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Some Characterization of L^r-Henstock-Kurzweil Integrable Functions

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abstract: In this article, we discuss few properties of L^r-Henstock-Kurzweil (in short L^r-HK) integrable functions, introduced by Paul Musial in [8]. We re-defined L^r-bounded variations. We demonstrated that L^r-Henstock-Kurzweil integrable functions are Denjoy integrable.

Key Words: L^r-Henstock-Kurzweil integral, Absolute L^r-Henstock-Kurzweil integral, Denjoy integral.

Contents

1. Introduction and Preliminaries

R. A. Gordon in [4] defined the Denjoy-Dunford, Denjoy-Pettis, and Denjoy-Bochner integrals which are the extension of Dunford, Pettis, and Bochner integrals, respectively. Gordon established that a Denjoy-Dunford (Denjoy-Bochner) integrable function on $[a, b]$ is Dunford (Bochner) integrable in some interval $[a, b]$ and that for the spaces that do not contain copy c_0, a Denjoy-Pettis integrable function on $[a, b]$ is Pettis integrable on some sub interval of $[a, b]$. Major and minor functions were first introduced by de la Vallée Poussin in his study of the properties of the Lebesgue integral and those of functions additive of a set (see [12]). Entirely equivalent notions (of "Ober"- and "Unterfunktionen") were introduced independently by O. Perron [11], who based on them a new definition of integral, which does not require the theory of measure. Calderón & Zygmund first gave the notion of derivation in L^r and unlike the idea of the approximate derivative had proven to be quite effective in applications of Partial Differential Equation, area of surfaces, etc. (see [2]). L.Gordon defined the notion of Dini derivatives in metric L^r (briefly L^r-derivatives) also in his work Perron integral in L^r was discussed (see [6]). Gordon proved that AP-derivatives are equivalent to L^r derivatives. Paul M. Musial and Yoram Sagher introduced the L^r- Henstock-Kurzweil integral in [8]. P. Musial and F. Tulone obtained a norm on the space of HK_r -integrable functions, as well as the dual and completion of this space (see [10]) . Paul M. Musial defined the class of L^r-variational integrable functions and show that it is equivalent to the class of L^r- Henstock-Kurzweil integrable functions. They also define the class of functions of L^r-bounded variation (see [9]). In this paper we charecterize properties of L^r- Henstock-Kurzweil integrable functions define in $[a, b]$.

To make our presentation reasonably self-contained we recalling a few definitions and results in this section that we will use in our main section. Recalling a positive function $\delta : [a, b] \to (0, \infty)$ is a gauge (see [4]).

Definition 1.1. [4, Definition 9.3] A function $f : [a, b] \to \mathbb{R}$ is said to be Henstock-Kurzweil integrable on $[a, b]$ if there exists $A \in \mathbb{R}$ with the following property: given $\epsilon > 0$ there exists a gauge δ on $[a, b]$ such that

$$\left| \sum_{i=1}^{\mathcal{P}} f(\xi_i)|I_i| - A \right| < \epsilon$$

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for each δ-fine $\mathcal{P}-$partition $\{(I_i,\xi_i)\}_{i=1}^{\mathcal{P}}$ of $[a,b]$. We write A as $H\int_{[a,b]} f$

Recalling $I=[a,b]$ denote the family of all compact sub intervals $J\subset I$, a function $F: I\to X$ is additive if $F(J\cup L)=F(J)+F(L)$ for any non overlapping $J,L\in I$ such that $J\cup L\in I$. Recalling the space $L^r,\ 1\le r<\infty$, as

$$L^r\Big([a,b]\Big)=\Bigg\{f:\ \Big(\frac{1}{h}\int_a^b |f(x)-P(x)|^r dx\Big)^{\frac{1}{r}}<\epsilon,\ 0<h<\infty,\ for\ some\ polynomial\ \ P(x)\Bigg\}.$$

For detailed of $L^r,\ 1\le r<\infty$ one can follow [2,8,14].

Definition 1.2. *[8] Let $f\in L^r(I)$ for $1\le r<\infty$ and $I=(a,b)$. For all $x\in I$, $r-$ Dini derivative. The upper-right L^r- derivative:*

$$D_r^+f(x)=\inf\Bigg\{a:\ \Big(\frac{1}{h}\int_0^h [f(x+t)-f(x)-at]_+^r dt\Big)^{\frac{1}{r}}=o(h)\Bigg\}.$$

The lower-right L^r- derivate:

$$D_{+,r}f(x)=\sup\Bigg\{a:\ \Big(\frac{1}{h}\int_0^h [f(x+t)-f(x)-at]_-^r dt\Big)^{\frac{1}{r}}=o(h)\Bigg\}.$$

The upper-left L^r- derivate:

$$D_r^-f(x)=\inf\Bigg\{a:\ \Big(\frac{1}{h}\int_0^h [-f(x-t)+f(x)-at]_+^r dt\Big)^{\frac{1}{r}}=o(h)\Bigg\}$$

and the lower-left L^r- derivate:

$$D_{-,r}f(x)=\sup\Bigg\{a:\ \Big(\frac{1}{h}\int_0^h [-f(x-t)+f(x)-at]_-^r dt\Big)^{\frac{1}{r}}=o(h)\Bigg\}$$

Remark 1.3. $D_r^+f(x)=\inf\Bigg\{a:\ \int_0^h\Big(\frac{f(x+t)-f(x)}{t}-a\Big)_+^r dt=o(h)\Bigg\}$*, with similar results for the other r-Dini derivatives.*

Definition 1.4. *[8] For $1\le\infty$, a real valued function f is L^r- Henstock-Kurzweil integrable (in short HK_r- integrable) if there exists a function $F\in L^r[a,b]$ so that for any $\epsilon>0$ there exists a gauge function δ so that for all finite collections $\mathcal{P}=\{(x_i,[c_i,d_i])\}$ of non overlapping tagged intervals in $[a,b]$ with $P<\delta$, we have:*

$$\sum_{i=1}^n\Bigg(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}\Big|F(y)-F(x_i)-f(x_i)(y-x_i)\Big|^r dy\Bigg)^{\frac{1}{r}}<\epsilon. \tag{1.1}$$

The function f is said to be L^r-Henstock-Kurzweil integrable on the set $E\subset[a,b]$ if the function $f\chi_E$ is L^r-Henstock-Kurzweil integrable on $[a,b]$. We write

$$(L^r-H)\int_I f\chi_E=(L^r-H)\int_E f.$$

Recalling that a gauge δ is HK_r-appropriate for ϵ and for f if (1.1) holds for any δ-fine tagged partition $\mathcal{P}$. If f is HK_r-integrable on $[a,b]$, the following function is well defined for all $x\in[a,b]$:

$$F(x)=(HK_r)\int_a^x f(t)dt. \tag{1.2}$$

Let $f \in HK_r[a,b]$. The HK_r norm of f as follows:

$$||f||_{HK_r} = ||F||_r,$$

where F is the indefinite HK_r integral of f as defined in (1.2). The concept of absolute continuity which characterizes indefinite HK_r -integrals as follows:

Definition 1.5. [8, Definition 11] Let $1 \leq r < \infty$. We say that $F \in AC_r(E)$ if for all $\epsilon > 0$ there exists $\nu > 0$ and a gauge function $\delta(x)$ defined on E so that for all $\mathcal{P} = \{(x_i, [c_i, d_i])\} < \delta_E$ such that $\sum_{i=1}^{q}(d_i - c_i) < \nu$ we have

$$\sum_{i=1}^{q}(\frac{1}{d_i - c_i}\int_{c_i}^{d_i}|F(y) - F(x_i)|^r dy)^{\frac{1}{r}} < \epsilon$$

2. Bounded variation of L^r-Henstock-Kurzweil integral

Paul Musial in [9] gave the definition of L^r- bounded variation. They missed the coherent concept of $L^r[a,b]$.

Definition 2.1. [9] Let $1 \leq r \leq \infty$, let $f : [a,b] \to \mathbb{R}$ and let E be a measurable subset of $[a,b]$. We say that f is L^r- bounded variational on $E(f \in BV_r[E])$ if there exists $M > 0$ and a gauge $\delta > 0$ defined on E so that if $\mathcal{P} = \{(x_i, [c_i, d_i])\}_{i=1}^{n}$ is a finite collection of $\delta-$ fine tagged sub-intervals of $[a,b]$ having tags in E, then

$$\sum_{i=1}^{q}(\frac{1}{d_i - c_i}\int_{c_i}^{d_i}|F(y) - F(x_i)|^r dy)^{\frac{1}{r}} < M.$$

We re-write the definition of L^r-bounded variational as follows:

Definition 2.2. *Let $1 \leq r \leq \infty$, let $f : [a,b] \to \mathbb{R}$ and let E be a measurable subset of $[a,b]$. We say that f is L^r- bounded variational on $E(f \in BV_r[E])$ if there exists a function $F \in L^r([a,b])$ so that for any $M > 0$ and a gauge $\delta > 0$ defined on E so that if $\mathcal{P} = \{(x_i, [c_i, d_i])\}_{i=1}^{n}$ is a finite collection of δ- fine tagged sub-intervals of $[a,b]$ having tags in E, then*

$$\sum_{i=1}^{q}\left(\frac{1}{d_i - c_i}\int_{c_i}^{d_i}|F(y) - F(x_i)|^r dy\right)^{\frac{1}{r}} < M.$$

Paul Musial in [9] mentioned the sketch of proof of the following Theorem. We have given the full proof here so that we can use this Theorem in our results.

Theorem 2.3. [9, Theorem 2] If $f \in BV_r(E)$, then we can find $\{E_i\}_{i\geq 1}$ so that $E = \bigcup_{i=1}^{\infty} E_i$ and $f \in BV(E_i)$ for all i.

Proof. Let $f \in BV_r(E)$ then for a function $F \in L^r([a,b])$ there exists $M > 0$ and a gauge $\delta > 0$ defined on E so that $\mathcal{P} = \{(x_i, [c_i, d_i])\}_{i=1}^{n}$ is a finite collection of $\delta-$fine tagged sub intervals of $[a,b]$ having tags in E then

$$\sum_{i=1}^{n}(\frac{1}{d_i - c_i}\int_{c_i}^{d_i}|F(y) - F(x_i)|^r dy)^{\frac{1}{r}} < M. \tag{2.1}$$

Assume $F \in BV_r[a,b]$ and let $\epsilon > 0$, then for a gauge function δ defined on $[a,b]$ so that if $\mathcal{P} = \{(x_i, [c_i, d_i])\} < \delta$ such that the equation(2.1) holds.

The function F is L^r-continuous and so clearly approximately continuous, using the [8, Theorem 5] there exists $\mathcal{P}_i = \{(x_{i,j}, [c_{i,j}, d_{i,j}])\} < \delta$, where $[c_{i,j}, d_{i,j}] \subseteq [c_i, d_i]$ for all i and j, so that

$$\sum_{i=1}^{n} \frac{1}{d_{i,j} - c_{i,j}} \int_{c_{i,j}}^{d_{i,j}} |F(y) - F(x_{i,j})| dy \quad \geq \tfrac{1}{2} |F(d_i) - F(c_i)|.$$

Since $\mathcal{P} = \bigcup_{i=1}^{n} P_i$ is sub-ordinates to δ, we have

$$\sum_{i=1}^{n} |F(d_i) - F(c_i)| \leq \frac{1}{2} \sum_{i=1}^{n} \sum_{j}^{n} \frac{1}{d_{i,j} - c_{i,j}} \int_{c_{i,j}}^{d_{i,j}} |F(y) - F(x_{i,j})| dy$$
$$< \frac{1}{2} \epsilon.$$

So, $F \in BV(E_i)$. Hence we can find $f \in BV(E_i)$. □

3. L^r-Henstock-Kurzweil integral and properties

In this section we discuss few properties of L^r-Henstock-Kurzweil integrals in real space $\mathbb{R}$. The collection of all function that are L^r-Henstock integrable on $I = [a, b]$, will be denoted by $HK_r(I)$. In the beggining of the section, we discuss few properties of $BV_r[a, b]$.

Proposition 3.1. *1. Let $F \in BV_r[a, b]$ then F is bounded variation on every sub interval of $[a, b]$ and*

$$BV_r(F, [a, b]) = BV_r(F, [a, c]) + BV_r(F, [c, b])$$

for each $c \in (a, b)$.

2. If F is in $BV_r[a, c]$ and F is in $BV_r[c, b]$ then F is in $BV_r[a, b]$.

Theorem 3.2. *The function $F \in AC_r[a, b]$ is in $BV_r[a, b]$.*

Proof. Let $F \in AC_r[a, b]$ and let $\epsilon > 0$. There exists $\nu > 0$ and a gauge function δ defined on $[a, b]$ so that if $\mathcal{P} = \{(x_n, [c_n, d_n])\} < \delta$ and

$$\sum_{n=1}^{q} (d_n - c_n) < \nu$$

then $\sum\limits_{n=1}^{q} \frac{1}{d_n - c_n} \int_{c_n}^{d_n} |F(y) - F(x_n)| dy < \epsilon$. □

Theorem 3.3. *For $1 \leq r < \infty$, $BV_r[a, b] = BV[a, b]$.*

Proof. Let us assume $F \in BV[a, b]$. If $\{[c_i, d_i]\}$ is a finite collection of non overlapping intervals that have end points in E, there exists $M > 0$ such that

$$\sup \sum_{j=1}^{q} |F(d_j) - F(c_j)| < M.$$

This implies that for any $\nu > 0$ if $\sum_{j=1}^{q} (d_j - c_j) < \nu$ then

$$\sum_{j=1}^{q} (\max_{x \in [c_j, d_j]} F(x) - \min_{x \in [c_j, d_j]} F(x)) < M.$$

For any choice of $x_j \in [c_j, d_j]$,

$$\sum_{j=1}^{q} (\frac{1}{d_j - c_j} \int_{c_j}^{d_j} |F(y) - F(x_j)|^r dy)^{\frac{1}{r}} \leq \sup \sum_{j=1}^{q} (\max_{x \in [c_j, d_j]} F(x) - \min_{x \in [c_j, d_j]} F(x))$$
$$< M \ for \ any \ gauge \ function \ \delta.$$

So, $BV[a, b] \subseteq BV_r[a, b]$. also from the Theorem(2.3) $BV_r[a, b] \subseteq BV[a, b]$. Hence $BV_r[a, b] = BV[a, b]$. □

Theorem 3.4. *If $f : I = [a,b] \to \mathbb{R}$ are L^r-Henstock-Kurzweil integrable on I. If $f \geq 0$ a.e. on I then $(L^r - H)\int_I f \geq 0$.*

Proof. Let f be L^r-Henstock-Kurzweil integrable on $I = [a,b]$ then there exists a function $F \in L^r[I]$ such that for any $\epsilon > 0$ there exists a gauge function δ such that for all finite collection $\mathcal{P} = \{(x_i, [c_i, d_i])\}$ of non-overlapping tagged intervals in I with $\mathcal{P} < \delta$ implies

$$\sum_{i=1}^{n}\left(\frac{1}{d_i - c_i}\int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r dy\right)^{\frac{1}{r}} < \epsilon.$$

That is,

$$\sum_{i=1}^{n}\left(\frac{1}{d_i - c_i}\int_{c_i}^{d_i} |F(y - x_i) - S(f, \mathcal{P})|^r dy\right)^{\frac{1}{r}} < \epsilon.$$

Now by the [8, Theorem 21], $f \in L^1[a,b]$. From the fact of Lebesgue integral we get the proof. □

Remark 3.5. *The linearity of L^r-Henstock-Kurzweil integral (see [9]) and the Theorem 3.4, gives if $f \geq g$ a.e. on I then $(L^r - H)\int_I f \geq (L^r - H)\int_I g$.*

Lemma 3.6. *For $1 \leq r < \infty$, $ACG_r[a,b] = ACG[a,b]$.*

Proof. Let $E \subseteq [a,b]$. From the known fact that $ACG_r[a,b] = \bigcup AC_r[E_n]$ where $E = \bigcup_{n=1}^{\infty} E_n$. Also $AC_r[E_n] = AC[E_n]$. Therefore,

$$\begin{aligned} ACG_r[a,b] &= \bigcup AC[E_n] \\ &= ACG[E]. \end{aligned}$$

Consequently, $ACG_r[a,b] = ACG[a,b]$. □

We can find from the known fact that $HK_r(I)$ is contained in $L^1(I)$, then any function in $HK_r(I)$ is Denjoy integrable. That is:

Theorem 3.7. *Let $f : I = [a,b] \to \mathbb{R}$. For $1 \leq r < \infty$, if f is L^r-Henstock-Kurzweil integrable function is Denjoy integrable.*

Theorem 3.8. *Let $f : I \to \mathbb{R}$ be L^r-Henstock-Kurzweil integrable on I. Then $|f| \in HK_r(I)$ if and only if the indefinite integral $F(x) = \int_a^x f$ has $BV_r(I)$.*

Proof. The proof is immediate. Since f is in $HK_r(I)$, then f is in $L^1(I)$. Therefore, $F(x)$ is of bounded variation, which tell us that f is in $BV_r(I)$. See [1, Theorem 7.5]. □

Corollary 3.9. *Let $f : [a,b] \to \mathbb{R}$ be L^r-Henstock-Kurzweil integrable function on $[a,b]$. L^r-Henstock-Kurzweil integrable function are absolutely integrable function on $[a,b]$.*

Theorem 3.10. *The function $f : I = [a,b] \to \mathbb{R}$.*

1. *If f is L^r-Henstock-Kurzweil integrable then f is measurable.*
2. *If f is L^r-Henstock-Kurzweil integrable on $[a,b]$ and $f \geq 0$ a.e then f is Lebesgue integrable on $[a,b]$.*

Proof. For (1) Let f be L^r-Henstock-Kurzweil integrable on $I = [a, b]$ and F is the L^r-Henstock-Kurzweil integral of f, then [8, Theorem 14] there exists $F \in ACG_r[a, b]$ so that $F'_r = f$ a.e. so that I is the sum of a sequence $\{E_n\}$ of closed sets on each of which F is L^r-AC. Again [8, Theorem 15] gives F is AC. [13, Lemma 4.1 of Ch VII] there exists for each n a function E_n of bounded variation on I, which coincides with F on E_n. We therefore have a.e. on E_n the relation $f(x) = F'_r(x) = F'_n(x)$ where $F'_r(x)$ is L^r- derivative of F and since the derivative of a function is bounded variation is measurable and a.e. finite, it follows that f is measurable and a.e. finite on each E_n and consequently on the whole interval $I = [a, b]$.
For (2), follows [8, Theorem 21]. □

Corollary 3.11. *If $f : [a, b] \to \mathbb{R}$ be L^r-Henstock-Kurzweil integrable on $[a, b]$. The following are holds:*

a) If f is bounded on $[a, b]$ then f is clearly Lebesgue integrable on $[a, b]$.

b) If $f \geq 0$ a.e. is L^r-Henstock integrable on every measurable subset of $[a, b]$ then f is Lebesgue integrable on $[a, b]$.

Theorem 3.12. *Let $f : [a, b] \to \mathbb{R}$. If f is L^r-Henstock-Kurzweil integrable on $[a, b]$ then every perfect set in $[a, b]$ contains a portion on which f is Lebesgue integrable.*

Proof. Let f be L^r-Henstock-Kurzweil integrable on $[a, b]$ then the Theorem(3.7), f is Denjoy integrable on $[a, b]$. Using [4, Theorem 12(c)], we found every perfect set in $[a, b]$ a portion on which f is Lebesgue integrable. □

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