Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–14.
ISSN-0037-8712 in press
doi:10.5269/bspm.66957

Pythagorean Fuzzy Nil Radical of Pythagorean Fuzzy Ideal

A. Talhaoui, S. Melliani, I. Bakhadach and Sofyane Achik

ABSTRACT: In this work, we introduce the Pythagorean fuzzy nil radical of a Pythagorean fuzzy ideal of a commutative ring, we further provide the notion of Pythagorean fuzzy semiprime ideal, and we study some related properties. Finally, we give the relation between Pythagorean fuzzy semiprime ideals and the Pythagorean fuzzy nil radical of a commutative ring.

Key Words: Pythagorean fuzzy ring, Pythagorean fuzzy ideal, Pythagorean fuzzy nil radical, Pythagorean fuzzy semiprime ideals.

Contents

1. Introduction

The theory of Pythagorean fuzzy sets (PFS) is extremely important to deal with uncertainties and vagueness of the most problems occurring in the real world, and that's thanks to its flexibility of measuring the fuzziness and imprecision, by considering the membership grade μ and nonmembership grade ν, satisfying the condition $\mu^2 + \nu^2 \leq 1$. This model is introduced by Yager in 2013 [9] like a generalization of Atanassov's intuitionistic fuzzy sets(IFS) [1], which is an extension of Zadeh's theory [11]. The IFSs is useful in practical multiple attribute decision making (MADM) problems. In 2014 The Pythagorean fuzzy sets theory had been developed [10] then Zhang and Xu [12] established the idea of Pythagorean fuzzy number (PFN). Additionally, in 2016 Garg [5] considered the applications of PFSs in decision-making problems. And Several researchers studied Pythagorean algebraic structure. For example in 2021 S. Bhunia and G. Ghorai represented the notion of Pythagorean fuzzy subgroup [4]. And in 2022 they introduced the notions of (α, β) Pythagorean fuzzy subring and (α, β) Pythagorean fuzzy ideal of a ring [3].
In this paper we introduce the notion of a Pythagorean fuzzy nil radical of a Pythagorean fuzzy ideal of a commutative ring, secondly we define the concept of Pythagorean fuzzy semiprime ideal and investigate some of its properties. after all we establish the vital connection between a Pythagorean fuzzy semiprime ideal and the Pythagorean fuzzy nil radical of a Pythagorean fuzzy ideal.

2. Preliminaries

We would like to reproduce some definitions and results which were proposed earlier by the pioneers in this field.

Definition 2.1. *[11] A mapping of a non-empty set S into the closed unit interval $[0, 1]$ is a fuzzy subset of S The set of all fuzzy subsets of S is denoted by $F(S)$. If $A \subseteq S$, then χ_A stands for the characteristic function of A in S.*

2010 *Mathematics Subject Classification:* 35B40, 35L70.
Submitted November 22, 2022. Published March 15, 2023

Definition 2.2. *[7]*

A fuzzy subset λ of a group G is a fuzzy subgroup of G if

1- $\min\{\lambda(a),\ \lambda(b)\} \leq \lambda(ab)$ *, and*

2- $\lambda(a^{-1}) = \lambda(a)$ *for all a, b in G.*

Equivalently, $\min\{\lambda(a),\ \lambda(b)\} \leq \lambda(ab^{-1})$. *In that case, $\lambda(x) \leq \lambda(e)$ for all x in G.*

Definition 2.3. *A fuzzy subset λ of a ring R is a fuzzy left ideal of R if*

1- $\min\{\lambda(a),\ \lambda(b)\} \leq \lambda(a-b)$ *, and*

2- $\lambda(b) \leq \lambda(ab)$ *for all a, b in R.*

A fuzzy subset λ of a ring R is a fuzzy right ideal of R if

1- $\min\{\lambda(a),\ \lambda(b)\} \leq \lambda(a-b)$ *, and*

2- $\lambda(a) \leq \lambda(ab)$ *for all a, b in R.*

If λ is both a fuzzy left and fuzzy right ideal of R, then λ is a fuzzy ideal of R.

Definition 2.4. *[8] If A and B two fuzzy subsets of a groupoide D. Then the product $A \circ B$ is a fuzzy subset of D defined as follows:*

$$(A \circ B)(x) = \begin{cases} \sup\limits_{y \cdot z = x} \min(A(y), B(z)) & \text{for } y, z \in D, y \cdot z = x, \\ 0 & \text{for any } y, z \in D, y \cdot z \neq x. \end{cases}$$

If the composition in D is commutative and associative, the product $A \circ B$ is commutative and associative respectively.

Definition 2.5. *An ideal I of a ring R is said to be semiprime if, whenever a^n belongs to I for some a in R and some positive integer n, then a belongs to I.*

Theorem 2.6. *A commutative ring R is regular iff every ideal of R is semiprime.*

Definition 2.7. *[8] Let f be a mapping from a set S to a set T. Let λ be a fuzzy subset of S and μ be a fuzzy subset of T. Then the inverse image $f^{-1}(\mu)$ of μ is the fuzzy subset of S defined by $f^{-1}(\mu)(x) = \mu(f(x))$, $x \in S$.*

$$(f(\mu))(y) = \begin{cases} \sup\limits_{f(x)=y} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{if } f^{-1}(y) = \emptyset, \end{cases}$$

From now on, throughout this paper R will denote a commutative ring, unless otherwise specified.

Definition 2.8. *Let $(C, \circ)$ be a group and $\psi = (\mu, v)$ be a PFS of C. Then ψ is said to be a PFSG of C if the following conditions hold:*

i- $\mu^2(m \circ n) \geq \mu^2(m) \wedge \mu^2(n)$ *and* $v^2(m \circ n) \leq v^2(m) \vee v^2(n) \forall m, n \in C$,

ii- $\mu^2\left(m^{-1}\right) \geq \mu^2(m)$ *and* $v^2\left(m^{-1}\right) \leq v^2(m) \forall m \in C$.

Definition 2.9. *Assume $(W, +, .)$ is a ring and $\psi^* = (\mu^\alpha, \nu^\beta)$ is an (α, β) Pythagorean fuzzy set of W. Then ψ^* is said to be an (α, β) Pythagorean fuzzy subring (PFSR) if:*

i- $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ *and* $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ *for all* $w_1, w_2 \in W$.

ii- $\mu^\alpha(w_1.w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ *and* $\nu^\beta(w_1.w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ *for all* $w_1, w_2 \in W$

Definition 2.10. *Let $(W, +, .)$ be a ring and $\psi^* = (\mu^\alpha, \nu^\beta)$ is an (α, β) Pythagorean fuzzy set of W. Then ψ^* is an (α, β) Pythagorean fuzzy ideal(PFID) if:*

i- $\mu^\alpha(w_1 - w_2) \geq \mu^\alpha(w_1) \wedge \mu^\alpha(w_2)$ *and* $\nu^\beta(w_1 - w_2) \leq \nu^\beta(w_1) \vee \nu^\beta(w_2)$ *for all* $w_1, w_2 \in W$.

ii- $\mu^\alpha(w_1.w_2) \geq \mu^\alpha(w_1) \vee \mu^\alpha(w_2)$ *and* $\nu^\beta(w_1.w_2) \leq \nu^\beta(w_1) \wedge \nu^\beta(w_2)$ *for all* $w_1, w_2 \in W$.

3. Pythagorean fuzzy nil radical

In this section let R denote a commutative ring unless otherwise specified.

Definition 3.1. *Let $A = (\mu_A,\ \gamma_A)$ be a Pythagorean fuzzy ideal of R. The Pythagorean fuzzy nil radical of $A = (\mu_A,\ \gamma_A)$ is defined to be a Pythagorean fuzzy set $\sqrt{A} = (\mu_{\sqrt{A}},\ \gamma_{\sqrt{A}})$ in R defined by*

$$\mu_{\sqrt{A}}(x) = \bigvee_{n\geq 1} \mu_A(x^n)\ ,\ \gamma_{\sqrt{A}}(x) = \bigwedge_{n\geq 1} \gamma_A(x^n)$$

for all $x \in R$ and some $n \in \mathbb{N}$.

Remark 3.2. $\mu_{\sqrt{A}}(x) = \lim_n \mu_A(x^n)$(*resp.* $\gamma_{\sqrt{A}}(x) = \lim_n \gamma_A(x^n)$) .

Proof. for all $x \in R$ and $n \in \mathbb{N}$. $\mu_A^2(x^n) + \gamma_A^2(x^n)) \leq 1$
Then

$$\lim_n \mu_A^2(x^n) + \gamma_A^2(x^n)) \leq 1.$$

Then

$$\mu^2_{\sqrt{A}}(x) + \gamma^2_{\sqrt{A}}(x) \leq 1.$$

Then $\sqrt{A}$ is a P.F.S. □

Proposition 3.3. . *For every Pythagorean fuzzy ideals $A = (\mu_A,\ \gamma_A)$ and $B = (\mu_B,\ \gamma_B)$ of R, we have*

i) $A \subseteq \sqrt{A}$,

ii) $A \subset B$ *implies* $\sqrt{A} \subseteq \sqrt{B}$.

iii) $\sqrt{\sqrt{A}} = \sqrt{A}$.

Theorem 3.4. *For any Pythagorean fuzzy ideal $A = (\mu_A, \gamma_A)$ of R, $\sqrt{A} = (\mu_{\sqrt{A}},\ \gamma_{\sqrt{A}})$ is a Pythagorean fuzzy ideal of R.*

Proof. Let $x, y \in R$. Then

$$\begin{aligned}\mu^2_{\sqrt{A}}(x) \wedge \mu^2_{\sqrt{A}}(y) &= (\bigvee_{m\geq 1} \mu_A^2(x^m)) \wedge (\bigvee_{n\geq 1} \mu_A^2(x^m)) \\ &= \bigvee_{m\geq 1}\bigvee_{n\geq 1} (\mu_A^2(x^m) \wedge \mu_A^2(x^m)) \quad (3.1),\end{aligned}$$

$$\begin{aligned}\gamma^2_{\sqrt{A}}(x) \vee \gamma^2_{\sqrt{A}}(y) &= \bigwedge_{m>1} \gamma_A^2(x^m) \vee \bigwedge_{n>1} \gamma_A^2(y^n) \\ &= \bigwedge_{m>1} (\bigwedge_{n>1} (\gamma^2_{\sqrt{A}}(x^m) \vee \gamma^2_{\sqrt{A}}(y^n))) \quad (3.2).\end{aligned}$$

On the other hand, there exist $r, t \in R$ such that $(x + y)^{m+n} = rx^m + ty^n$.
Thus

$$\begin{aligned}\mu_A^2(x^m) \wedge \mu_A^2(y^n) &\leq (\mu_A^2(x^m) \vee \mu_A^2(r)) \wedge (\mu_A^2(y^n) \vee \mu_A^2(t)) \\ &\leq \mu_A^2(rx^m) \wedge \mu_A^2(ty^n) \\ &\leq \mu_A^2(rx^m + ty^n) \\ &\leq \mu^2_{\sqrt{A}}(x + y).\end{aligned}$$

So

$$\mu^2_{\sqrt{A}}(x) \wedge \mu^2_{\sqrt{A}}(y) \leq \mu^2_{\sqrt{A}}(x+y).$$

Similarly we can show that

$$\gamma^2_{\sqrt{A}}(x) \vee \gamma^2_{\sqrt{A}}(y) \geq \gamma^2_{\sqrt{A}}(x+y). \quad (3.4)$$

Noticing that

$$\mu^2_A(x-y) \geq \mu^2_A(x) \wedge \mu^2_A(y) \Leftrightarrow \mu^2_A(x+y) \geq \mu^2_A(x) \wedge \mu^2_A(y)\ ,$$

and

$$\gamma^2_A(x-y) \leq \gamma^2_A(x) \vee \gamma^2_A(y) \Leftrightarrow \gamma^2_A(x+y) \leq \gamma^2_A(x) \vee \gamma^2_A(y)\ ,$$

it follows from (3.1) and (3.2) that

$$\mu^2_{\sqrt{A}}(x+y) \geq \mu^2_{\sqrt{A}}(x) \wedge \mu^2_{\sqrt{A}}(y) \quad \text{and} \quad \gamma^2_{\sqrt{A}}(x+y) \leq \gamma^2_{\sqrt{A}}(x) \vee \gamma^2_{\sqrt{A}}(y)$$

Next we have

$$\mu^2_{\sqrt{A}}(x) \vee \mu^2_{\sqrt{A}}(y) = \bigvee_n \mu^2_A(x^n) \vee \bigvee_n \mu^2_A(y^n) = \bigvee_n (\mu^2_A(x^n) \vee \mu^2_A(y^n))\ , \quad (3.5)$$

$$\gamma^2_{\sqrt{A}}(x) \wedge \gamma^2_{\sqrt{A}}(y) = \bigwedge_n \gamma^2_A(x^n) \wedge \bigwedge_n \gamma^2_A(y^n) = \bigwedge_n (\gamma^2_A(x^n) \wedge \gamma^2_A(y^n))\ . \quad (3.6)$$

Since

$$\mu^2_A(x^n) \vee \mu^2_A(y^n) \leq \mu^2_A(x^n y^n) = \mu^2_A((xy)^n) \leq \bigvee_{k\geq 1} \mu^2_A((xy)^k) = \mu^2_{\sqrt{A}}(xy)$$

and

$$\gamma^2_A(x^n) \wedge \gamma^2_A(y^n) \geq \gamma^2_A(x^n y^n) = \gamma^2_A((xy)^n) \geq \bigwedge_{k>1} \gamma^2_A((xy)^k) = \gamma^2_{\sqrt{A}}(xy)\ ,$$

From (3.5) and (3.6), we get
$\mu^2_{\sqrt{A}}(xy) \geq \mu^2_{\sqrt{A}}(x) \vee \mu^2_{\sqrt{A}}(y)$ and $\gamma^2_{\sqrt{A}}(xy) \leq \gamma^2_{\sqrt{A}}(x) \wedge \gamma^2_{\sqrt{A}}(y)$. □

Definition 3.5. *Let $A = (\mu_A,\ \gamma_A)$ and $B = (\mu_B,\ \gamma_B)$ be Pythagorean fuzzy sets in a ring R (not necessarily commutative). The Pythagorean intrinsic product of $A = (\mu_A,\ \gamma_A)$ and $B = (\mu_B,\ \gamma_B)$ is defined to be the Pythagorean fuzzy set $A * B = (\mu_{A*B},\ \gamma_{A*B})$ in R given by*

$$\mu_{A*B}(x) := \vee\{ \bigwedge_{1\leq i\leq k} \mu_A(a_i) \wedge \mu_B(b_i) : \sum_{i=1}^{k} a_i b_i = x,\ k \in \mathbb{N}\},$$

$$\gamma_{A*B}(x) := \wedge\{ \bigvee_{1\leq i\leq k} \gamma_A(a_i) \vee \gamma_B(b_i) : \sum_{i=1}^{k} a_i b_i = x,\ k \in \mathbb{N}\},$$

if we can express $x = \sum_{i=1}^{k} a_i b_i$ for some $a_i, b_i \in R$, where each $a_i b_i \neq 0$ and $k \in \mathbb{N}$. Otherwise, we define $A * B = 0_\sim$, i.e., $\mu_{A*B}(x) = 0$ and $\gamma_{A*B}(x) = 1$.

Obviously the product $A * B$ is commutative if R is a commutative ring.

Proof. let's show that $A * B$ is a Pythagorean fuzzy set.

$$\mu^2_{A*B}(x) + \gamma^2_{A*B}(x) = \vee(\bigwedge_{1\leq i\leq k} \mu^2_A(a_i) \wedge \mu^2_B(b_i)) + \wedge(\bigvee_{1\leq i\leq k} \gamma^2_A(a_i) \vee \gamma^2_B(b_i))\ .$$

we take

$$\vee(\bigwedge_{1\leq i\leq k} \mu^2_A(a_i) \wedge \mu^2_B(b_i)) = (\mu^2_A(a_j) \wedge \mu^2_B(b_j)).$$

Then

$$\mu^2_{A*B}(x) + \gamma^2_{A*B}(x) \le (\mu^2_A(a_j) \wedge \mu^2_B(b_j)) + (\gamma^2_A(a_j) \vee \gamma^2_B(b_j)).$$

Suppose that

$$\gamma^2_B(b_j) = (\gamma^2_A(a_j) \vee \gamma^2_B(bj).$$

then

$$\mu^2_{A*B}(x) + \gamma^2_{A*B}(x) \le \mu^2_B(b_j) + \gamma^2_B(b_j) \le 1.$$

So $A * B$ is a Pythagorean fuzzy set. □

Theorem 3.6. *If $A = (\mu_A,\ \gamma_A)$ and $B = (\mu_B,\ \gamma_B)$ are Pythagorean fuzzy ideals of R, then so is $A * B = (\mu_{A*B},\ \gamma_{A*B})$.*

Proof. For any $x, y \in R$, we have

$$\mu^2_{A*B}(x-y) = \vee\{\bigwedge_{1\le i\le k} \mu^2_A(\alpha_i) \wedge \mu^2_B(\beta_i) : x - y = \sum_{i=1}^{k} \alpha_i\beta_i, k \in \mathbb{N}\}$$

$$\ge \vee\{(\bigwedge_{1\le i\le m} \mu^2_A(a_i) \wedge \mu^2_B(b_i)) \wedge (\bigwedge_{1\le i\le n} \mu^2_A(-c_i) \wedge \mu^2_B(d_i)) : x = \sum_{i=1}^{m} a_ib_i,\ -y = \sum_{i=1}^{m} -c_id_i, m, n \in \mathbb{N}\}$$

$$= \vee\{(\bigwedge_{1\le i\le m} \mu^2_A(a_i) \wedge \mu^2_B(b_i)) \wedge (\bigwedge_{1\le i\le n} \mu^2_A(c_i) \wedge \mu^2_B(d_i)) : x = \sum_{i=1}^{m} a_ib_i, y = \sum_{i=1}^{m} c_id_i, m, n \in \mathbb{N}\}$$

$$= \vee\{\bigwedge_{1\le i\le m} \mu^2_A(a_i) \wedge \mu^2_B(b_i) : x = \sum_{i=1}^{m} a_ib_i, m \in \mathbb{N}\} \wedge \vee\{\bigwedge_{1\le i\le n} \mu^2_A(c_i) \wedge \mu^2_B(d_i) : y = \sum_{i=1}^{m} c_id_i, n \in \mathbb{N}\}$$

$$= \mu^2_{A*B}(x) \wedge \mu^2_{A*B}(y)$$

$$\gamma^2_{A*B}(x-y) = \wedge\{\bigvee_{1\le i\le k} \gamma^2_A(\alpha_i) \vee \gamma^2_B(\beta_i) : x - y = \sum_{i=1}^{k} \alpha_i\beta_i, k \in \mathbb{N}\}$$

$$\le \wedge\{(\bigvee_{1\le i\le m} \gamma^2_A(a_i) \vee \gamma^2_B(b_i)) \vee (\bigvee_{1\le i\le n} \gamma^2_A(-c_i) \vee \gamma^2_B(d_i)) : x = \sum_{i=1}^{m} a_ib_i,\ -y = \sum_{i=1}^{m} -c_id_i, m, n \in \mathbb{N}\}$$

$$= \wedge\{(\bigvee_{1\le i\le m} \gamma^2_A(a_i) \vee \gamma^2_B(b_i)) \vee (\bigwedge_{1\le i\le n} \gamma^2_A(c_i) \vee \gamma^2_B(d_i)) : x = \sum_{i=1}^{m} a_ib_i, y = \sum_{i=1}^{m} c_id_i,\ m, n \in \mathbb{N}\}$$

$$= \wedge\{\bigvee_{1\le i\le m} \gamma^2_A(a_i) \vee \gamma^2_B(b_i) : x = \sum_{i=1}^{m} a_ib_i, m \in \mathbb{N}\} \vee \wedge\{\bigvee_{1\le i\le n} \gamma^2_A(c_i) \vee \gamma^2_B(d_i) : y = \sum_{i=1}^{m} c_id_i, n \in \mathbb{N}\}$$

$$= \gamma^2_{A*B}(x) \vee \gamma^2 a_{A*B}(y)\ .$$

Also, we have

$$\begin{aligned}
\mu^2_{A*B}(x) &= \vee\{\bigwedge_{1\le i\le m} \mu^2_A(a_i) \wedge \mu^2_B(b_i) : x = \sum_{i=1}^{m} a_ib_i, m \in \mathbb{N}\} \\
&\le \vee\{\bigwedge_{1\le i\le m} \mu^2_A(a_i) \wedge \mu^2_B(b_iy)\ :\ xy = \sum_{i=1}^{m} a_i(b_iy),\ m,\ \in \mathbb{N}\} \\
&\le \vee\{\bigwedge_{1\le i\le k} \mu^2_A(\alpha_i) \wedge \mu^2_B(\beta_i) : xy = \sum_{i=1}^{k} \alpha_i\beta_i,\ k,\ \in \mathbb{N}\} = \mu^2_{A*B}(xy).
\end{aligned}$$

and

$$\gamma^2_{A*B}(x) = \vee\{_{1\leq} \bigvee_{i\leq m} \gamma^2_A(a_i) \vee \gamma^2_B(b_i) : x = \sum_{i=1}^{m} a_i b_i, m \in \mathbb{N}\}$$

$$\geq \wedge\{_{1\leq} \bigvee_{i\leq m} \gamma^2_A(a_i) \vee \gamma^2_B(b_i y) \; : \; xy = \sum_{i=1}^{m} a_i(b_i y), \; m, \; \in \mathbb{N}\}$$

$$\geq \wedge\{_{1\leq} \bigvee_{i\leq k} \gamma^2_A(\alpha_i) \vee \gamma^2_B(\beta_i) : xy = \sum_{i=1}^{k} \alpha_i \beta_i, \; k, \; \in \mathbb{N}\} = \gamma^2_{A*B}(xy) \; .$$

Hence

$$\mu^2_{A*B}(x) \leq \mu^2_{A*B}(xy) \;\; and \;\; \gamma^2_{A*B}(x) \geq \gamma^2_{A*B}(xy).$$

Similarly,we get

$$\mu^2_{A*B}(y) \leq \mu^2_{A*B}(xy) \quad \text{and} \quad \gamma^2_{A*B}(y) \geq \gamma^2_{A*B}(xy).$$

Therefore $A * B = (\mu_{A*B}, \; \gamma_{A*B})$ is a Pythagorean fuzzy ideal of R. □

Theorem 3.7. *If $A = (\mu_A, \; \gamma_A)$ and $B = (\mu_B, \; \gamma_B)$ are Pythagorean fuzzy ideals of R, then*

$$\sqrt{A * B} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}.$$

Proof. Let $x \in R$ such that

$x = \sum_{i=1}^{m} a_i b_i$ where $a_i b_i \neq 0$ in R. (3.7)

Then

$$\mu^2_A(a_i) \wedge \mu^2_B(b_i) \leq \mu^2_A(a_i) \leq \mu^2_A(a_i b_i)$$

and

$$\gamma^2_A(a_i) \vee \gamma^2_B(b_i) \geq \gamma^2_A(a_i) \geq \gamma^2_A(a_i b_i) \qquad for 1 \leq i \leq m.$$

Thus

$$\bigwedge_{1\leq i\leq m} \mu^2_A(a_i) \wedge \mu^2_B(b_i) \;\; \leq \;\; \bigwedge_{1\leq i\leq m} \mu^2_A(a_i b_i) \;\; \leq \;\; \mu^2_A(\sum_{i=1}^{m} a_i b_i) = \mu^2_A(x),$$

$$\bigvee_{1\leq i\leq m} \gamma^2_A(a_i) \vee \gamma^2_B(b_i) \;\; \geq \;\; \bigvee_{1\leq i\leq m} \gamma^2_A(a_i b_i) \;\; \geq \;\; \gamma^2_A(\sum_{i=1}^{m} a_i b_i) = \gamma^2_A(x).$$

Taking the supremum and infinimum, respectively, over all expressions like (3.7), we have

$$\mu^2_{A*B}(x) \leq \mu^2_A(x) \quad \text{and} \quad \gamma^2_{A*B}(x) \geq \gamma^2_A(x).$$

Similarly,

$$\mu^2_{A*B}(x) \leq \mu^2_B(x) \quad \text{and} \quad \gamma^2_{A*B}(x) \geq \gamma^2_B(x).$$

Hence $\mu^2_{A*B}(x) \leq \mu^2_A(x) \wedge \mu^2_B(x)$ and $\gamma^2_{A*B}(x) \geq \gamma^2_A(x) \vee \gamma^2_B(x)$ for all $x \in R$, that is, $A * B \subseteq A \cap B$. Using Proposition 3.3, we obtain $\sqrt{A * B} \subseteq \sqrt{A \cap B}$.
Now, for any $x \in R$, we have

$$\mu^2_{\sqrt{A*B}}(x) = \bigvee_{k} \mu^2_{A*B}(x^k) \geq \mu^2_{A*B}(x^{2n}) \geq \mu^2_A(x^n) \wedge \mu^2_B(x^n) = \mu^2_{A\cap B}(x^n) \; ,$$

$$\gamma^2_{\sqrt{A*B}}(x) = \bigwedge_k \gamma^2_{A*B}(x^k) \leq \gamma^2_{A*B}(x^{2n}) \leq \gamma^2_A(x^n) \vee \gamma^2_B(x^n) = \gamma^2_{A\cap B}(x^n)$$

for all $n \geq 1$. Taking the supremum and infinimum, respectively, over all $n \geq 1$, we get $\mu^2_{\sqrt{A\cap B}}(x) \leq \mu^2_{\sqrt{A*B}}(x)$ and $\gamma^2_{\sqrt{A\cap B}}(x) \geq \gamma^2_{\sqrt{A*B}}(x)$.
This proves the first equality.
Proposition 3.3(ii) implies $\sqrt{A\cap B} \subseteq \sqrt{A}$ and $\sqrt{A\cap B} \subseteq \sqrt{B}$, and so $\sqrt{A\cap B} \subseteq \sqrt{A} \cap \sqrt{B}$. On the other hand, let $x \in R$.
Then

$$(\mu^2_{\sqrt{A}} \wedge \mu^2_{\sqrt{B}})(x) = \vee\mu^2_A(x^m) \wedge \vee\mu^2_B(x^n) = \bigvee_m (\bigvee_n \mu^2_A(x^m) \wedge \mu^2_B(x^n)) \ ,$$

$$(\gamma^2_{\sqrt{A}} \vee \gamma^2_{\sqrt{B}})(x) = \bigwedge_m \gamma^2_A(x^m) \vee \bigwedge_n \gamma^2_B(x^n) = \bigwedge_m (\bigwedge_n \gamma^2_A(x^m) \vee \gamma^2_B(x^n)) \ .$$

Now let m and n be any positive integers. Then

$$\mu^2_A(x^m) \wedge \mu_B(x^n) \leq \mu^2_A(x^{mn}) \wedge \mu^2_B(x^{mn}) = (\mu^2_A \wedge \mu^2_B)(x^{mn}) \leq \bigvee_{k\geq 1} (\mu^2_A \wedge \mu^2_B)(x^k) = \mu^2_{\sqrt{A\cap B}}(x)$$

$$\gamma^2_A(x^m) \vee \gamma^2_B(x^n) \geq \gamma^2_A(x^{mn}) \vee \gamma^2_B(x^{mn}) = (\gamma^2_A \vee \gamma^2_B)(x^{mn}) \geq \bigwedge_{k>1} (\gamma^2_A \vee \gamma^2_B)(x^k) = \gamma^2_{\sqrt{A\cap B}}(x) \ .$$

Hence $\sqrt{A}\sqrt{B} \subseteq \sqrt{A\cap B}$. This proves the second equality. □

Definition 3.8. *Let $A = (\mu_A,\ \gamma_A)$ and $B = (\mu_B,\ \gamma_B)$ be a Pythagorean fuzzy sets in a ring R (not necessarily commutative). The Pythagorean sum of $A = (\mu_A, \gamma_A)$ and $B = (\mu_B, \gamma_B)$ is defined to be the Pythagorean fuzzy set $A \oplus B = (\mu_{A\oplus B},\ \gamma_{A\oplus B})$ in R given by*

$$\mu_{A\oplus B}(x) := \begin{cases} \bigvee\limits_{x=y+z} (\mu_A(y) \wedge \mu_B(z)) & if \quad x = y+z, \\ 0 & otherwise. \end{cases}$$

$$\gamma_{A\oplus B}(x) := \begin{cases} \bigwedge\limits_{x=y+z} \{\gamma_A(y) \vee \gamma_B(z)\} & if \quad x = y+z, \\ 1 & otherwise. \end{cases}$$

Proof. let's show that $A \oplus B$ is a P.F.S

$$\mu^2_{A\oplus B}(\alpha) + \gamma^2_{A\oplus B}(\beta) = \bigvee_{x=y+z} \{\mu^2_A(y) \wedge \mu^2_B(z)\} + \bigwedge_{x=y+z} \{\gamma^2_A(y) \vee \gamma^2_B(z)\}$$

Let y_1 and z_1 such as

$$\mu^2_A(y_1) \wedge \mu^2_B(z_1) = \bigvee_{x=y+z} \{\mu^2_A(y) \wedge \mu^2_B(z)\}$$

Then

$$\mu^2_{A\oplus B}(\alpha) + \gamma^2_{A\oplus B}(\beta) \leq (\mu^2_A(y_1) \wedge \mu^2_B(z_1)) + (\gamma^2_A(y_1) \vee \gamma^2_B(z_1))$$

We suppose that

$$\gamma^2(y_1) = \gamma^2_A(y_1) \vee \gamma^2_B(z_1)$$

So

$$\mu^2_{A\oplus B}(\alpha) + \gamma^2_{A\oplus B}(\beta) \leq \mu^2(y_1) + \gamma^2(z_1) \leq 1.$$

□

Theorem 3.9. *If $A = (\mu_A,\ \gamma_A)$ and $B = (\mu_B,\ \gamma_B)$ are Pythagorean fuzzy ideals of R, then so is $A \oplus B = (\mu_{A\oplus B},\ \gamma_{A\oplus B})$.*

Proof. For any $x, y \in R$, we haye

$$\begin{aligned}
\mu^2_{A\oplus B}(x) \wedge \mu^2_{A\oplus B}(y) &= \bigvee\{\mu^2_A(a)\wedge\mu^2_B(b) : x = a+b\} \wedge \bigvee\{\mu^2_A(c)\wedge\mu^2_B(d) : y = c+d\} \\
&= \bigvee\{(\mu^2_A(a)\wedge\mu^2_B(b))\wedge(\mu^2_A(c)\wedge\mu^2_B(d)) \ : \ x = a+b, y = c+d\} \\
&= \bigvee\{(\mu^2_A(a)\wedge\mu^2_B(b))\wedge(\mu^2_A(-c)\wedge\mu^2_B(-d)) \ : \ x = a+b, \ -y = -c-d\} \\
&= \bigvee\{(\mu^2_A(a)\wedge\mu^2_A(-c))\wedge(\mu^2_B(b)\wedge\mu^2_B(-d)) \ : \ x = a+b, \ -y = -c-d\} \\
&\le \bigvee\{(\mu^2_A(a-c)\wedge\mu^2_B(b-d)) \ : \ x-y = (a-c)+(b-d)\} \\
&= \mu^2_{A\oplus B}(x-y)
\end{aligned}$$

and

$$\begin{aligned}
\gamma^2_{A\oplus B}(x) \vee \gamma^2_{A\oplus B}(y) &= \bigwedge\{\gamma^2_A(a)\vee\gamma^2_B(b) \ : \ x = a+b\} \vee \bigwedge\{\gamma^2_A(c)\vee\gamma^2_B(d) \ : \ y = c+d\} \\
&= \bigwedge\{(\gamma^2_A(a)\vee\gamma^2_B(b))\vee(\gamma^2_A(c)\vee\gamma^2_B(d)) \ : \ x = a+b, y = c+d\} \\
&= \bigwedge\{(\gamma^2_A(a)\vee\gamma^2_B(b))\vee(\gamma^2_A(-c)\vee\gamma^2_B(-d)) \ : \ x = a+b, \ -y = -c-d\} \\
&= \bigwedge\{(\gamma^2_A(a)\vee\gamma^2_A(-c))\vee(\gamma^2_B(b)\vee\gamma^2_B(-d)) \ : \ x = a+b, \ -y = -c-d\} \\
&\ge \bigwedge\{(\gamma^2_A(a-c)\vee\gamma^2_B(b-d)) \ : \ x-y = (a-c)+(b-d)\} \\
&= \gamma^2_{A\oplus B}(x-y) \ .
\end{aligned}$$

Also, we have

$$\begin{aligned}
\mu^2_{A\oplus B}(x) &= \vee\{\mu^2_A(a)\wedge\mu^2_B(b) \ : \ x = a+b\} \\
&\le \vee\{\mu^2_A(ay)\wedge\mu^2_B(by) \ : \ xy = ay+by\} \\
&\le \vee\{\mu^2_A(\alpha)\wedge\mu^2_B(\beta) : xy = \alpha+\beta\} \\
&= \mu^2_{A\oplus B}(xy)
\end{aligned}$$

and

$$\begin{aligned}
\gamma^2_{A\oplus B}(x) &= \wedge\{\gamma^2_A(a)\vee\gamma^2_B(b) \ : \ x = a+b\} \\
&\ge \wedge\{\gamma^2_A(ay)\vee\gamma^2_B(by) \ : \ xy = ay+by\} \\
&\ge \vee\{\mu^2_A(\alpha)\vee\gamma^2_B(\beta) : xy = \alpha+\beta\} \\
&= \gamma^2_{A\oplus B}(xy) \ .
\end{aligned}$$

Hence

$$\mu^2_{A\oplus B}(x) \le \mu^2_{A\oplus B}(xy) \ \ and \ \ \gamma^2_{A\oplus B}(x) \ge \gamma^2_{A\oplus B}(xy).$$

Similarly ,we get

$$\mu^2_{A\oplus B}(y) \le \mu^2_{A\oplus B}(xy) \ \ and \ \ \gamma^2_{A\oplus B}(y) \ge \gamma^2_{A\oplus B}(xy).$$

Therefore $A\oplus B = (\mu_{A\oplus B}, \ \gamma_{A\oplus B})$ is a Pythagorean fuzzy ideal of R. □

Theorem 3.10. *If $A = (\mu_A, \ \gamma_A)$ and $B = (\mu_B, \ \gamma_B)$ are Pythagorean fuzzy ideals of R, then*

$$\sqrt{A}\oplus\sqrt{B} \subseteq \sqrt{\sqrt{A}\oplus\sqrt{B}} = \sqrt{A\oplus B}.$$

Proof. The first inclusion follows from Proposition 3.3(i). Since $A \subseteq \sqrt{A}$ and $B \subseteq \sqrt{B}$ by Proposition 3.3(i), it follows from Proposition 3.3(ii) that $\sqrt{A\oplus B} \subseteq \sqrt{\sqrt{A}\oplus\sqrt{B}}$.
Let $x \in R$.
Then

$$\begin{aligned}\mu^2_{\sqrt{A}\oplus\sqrt{B}}(x) &= \bigvee_{x=a+b}\{(\vee(a^m))\wedge(\vee(b^n))\}\\ &= \bigvee_{x=a+b}(\bigvee_{n}\bigvee_{m}\{\mu^2_A(a^m)\wedge\mu^2_B(b^n)\}), (3.8)\end{aligned}$$

$$\begin{aligned}\gamma^2_{\sqrt{A}\oplus\sqrt{B}}(x) &= \bigwedge_{x=a+b}\{\gamma^2_{\sqrt{A}}(a)\wedge\gamma^2_{\sqrt{B}}(b)\}\\ &= \bigwedge_{x=a+b}\{(\bigwedge_{m>1}\gamma^2_A(a^m))\vee(\bigwedge_{n>1}\gamma^2_B(b^n))\}\\ &= \bigwedge_{x=a+b}(\bigwedge_{m>1}\bigwedge_{n>1}\{\gamma^2_A(a^m)\vee\gamma^2_B(b^n)\}).(3.9)\end{aligned}$$

Now let $x = a + b$ for $a, b \in R$ and let m and n be any positive integers. Since R is commutative, we have $x^{m+n} = ta^m + rb^n$ for some $t, r \in R$. Hence

$$\begin{aligned}\mu^2_A(a^m)\wedge\mu^2_B(b^n) &\le \mu^2_A(ta^m)\wedge\mu^2_B(rb^n)\\ &\le \mu^2_{A\oplus B}(ta^m+rb^n) = \mu^2_{A\oplus B}(x^{m+n})\\ &\le \bigvee_{k\ge1}\mu^2_{A\oplus B}(x^k) = \mu^2_{\sqrt{A\oplus B}}(x)\end{aligned}$$

$$\begin{aligned}\gamma^2_A(a^m)\vee\gamma^2_B(b^n) &\ge \gamma^2_A(ta^m)\vee\gamma^2_B(rb^n)\\ &\ge \gamma^2_{A\oplus B}(ta^m+rb^n) = \gamma^2_{A\oplus B}(x^{m+n})\\ &\ge \bigwedge_{k\ge1}\gamma^2_{A\oplus B}(x^k) = \gamma^2_{\sqrt{A\oplus B}}(x)\ .\end{aligned}$$

It follows from (3.8) and (3.9) that
$\mu^2_{\sqrt{A\oplus B}}(x) \ge \mu^2_{\sqrt{A}\oplus\sqrt{B}}(x)$ and $\gamma^2_{\sqrt{A\oplus B}}(x) \le \gamma^2_{\sqrt{A}\oplus\sqrt{B}}(x)$ for all $x \in R$.
Hence $\sqrt{A}\oplus\sqrt{B} \subseteq \sqrt{\sqrt{A}\oplus\sqrt{B}} = \sqrt{A\oplus B}$. □

Theorem 3.11. *Let $f : R \to S$ be a ring homomorphism. If $A = (\mu_A,\ \gamma_A)$ and $B = (\mu_B,\ \gamma_B)$ are Pythagorean fuzzy ideals of R and S respectively, then*

i) $\sqrt{f^{-1}(B)} = f^{-1}(\sqrt{B})$.

ii) $\sqrt{f(A)} = f(\sqrt{A})$ *provided $A = (\mu_A,\ \gamma_A)$ is f-invariant, that is, $f(x) = f(y)$ implies $\mu_A(x) = \mu_A(y)$ and $\gamma_A(x) = \gamma_A(y)$, and f is onto.*

Proof. (i) For any $x \in R$, we have

$$\begin{aligned}\mu^2_{\sqrt{f^{-1}(B)}}(x) = \bigvee_n \mu^2_{f^{-1}(B)}(x^n) &= \bigvee_n \mu^2_B(f(x^n))\\ &= \bigvee_n \mu^2_B((f(x))^n) = \mu^2_{\sqrt{B}}(f(x)) = \mu^2_{f^{-1}(\sqrt{B})}(x)\ ,\end{aligned}$$

$$\begin{aligned}\gamma^2_{\sqrt{f^{-1}(B)}}(x) = \bigwedge_n \gamma^2_{f^{-1}(B)}(x^n) &= \bigwedge_n \gamma^2_B(f(x^n))\\ &= \bigwedge_n \gamma^2_B((f(x))^n) = \gamma^2_{\sqrt{B}}(f(x)) = \gamma^2_{f^{-1}(\sqrt{B})}(x)\ .\end{aligned}$$

Hence $\sqrt{f^{-1}(B)} = f^{-1}(\sqrt{B})$.

(ii) Let $y \in S$. Since f is onto, there exists $x \in R$ such that $f(x) = y$.
Furthermore, $f^{-1}(f(A)) = A$, i.e., $\mu_{f^{-1}(f(A))} = \mu_A$ and $\gamma_{f^{-1}(f(A))} = \gamma_A$, as $A = (\mu_A,\ \gamma_A)$ is f-invariant. Let $(\epsilon,\ \delta) > 0_\sim$, i.e., $\epsilon > 0$ and $\delta < 1$. Then there exists $m \in \mathbb{N}$ such that

$$\begin{aligned}\mu^2_{\sqrt{f(A)}}(y) - \epsilon &< \mu^2_{f(A)}(y^m) = \mu^2_{f(A)}(f(x^m)) = \mu^2_{f^{-1}(f(A))}(x^m) = \mu^2_A(x^m)\\ &\leq \mu^2_{\sqrt{A}}(x) \leq \bigvee_{t\in f^{-1}(y)} \mu^2_{\sqrt{A}}(t) = \mu^2_{f(\sqrt{A})}(y)\end{aligned}$$

and

$$\begin{aligned}\gamma^2_{\sqrt{f(A)}}(y) + \delta &> \gamma^2_{f(A)}(y^m) = \gamma^2_{f(A)}(f(x^m)) = \gamma^2_{f^{-1}(f(A))}(x^m) = \gamma^2_A(x^m)\\ &\geq \gamma^2_{\sqrt{A}}(x) \geq \bigwedge_{t\in f-1(y)} \gamma^2_{\sqrt{A}}(t) = \gamma_{f(\sqrt{A})}(y)\ .\end{aligned}$$

Since, (ϵ, δ) is arbitrary, we have $\mu^2_{\sqrt{f(A)}}(y) \leq \mu^2_{f(\sqrt{A})}(y)$
and $\gamma^2_{\sqrt{f(A)}}(y) \geq \gamma^2_{f(\sqrt{A})}(y)$ for all $y \in S$.
On the other hand, there exists $x_0 \in R$ such that $f(x_0) = y$,
So

$$\mu^2_{f(\sqrt{A})}(y) - \epsilon < \mu^2_{\sqrt{A}}(x_0) = \bigvee_n \mu^2_A(x_0^n)$$

and

$$\gamma^2_{f(\sqrt{A})}(y) + \delta > \gamma^2_{\sqrt{A}}(x_0) = \bigwedge_n \gamma^2_A(x_0^n)\ .$$

Also, there exists $k \in \mathbb{N}$ such that

$$\mu^2_{f(\sqrt{A})}(y) - \epsilon < \mu^2_A(x_0^k) \leq \bigvee_{t\in f^{-1}(y^k)} \mu^2_A(t) = \mu^2_{f(A)}(y^k) \leq \bigvee_n \mu^2_{f(A)}(y^n) = \mu^2_{\sqrt{f(A)}}(y)$$

and

$$\gamma^2_{f(\sqrt{A})}(y) + \delta > \gamma^2_A(x_0^k) \geq \bigwedge_{t\in f^{-1}(y^k)} \gamma^2_A(t) = \gamma^2_{f(A)}(y^k) \geq \bigwedge_n \gamma^2_{f(A)}(y^n) = \gamma^2_{\sqrt{f(A)}}(y)\ .$$

Since $(\epsilon,\ \delta)$ is arbitrary, we get $\mu^2_{f(\sqrt{A})}(y) \leq \mu^2_{\sqrt{f(A)}}(y)$ and $\gamma^2_{f(\sqrt{A})}(y) \geq \gamma^2_{\sqrt{f(A)}}(y)$ for all $y \in S$.
Hence $\sqrt{f(A)} = f(\sqrt{A})$.

□

Definition 3.12. *A Pythagorean fuzzy ideal $A = (\mu_A,\ \gamma_A)$ of R is said to be semiprime if $\sqrt{A} = A$.*
One can verify that an arbitrary intersection of semiprime Pythagorean fuzzy ideals of R is a semiprime Pythagorean fuzzy ideal of R.

Example 3.13. *Let A be a semiprime ideal of R. Define an Pythagorean fuzzy set $A = (\mu_A,\ \gamma_A)$ in R by* $\mu_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ \frac{\sqrt{3}}{2} & \text{otherwise,}\end{cases}$

$\gamma_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ \frac{1}{2} & \text{otherwise.}\end{cases}$

Then $A = (\mu_A,\ \gamma_A)$ is a semiprime Pythagorean fuzzy ideal of R.

Theorem 3.14. *If $A = (\mu_A,\ \gamma_A)$ is an Pythagorean fuzzy ideal of R, then the following assertions are equivalent:*

i) $A = (\mu_A,\ \gamma_A)$ is a semiprime Pythagorean fuzzy ideal of R.

ii) $\mu_A^2(x^n) = \mu_A^2(x)$ *and* $\gamma_A^2(x^n) = \gamma_A^2(x)$ *for all* $x \in R$ *and all* $n \geq 1$.

iii) $\mu_A^2(x^m) = \mu_A^2(x)$ *and* $\gamma_A^2(x^m) = \gamma_A^2(x)$ *for all* $x \in R$ *and some* $m \geq 2$.

Proof. (i) $\Rightarrow$ (ii).
Let $x \in R$ and $n \in \mathbb{N}$.
Then

$$\mu_A^2(x) \leq \mu_A^2(x^n) \leq \bigvee_n \mu_A^2(x^n) = \mu_{\sqrt{A}}^2(x) = \mu_A^2(x)$$

and

$$\gamma_A^2(x) \geq \gamma_A^2(x^n) \geq \bigwedge_n \gamma_A^2(x^n) = \gamma_{\sqrt{A}}^2(x) = \gamma_A^2(x)$$

which imply that $\mu_A^2(x^n) = \mu_A^2(x)$ and $\gamma_A^2(x^n) = \gamma_A^2(x)$.
(ii) $\Rightarrow$ (iii) . Obvious.
(iii) $\Rightarrow$ (i) . Let $x \in R$ and let $m < n$. Then $m^k > n$ for some $k \in \mathbb{N}$. Hence

$$\mu_A^2(x) = \mu_A^2(x^m) = \mu_A^2(x^{m^2}) = \cdots = \mu_A^2(x^{m^k}) \geq \mu_A^2(x^n) = \mu_A^2(x)$$

and

$$\gamma_A^2(x) = \gamma_A^2(x^m) = \gamma_A^2(x^{m^2}) = \cdots = \gamma_A^2(x^{m^k}) \leq \gamma_A^2(x^n) = \gamma_A^2(x) \ ,$$

and so $\mu_A^2(x) = \mu_A^2(x^n)$ and $\gamma_A^2(x) = \gamma_A^2(x^n)$ for all $n > m$. It follows that $\mu_{\sqrt{A}}(x) = \lim_n \mu_A(x^n) = \mu_A(x)$ and $\gamma_{\sqrt{A}}(x) = \lim_n \gamma_A(x^n) = \gamma_A(x)$ so that $\sqrt{A} = A$. Thus $A = (\mu_A,\ \gamma_A)$ is a semiprime Pythagorean fuzzy ideal of R. $\square$

Theorem 3.15. *Let* $A = (\mu_A,\ \gamma_A)$ *be a PFS in* R *such that* $A(0) = (\alpha,\ \beta)$ *, that is,* $\mu_A(0) = \alpha$ *and* $\gamma_A(0) = \beta$*. If* $A = (\mu_A,\ \gamma_A)$ *is a Pythagorean fuzzy semiprime ideal of* R*, then* $A^{-1}([\delta_\alpha,\ \alpha]) \times [\beta,\ \theta_\beta])$ *is a semiprime ideal of* R *for all* $\delta_\alpha \in]0,\ \alpha]$ $\ \delta_\beta \in [\beta,\ 1[$ *.*

Proof. Let $\delta_\alpha \in]0,\ \alpha]$ and $\delta_\beta \in [\beta,\ 1[$.
Suppose that $A = (\mu_A,\ \gamma_A)$ is a Pythagorean fuzzy semiprime ideal of R. Let $x, y \in A^{-1}([\delta_\alpha,\ \alpha]) \times [\beta,\ \theta_\beta])$
Then $\mu_A(x), \mu_A(y) \in [\delta_\alpha,\ \alpha]$, then

$$\delta_\alpha^2 \leq \mu_A^2(x) \wedge \mu_A^2(y) \leq \mu_A^2(x-y) \leq \mu_A^2(0) = \alpha^2$$

and $\delta_\alpha^2 \leq \mu_A^2(x) \leq \mu_A^2(rx) \leq \mu_A^2(0) = \alpha^2$ for all $r \in R$.
Similarly, $\gamma_A(x), \gamma_A(y) \in [\beta, \theta_\beta]$, so

$$\gamma_A^2(0) = \beta \leq \gamma_A^2(x) \vee \gamma_A^2(y) \leq \gamma_A^2(x-y) \leq \theta_\beta^2$$

and $\delta_\alpha^2 \leq \mu_A^2(x) \leq \mu_A^2(rx) \leq \mu_A^2(0) = \alpha^2$ for all $r \in R$.
Then

$$\beta^2 = \gamma_A^2(0) \leq \gamma_A^2(x-y) \leq \gamma_A^2(x) \vee \gamma_A^2(y) \leq \delta_\beta^2,$$

$$\beta^2 = \gamma_A^2(0) \leq \gamma_A^2(rx) \leq \gamma_A^2(x) \leq \delta_\beta^2,$$

$$\beta^2 = \gamma_A^2(0) \leq \gamma_A^2(xr) \leq \gamma_A^2(x) \leq \delta_\beta^2.$$

Hence $x - y, xr \in A^{-1}([\delta_\alpha,\ \alpha]) \times [\beta,\ \theta_\beta])$ Therefore $A^{-1}([\delta_\alpha,\ \alpha]) \times [\beta,\ \theta_\beta])$ is an ideal of R.
Now assume that $a^n \in A^{-1}([\delta_\alpha,\ \alpha]) \times [\beta,\ \theta_\beta])$ for some $a \in R$.
Using Theorem 3.14,we have $\delta_\alpha^2 \leq \mu_A^2(a^n) = \mu_A^2(a) \leq \alpha^2$ (resp. $\beta^2 \leq \gamma_A^2(a^n) = \gamma_A^2(a) \leq \delta_\beta^2$) , and so $a \in A^{-1}([\delta_\alpha,\ \alpha]) \times [\beta,\ \theta_\beta])$
Consequently, $A^{-1}([\delta_\alpha,\ \alpha]) \times [\beta,\ \theta_\beta])$ is a semiprime ideal of R. $\square$

Theorem 3.16. *If* R *is regular, then every Pythagorean fuzzy ideal of* R *is semiprime.*

Proof. Let $A = (\mu_A,\ \gamma_A)$ be a Pythagorean fuzzy ideal of a regular ring R. Given $x \in R$, there exists $a \in R$ such that $x = xax = x^2a$. Repeated application of this result leads to $x = x^n a^{n-1}$ for all $n \geq 2$. Then

$$\mu_A^2(x) = \mu_A^2(x^n a^{n-1}) \geq \mu_A^2(x^n) \geq \mu_A^2(x)\ ,$$

$$\gamma_A^2(x) = \gamma_A^2(x^n a^{n-1}) \leq \gamma_A^2(x^n) \leq \gamma_A^2(x)$$

and so $\mu_A^2(x^n) = \mu_A^2(x)$ and $\gamma_A^2(x^n) = \gamma_A^2(x)$ for all $x \in R$ and $n \geq 1$. It follows from Theorem 9 that $A = (\mu_A,\ \gamma_A)$ is a semiprime Pythagorean fuzzy ideal of R. □

4. Pythagorean Fuzzy prime ideal

Definition 4.1. *A Pythagorean fuzzy ideal $A = (\mu_A, \gamma_A)$ of a ring R (not necessarily commutative) is Pythagorean fuzzy prime if*

$\max\{\mu_A(x),\ \mu_A(y)\} = \mu_A(xy)$ *and* $\min\{\gamma_A(x),\ \gamma_A(y)\} = \gamma_A(xy)$ *for all* $x, y \in R$.

From Theorem 3.14, it follows that a Pythagorean fuzzy prime ideal of R is a semiprime Pythagorean fuzzy ideal of R.

Lemma 4.2. *[6] Let A be a Pythagorean fuzzy subset of a ring R (not necessarily commutative) with $A(0) = (\alpha, \beta)$. Then the following four statements are equivalent:*

(a) A is a Pythagorean fuzzy ideal of R,

(b) $A^{-1}([\tau,\ \alpha] \times [\beta, \theta])$ is an ideal of R, whenever $0 < \tau \leq \alpha$, and $\beta \leq \theta < 1$,

(c) $A^{-1}([\tau,\ \alpha] \times [\beta, \theta])$ is an ideal of R, whenever $(\tau, \theta) \in Im(A)$ with $\tau \neq 0$,

(d) $A^{-1}(]\tau,\ \alpha] \times [\beta, \theta[)$ is an ideal of R, whenever $0 < \tau < \alpha$, and $\beta < \theta < 1$.

Theorem 4.3. *If A be a Pythagorean fuzzy ideal of a ring R $A(0) = (\alpha; \beta)$(not necessarily commutative) with . Then the following four statements are equivalent:*

(a) A is a Pythagorean fuzzy prime ideal of R,

(b) $A^{-1}([\tau,\ \alpha] \times [\beta, \theta])$ is a prime ideal of R, whenever $0 < \tau \leq \alpha$, and $\beta \leq \theta < 1$,

(c) $A^{-1}([\tau,\ \alpha] \times [\beta, \theta])$ is a prime ideal of R, whenever $(\tau; \theta) \in Im(A)$ with $\tau \neq 0$ and $\theta \neq 1$,

(d) $A^{-1}(]\tau,\ \alpha] \times [\beta, \theta[)$ is a prime ideal of R, whenever $0 < \tau < \alpha$, and $\beta < \theta < 1$.

Proof. By Lemma 4.2, the concerned level subsets are all ideals of R.

(a) ⇒ (b) . Let $ab \in A^{-1}([\tau,\ \alpha] \times [\beta, \theta]), a, b \in R$. We have

$\tau \leq \mu_A^2(ab) = \max\{\mu^2(a),\ \mu^2(b)\} \leq \alpha^2$.

and $\beta^2 \leq \gamma_A^2(ab) = \min\{\gamma^2(a),\ \gamma^2(b)\} \leq \theta$.

Therefore, either

$$\tau^2 \leq \mu_A^2(a) \leq \alpha^2 \quad \text{or} \quad \tau^2 \leq \mu_A^2(b) \leq \alpha^2,$$

and

$$\beta^2 \leq \gamma_A^2(a) \leq \theta^2 \quad \text{or} \quad \beta^2 \leq \gamma_A^2(b) \leq \theta^2,$$

hence, either

$$a \in A^{-1}(]\tau,\ \alpha] \times [\beta, \theta[) \quad \text{or} \quad b \in A^{-1}(]\tau,\ \alpha] \times [\beta, \theta[)$$

. Obvious.

(c) ⇒ (d) .

$$A^{-1}(]\tau,\ \alpha] \times [\beta, \theta[) = \bigcup_{\tau < \delta \leq \alpha, \delta \in \mathrm{Im}(\lambda)} A^{-1}([\phi,\ \alpha] \times [\beta, \rho]),$$

which is a prime ideal of R.

$(d) \Rightarrow (a)$. By Lemma 4.2, A is already a Pythagorean fuzzy ideal of R. If possible, let there be $a, b \in R$ such that

$\max\{\mu_A(a),\ \mu_A(b)\} < \mu_A(ab)$.

Let $\mu_A(b) \leq \mu_A(a) = \tau$.

Then $ab \in \mu_A^{-1}]\tau, \alpha]$. By (d), either $a \in \mu_A^{-1}]\tau, \alpha]$, or $b \in \mu_A^{-1}]\ \tau, \alpha]$, which implies either $\tau < \mu_A(a) = \tau$, or $\tau < \mu_A(b) \leq \lambda(a) = \tau$,

both of which are false. Therefore, A is a Pythagorean fuzzy prime ideal of R. □

The following theorem demonstrates the relation between the definition of the Pythagorean fuzzy nil radical of a Pythagorean fuzzy ideal of R, as given in Definition 3.1, and the definition of a Pythagorean fuzzy semi-prime ideal of R, as given in Definition 4.1.

Theorem 4.4. *If A is a Pythagorean fuzzy ideal of R, then $\sqrt{A} = \wedge\{\mu : A \leq \mu, \mu$ is a Pythagorean fuzzy semiprime ideal of $R.\}$*

Proof. Let $\{a_i\}_i$ be the set of all Pythagorean fuzzy semiprime ideals of R containing A.
Let $\sigma = \bigwedge_i \mu_i$. First, by Proposition 2 (ii), we obtain

$\sqrt{A} \leq \sqrt{a_i} = a_j$ for all i,

and hence we get

$\sqrt{A} \leq \wedge\mu_i = \sigma$.

On the other hand, by Theorem 3.4 and Proposition 3.3, $\sqrt{A}$ is a Pythagorean fuzzy semiprime ideal of R containing A. In other words, $\sqrt{\lambda}$ is one of the a_i. It follows that $\sigma \leq \sqrt{A}$. This completes the theorem.

□

5. Conclusion

We have introduced the concepts of Pythagorean fuzzy nil radical of a Pythagorean fuzzy ideal of a commutative ring. And defined a Pythagorean fuzzy semiprime ideal then investigated related properties. and finally we gave the relation between the semiprime Pythagorean fuzzy ideals and the Pythagorean fuzzy nil radical of a commutative ring. Based on these results, we will study Pythagorean fuzzy maximal ideals, Pythagorean fuzzy (Jacobson) radicals and Pythagorean fuzzy prime radicals in a ring.

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A. Talhaoui
, S. Melliani,
I. Bakhadach (Corresponding author),
Sofyane Achik,
Laboratory of Applied Mathematics and Scientific Calculus,
Sultan Moulay Slimane University, BP 523, 23000, Beni Mellal,
Morocco.
E-mail address: ml.talhaoui@gmail.com
E-mail address: i.bakhadach@usms.ma
E-mail address: s.melliani@usms.ma
E-mail address: SofyaneBenAbdou@gmail.com