=1}^{n} \lambda^k f_k(x) & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y^0(x) + \sum_{k=1}^{n} \tau^k y_k^0(x) & \text{in } \Omega. \end{array} \right. \tag{1}$$

The function $g : \mathbb{R} \to \mathbb{R}$ is nonlinear in C^1, n is a fixed number in $\mathbb{N}^*$, the source term has two parts, the first is f_0 known in $L^2(\Omega)$, the second is unknown for all $k \in \{1, ..., n\}$ and $f_k \in L^2(\Omega)$, called pollution terms.

Also, in the initial condition is divided into tow parts, y^0 is known in $L^2(\Omega)$, and $y_k^0 \in L^2(\Omega)$ for all $k \in \{1, ..., n\}$, are unknown called missing data. The real parameters λ and τ are small. Denote by $y(\lambda, \tau) = y(x, t; \lambda, \tau) \in L^2(Q)$ the unique solution of (1).

Moreover, suppose one gets a state observation on a non empty subregion $O \subset \Omega$ given by

$$y(\lambda, \tau) = y_{obs} \text{ in } O. \tag{2}$$

Problem. Can we obtain information about pollution terms without taking the missing terms into account?

Fix a function $h \in L^2(O \times (0, T))$ and take an average value, to know if something is happening. Define

$$A(\lambda, \tau) = \int_0^T \int_\Omega h y(x, t; \lambda, \tau)\, dx dt,$$

and one seeks to determine the pollution term independently of the term in, at first order for example. But, there is in general no reason to say that, $A(\lambda, \tau)$ is independent of τ. In other words, there is no reason to get

$$\frac{\partial A}{\partial \tau}(0,0) = \int_0^T \int_O h \frac{\partial y}{\partial \tau}(0,0)\, dx dt.$$

Another idea is to introduce a functional given by

$$S(\lambda, \tau) = \int_0^T \int_\Omega (h\chi_O + v\chi_\omega)\, y(x, t; \lambda, \tau)\, dx dt, \tag{3}$$

where $v \in L^2(\omega \times (0, T))$ is a control function and ω is a non empty subset of Ω, it will be fixed later [6].

Let's introduce the definition of a sentinel of nth-order insensitivity as follows

Definition 2.1. *The functional* $\mathrm{S}(\lambda, \tau)$ *is said to be a sentinel with insensitivity of order* n *if*

$$\forall\ k \in \{1, ..., n\} : \frac{\partial^k \mathrm{S}}{\partial \tau^k}(0,0) = 0, \text{ for all } y_k^0 \in L^2(\Omega), \tag{4}$$

with

$$\|v\|_{L^2(\omega \times (0,T))} = \min \|w\|_{L^2(\omega \times (0,T))}. \tag{5}$$

Note that when $n = 1$, i.e., the pollution term and the incomplete data have the forms $f_0 + \lambda f_1$ and $y^0(x) + \tau y_1^0(x)$, respectively, the definition 1 exactly matches the original definition of sentinels given by Lions in [6], this means that the definition 1 is a generalization of the classical one.

3. Existence of nth-order insensitive sentinel is equivalent to a null controllability problem

In this subsection, we prove that an nth-order insensitive sentinel exists if and only if a null controllability property for the adjoint state is verified.

Proposition 3.1. *The sentinel with insensitivity of order n defined by $(3)-(5)$ exists iff the adjoint state q solution to*

$$\left|\begin{array}{ll} -\frac{\partial q}{\partial t}-\Delta q+g'(y\,(0,0))q=h\chi_O+v\chi_\omega & \text{in } Q, \\ q=0 & \text{on } \Sigma, \\ q\,(x,T)=0 & \text{in } \Omega, \end{array}\right. \tag{6}$$

verifies the following null controllability property

$$q\,(x,0)=0 \text{ in } \Omega, \tag{7}$$

with $y\,(0,0)=y\,(\lambda,\tau)|_{\lambda=\tau=0}$.

Proof. In fact, the nth-order insensitivity condition (4) writes

$$\int_0^T\int_\Omega (h+\chi_O v)\,D_\tau^k y\,dxdt=0 \text{ for every } y_k^0\in L^2(\Omega)\,,$$

where $D_\tau^k y=\frac{\partial^k y}{\partial\tau^k}(x,t;0,0)$ is a solution to the following linear equation

$$\left|\begin{array}{ll} \frac{\partial}{\partial t}D_\tau^k y-\Delta D_\tau^k y+g'(y\,(0,0))D_\tau^k y=0 & \text{in } Q, \\ D_\tau^k y=0 & \text{on } \Sigma, \\ D_\tau^k y\,(x,0)=k!y_k^0\,(x) & \text{in } \Omega. \end{array}\right.$$

For every $1\le k\le n$, use Green formula, and (6) to find that

$$\begin{aligned} &\int_0^T\int_\Omega \left(\frac{\partial}{\partial t}D_\tau^k y-\Delta D_\tau^k y+g'(y\,(0,0))D_\tau^k y\right)q\,(x,t)\,dxdt \\ =\ &\int_0^T\int_\Omega \left(-\frac{\partial q}{\partial t}-\Delta q+g'(y\,(0,0))q\right)D_\tau^k y\,dxdt-k!\int_\Omega q\,(x,0)\,y_k^0dx \\ =\ &0 \end{aligned}$$

in other words

$$\int_\Omega q\,(x,0)\,y_k^0dx=0 \text{ for all } y_k^0\in L^2(\Omega)\,,$$

i.e.

$$q\,(x,0)=0 \text{ a.e. in } \Omega.$$

4. Solving the controllability problem $(6)-(7)$

Let's define the differential operator $L=\frac{\partial}{\partial t}-\Delta+g'(y\,(0,0))I_d$ with the adjoint given by $L^*=-\frac{\partial}{\partial t}-\Delta+g'(y\,(0,0))I_d$.

Theorem 4.1. *Let be $r\in V=\left\{\varphi\in C^\infty\left(\overline{Q}\right) \text{ such that: } \varphi=0 \text{ and } \frac{d\varphi}{dt}=0 \text{ on } \Sigma\right\}$, then there exists a positive constant $C=C\,(\Omega,\omega,T,g'(y\,(0,0)))$ such that*

$$\int_0^T\int_\Omega \frac{1}{\theta^2}|r|^2\,dxdt\le C\left[\int_0^T\int_\Omega |Lr|^2\,dxdt+\int_0^T\int_\omega |r|^2\,dxdt\right], \tag{8}$$

where $\theta\in C^2(Q)$ positive with $\frac{1}{\theta}$ bounded.

Proof. See [9].

The Carleman estimate (8) allows us to introduce the following inner product on V the Hilbert space completion of V by

$$a\left(r,s\right)=\int_{0}^{T}\int_{\Omega}\left\{Lr.Ls+\chi_{\omega}rs\right\}dxdt, \tag{9}$$

with its associated norm $\|.\|_{V}=\sqrt{a\left(.,.\right)}$.

Remark 4.2. *We can see* V *as a subspace of a weighted Sobolev space.*

In fact, let $H_{\theta}\left(Q\right)$ be the weighted Hilbert space defined by

$$H_{\theta}\left(Q\right)=\left\{\varphi\in L^{2}\left(Q\right)\ \text{such that}\ \int_{0}^{T}\int_{\Omega}\frac{1}{\theta^{2}}\left|\varphi\right|^{2}dxdt<\infty\right\}$$

endowed with the natural norm

$$\|.\|_{H_{\theta}\left(Q\right)}=\left(\int_{0}^{T}\int_{\Omega}\frac{1}{\theta^{2}}\left|.\right|^{2}dxdt\right)^{\frac{1}{2}},$$

and we can see that V *is embedded continuously in $H_{\theta}\left(Q\right)$ because of the Carleman inequality* (8).

Proposition 4.3. *The pair $\left(q,v\right)$ given by*

$$q=L\widetilde{r},\ \ v=-\widetilde{r}\chi_{\omega}$$

is a solution of the null controllability problem (6) (7)*, where $\widetilde{r}$ is the unique solution of the variational equation*

$$a\left(r,s\right)=l\left(s\right)\ \text{for all } s \text{ in } \mathrm{V},$$

and $l\left(s\right)=\int_{0}^{T}\int_{\Omega}h\chi_{O}sdxdt.$

Proof. By application of the Lax–Milgram theorem and using Carleman inequality (9) to prove the coercivity of the inner product (9). Integrate by parts to get the null controllability property (7).

5. Information given by nth-order insensitive sentinel (pollution term detection)

First, denote $D_{\lambda}^{k}y=\frac{\partial^{k}y}{\partial\lambda^{k}}\left(x,t;0,0\right)$, which verifies the following linear equation

$$\left|\begin{array}{c}\frac{\partial}{\partial t}D_{\lambda}^{k}y-\Delta D_{\lambda}^{k}y+g'(y\left(0,0\right))D_{\lambda}^{k}y=k!f_{k}\ \text{ in } Q,\\ D_{\lambda}^{k}y=0 \text{ on } \Sigma,\\ D_{\lambda}^{k}y\left(x,0\right)=0 \text{ in } \ \Omega.\end{array}\right.$$

Also, note that for all k_1,k_2 in $\{1,...,n\}$ such that $k_1+k_2=k$, it's clear that $\frac{\partial^{k}y}{\partial\lambda^{k_1}\partial\tau^{k_2}}\left(x,t;0,0\right)$ is a solution to

$$\left|\begin{array}{c}\frac{\partial}{\partial t}\left(\frac{\partial^{k}y}{\partial\lambda^{k_1}\partial\tau^{k_2}}\right)-\Delta\left(\frac{\partial^{k}y}{\partial\lambda^{k_1}\partial\tau^{k_2}}\right)+g'(y\left(0,0\right))\frac{\partial^{k}y}{\partial\lambda^{k_1}\partial\tau^{k_2}}=0\ \text{ in } Q,\\ \frac{\partial^{k}y}{\partial\lambda^{k_1}\partial\tau^{k_2}}=0 \text{ on } \Sigma,\\ \frac{\partial^{k}y}{\partial\lambda^{k_1}\partial\tau^{k_2}}\left(x,0\right)=0 \text{ in } \ \Omega.\end{array}\right.$$

From uniqueness property of linear parabolic PDEs, we deduce that $\frac{\partial^k y}{\partial\lambda^{k_1}\partial\tau^{k_2}}(0,0) = 0$ in Q, this implies that

$$\frac{\partial^k S}{\partial\lambda^{k_1}\partial\tau^{k_2}}(0,0) = \int_0^T \int_\Omega (h + \chi_O v) \frac{\partial^k y}{\partial\lambda^{k_1}\partial\tau^{k_2}}(0,0)\, dxdt = 0. \tag{7}$$

Now, a Taylor expansion to the order n for $S(\lambda,\tau)$, with the insensitivity condition (4), and (7) gives

$$\begin{aligned} S(\lambda,\tau) - S(0,0) &= \sum_{k=1}^{n} \frac{\lambda^k}{k!} \frac{\partial^k S}{\partial\lambda^k}(0,0) + O\left(\lambda^{n+1}, \tau^{n+1}\right) \\ &= \sum_{k=1}^{n} \frac{\lambda^k}{k!} \int_0^T \int_\Omega (h + \chi_O v)\, D_\lambda^k y dxdt + O\left(\lambda^{n+1}, \tau^{n+1}\right) \end{aligned}$$

with (5), we obtain

$$S(\lambda,\tau) - S(0,0) = \int_0^T \int_\Omega \left(\sum_{k=1}^{n} \lambda^k f_k \right) q dxdt + O\left(\lambda^{n+1}, \tau^{n+1}\right)$$

Note that $S(\lambda,\tau)$ is defined by the state observation (2) and $S(0,0)$ is given from the fully defined state $y(0,0)$, then, we have got information about the pollution term.

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