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On the Nil Graph of Module Over Ring

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ABSTRACT: Let M be an unital left module over a ring R with unity. We define an undirected *nil graphs* for the module M as a graph whose vertex set is $M^* = M - \{0\}$ and any two distinct vertices x and y, in these graphs, are adjacent if and only if there exist $r \in R$ such that $r^2(x+y) = 0$ and $r(x+y) \neq 0$. In this paper, we study the graph's adjacency, diameter, radius, and eulerian and hamiltonian properties. We also defined another nil graph $\Gamma_N^*(M)$, in which we reduced the vertex set to $N(M^*)$, set of all non-zero nil elements of the module, and keep the adjacency relation same as that of $\Gamma_N(M)$. We investigate the adjacency, diameter, radius, eulerian and hamiltonian properties of the graph $\Gamma_N^*(\mathbb{Z}_{p^n})$ and compare these properties among both the graphs.

Key Words: Nil-graph, Diameter, Radius, Eulerian graph, Hamiltonian graph.

Contents

1. Introduction and preliminaries

The graphs associated with various algebraic structures have gotten a lot of attention and importance in the last few decades due to their two-way utilities. Their properties and theories help to characterize each other. There are many graphs defined on groups, rings, and some other algebraic structures. *Zero divisor graph of ring* defined by Beck [4] has been extensively studied by many authors including [1,2], and has yielded numerous results and properties.

Chen [6] assigned a new kind of graph to rings which was further studied by many authors, including [8,9,10] and later named as Nil-graph of a ring.

Ai-Hua Li and Qi-Sheng Li [8,9] studied a new kind of graph structure for non-reduced rings and Von-Neumann regular rings. The undirected graph $\Gamma_N(R)$ is defined as the graph in which two nonzero elements of R, x and y, are adjacent if and only if xy is a nil-element. If R is a von Neumann regular ring or a commutative ring, then $\Gamma_N(R)$ is connected, the diameter of $\Gamma_N(R)$ is at most 3, and the girth of $\Gamma_N(R)$ is not more than 4. Furthermore, if R is non-reduced, the girth of $\Gamma_N(R)$ is 3 or ∞. They also demonstrated that the edge chromatic number of $\Gamma_N(R)$ is equal to the maximum degree of $\Gamma_N(R)$ for a finite commutative ring R unless R is a nilpotent ring with even order.

In his paper, M. Behboodi [5] has generalized the concept of zero-divisors to modules over the commutative ring with unity. He associated three undirected simple graphs to module M over the ring R with unity and considered an annihilator I_x of a factor M/Rx, where he defined an element $x \in R$ to be a zero divisor if and only if $I_x I_y R = 0$ for some non-zero $y \in R$.

Safaeeyan et.al. [11] introduced a new generalization of the classic zero-divisor graph $\Gamma(M)$. They defined the zero-divisor graph for module M in [11], where $M = R$. Many results have been generalized for modules that have been established for the zero divisor graph for the commutative rings.

Anderson and Badawi [1] introduced the total graph of commutative ring R, in which the ring elements were assumed to be the total graph's vertices, and three induced subgraphs were investigated. Later, Atani and Habibi introduced and extended the study of the notion of the total torsion element graph of a

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module over a commutative ring in [3]. They considered all elements of M as vertices. Any two distinct vertices $m,\ n$ in the graph are adjacent if and only if $m+n \in T(M)$, where $T(M)$ is the set of torsion elements of M(unitary) over a commutative ring R. This graph was denoted by $T(\Gamma(M))$.

In their paper "Generalization of nilpotency of ring elements to module elements", Ssevviiri and Groenewald introduced the notion of nilpotency in module theory [12]. Using that notion and the definition of nilpotent elements, we defined the Nil graph for a module.

We defined it as a nil graph of a module over a ring in this paper and investigated some properties of the two types of nil graphs, $\Gamma_N(\mathbb{Z}_{p^n})$ and $\Gamma^*_N(\mathbb{Z}_{p^n})$ based on the vertex set of the graph as an initial study up to $\mathbb{Z}_{p^n} - module$, and throughout our study we have considered $R = M$. In this paper, we have studied some basic properties of these graphs such as adjacency, diameter, radius, eulerian and hamiltonian properties.

A *graph* G is composed of two sets, V, a non-empty set of elements known as the set of vertices, and E, a set of ordered or unordered pairs of distinct vertices known as the set of edges. The symbols $V(G)$ and $E(G)$ are commonly used to represent the vertex-set and edge-set of G.

If an edge e in G connects two vertices u and v, edge e is said to be *incident* on each of the vertices, and the two vertices are said to be *adjacent*. A vertex of a graph that is not adjacent to any other vertex is called an *isolated* vertex, and a graph containing only isolated vertices is called a *null graph*. In the graph $G = (V, E)$, if each edge $e \in E$ is associated with an ordered pair of vertices, then G is called a *directed graph*. If each edge is associated with an unordered pair of vertices, then G is called an *undirected graph*.

The *degree* of a vertex is the number of edges incident to it, with the exception that a loop at a vertex contributes two to the degree of that vertex. A *walk* is an alternating sequence of vertices and edges that starts and ends with a vertex such that each edge is incident on the vertices immediately preceding and following it. A *trail* is a walk where all the edges or lines are distinct, and a *path* is a walk where all the vertices are distinct. A simple graph in which the degree of every vertex is the same is called a *regular* graph. The *girth* of G is the length of the shortest cycle in G, denoted by $g(G)$. The *circumference* of G is the length of the longest cycle in G denoted by $c(G)$. The length of the shortest $u-v$ path is the *distance* between two vertices u and v in G, denoted by $d(u, v)$. If there is no such path between u and v then $d(u, v) = \infty$.

The length of the longest geodesic in a graph G is called *diameter* and is denoted by $diam(G)$. In a connected graph G, the *eccentricity* of a vertex $v \in V$ is the maximum of the distances of v from all other vertices. It is denoted by $e(v)$ and can also be defined as follow $e(v) = max\{d(v, u)|\forall\ u \in V\}$. The *radius* of a graph G is the minimum eccentricity of the vertices and is denoted by $r(G)$. The *centre* of a graph denoted by $c(G)$ can be defined as $C(G) = \{v|e(v) = r(G),\ for\ some\ v \in G\}$. In a graph G, the *neighbourhood* of a vertex v is the subgraph induced by all vertices adjacent to v and it will be denoted by $nh(v)$. A graph G is said to be an Euler graph or *Eulerian* if G has a closed walk that traverses each line exactly once, goes through all vertices and ends at the starting vertex. A graph is said to be a *Hamiltonian* graph if it consists of a cycle passing through all vertices of the graph and this cycle is called a Hamiltonian cycle.

Definition 1.1. (Module over ring) *A left $R-module$ M over a ring R is an abelian group $(M, +)$ with respect to the operation $'+'$ (called addition) and has a scalar product, i.e., $R \times M \to M$ defined as $(r,\ x) \in R \times M \to r.x \in M$, $\forall r \in R$ and $\forall x \in M$ which satisfies the following axioms*

(i) $r.(s.x) = (rs).x\ ;\ \forall r, s \in R$ *and* $x \in M$.

(ii) $r.(x + y) = r.x + r.y\ ;\ \forall r \in R$ *and* $x, y \in M$.

(iii) $(r + s).x = r.x + s.x\ ;\ \forall r, s \in R$ *and* $x \in M$.

(iv) *If R has identity* 1, *then* $1.x = x\ ;\ \forall x \in M$.

Definition 1.2. (Nilpotent element of module) [12] *An element m is said to be a nilpotent in the module M over a ring R if there exist $a \in R$ such that $a^k m = 0$ and $a.m \neq 0$ for some $k \in N$.*

Let $N(M^*)$ be a collection of all non-zero nilpotent elements of the module M over a ring R such that $N(M^*) = \{m \in M^* : a^k.m = 0\ but\ a.m \neq 0\ for\ some\ k \in N\}$ where $M^* = M - \{0\}$.

Also, let $N^c(M^*)$ be the complement of the set $N(M^*)$, i.e., $N^c(M^*) = \{x \in M^* : x \notin N(M^*)\}$.

Example 1.3. *We consider $R = \mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $M = R$. Also considering $M^* = \{1, 2, 3, 4, 5, 6, 7\}$. For $1 \in M$, 1 is a nilpotent, since there exist $2 \in R$ such that for some $k \in N$ (say $k = 3$), $a^k.m = 2^3.1 = 8 \equiv 0(mod\ 8)$ and $2.1 \neq 0$. Therefore, $1 \in M$ is a nilpotent element. In a similar way, $2, 3, 5, 6, 7 \in M$ are nilpotent elements of M. Thus, $N(M^*) = \{1, 2, 3, 5, 6, 7\}$ and $N^c(M^*) = 4$.*

2. Nil Graph of Module over Ring

In this section we define two graph structures on the elements of the module based on the definitions of nilpotent elements given by Ssevviiri and Groenewald [12]. We defined two nil graphs, one by taking all the non-zero nilpotent elements and another by taking all the non-zero elements as the vertex set respectively and restricted the index of r to 2 which give us a graph that is a subgraph of a nilpotent graph.

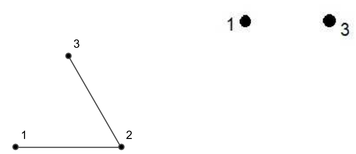
Definition 2.1. (Nil Graph $\Gamma_N(M)$) *In Nil Graph $\Gamma_N(M)$, the set of all nonzero elements of M is considered as the vertex set, and two vertices m_1 and m_2 are adjacent iff $r^2(m_1 + m_2) = 0$ and $r(m_1 + m_2) \neq 0$ for some $r \in R$.*

Thus $V(\Gamma_N(M)) = M^*$ and $E(\Gamma_N(M)) = \{(m_1, m_2) : r^2(m_1 + m_2) = 0$ and $r(m_1 + m_2) \neq 0$ for some $r \in R\}$.

Definition 2.2. (Nil Graph $\Gamma_N^*(M)$) *The set of all nonzero nil elements of M is considered as the vertex set in the Nil Graph $\Gamma_N^*(M)$, and two vertices m_1 and m_2 are adjacent in $\Gamma_N^*(M)$ iff there exists $r \in R$ such that $r^2(m_1 + m_2) = 0$ and $r(m_1 + m_2) \neq 0$.*

Thus $V(\Gamma_N^*(M)) = N(M^*)$ and $E(\Gamma_N^*(M)) = \{(m_1, m_2) : r^2(m_1 + m_2) = 0$ and $r(m_1 + m_2) \neq 0$ for some $r \in R\}$.

Example 2.3. *Let $R = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ and $M = R$. Here, $M^* = M - \{0\} = \{1, 2, 3, \}$ and $N(M^*) = \{1, 3\}$. Then the graph $\Gamma_N(\mathbb{Z}_4)$ is a path and $\Gamma_N^*(\mathbb{Z}_4)$ is a totally disconnected graph as shown in the* **Figure 1**.



(a) (b)

Figure 1: (a) $\Gamma_N(\mathbb{Z}_4)$ $\Gamma_N^*(\mathbb{Z}_4)$

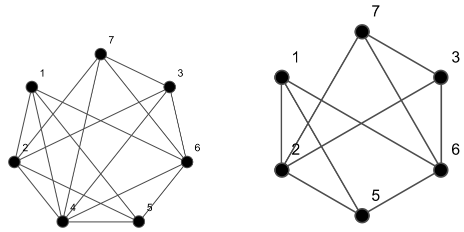
Example 2.4. *Let $R = \mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and $M = R$. Here, $M^* = M - \{0\} = \{1, 2, 3, 4, 5, 6, 7\}$ and $N(M^*) = \{1, 2, 3, 5, 6, 7\}$. Then the graph $\Gamma_N(M)$ and $\Gamma_N^*(M)$ are shown in the* **Figure 2**.

Example 2.5. *The graph $\Gamma_N(\mathbb{Z}_9)$ is connected but $\Gamma_N^*(\mathbb{Z}_9)$ is disconnected, as shown in* **Figure 3**.

3. Properties of Nil Graph $\Gamma_N(\mathbb{Z}_{p^n})$ and $\Gamma_N^*(\mathbb{Z}_{p^n})$

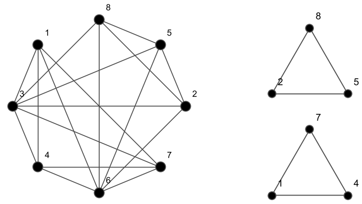
We have studied several characteristics of the graphs $\Gamma_N(\mathbb{Z}_{p^n})$ and $\Gamma_N(\mathbb{Z}_{p^n})$ in this section. The diameter, girth, and radius of these graphs are discussed. The hamiltonian and eulerian qualities have also been investigated for both the graphs.

Theorem 3.1. *$\Gamma_N^*(M)$ is subgraph of $\Gamma_N(M)$.*



(a) (b)

Figure 2: (a) $\Gamma_N(\mathbb{Z}_8)$ $\Gamma^*_N(\mathbb{Z}_8)$



(a) (b)

Figure 3: (a) $\Gamma_N(\mathbb{Z}_9)$ and (b) $\Gamma^*_N(\mathbb{Z}_9)$

Proof. We know that $V(\Gamma^*_N(M)) = N(M^*)$ and $V(\Gamma_N(M)) = M^*$ from defintion. It is obvious that for any arbitrary module M, $N(M^*) \subset M^* \Rightarrow V(\Gamma^*_N(M)) \subset V(\Gamma_N(M))$.

Now, as the adjacency relation for both the graphs is defined in the same way, so $E(\Gamma^*_N(M)) \subset E(\Gamma_N(M))$.

As an outcome, we can conclude that $\Gamma^*_N(M)$ is a subgraph of $\Gamma_N(M)$ and $\Gamma^*_N(M)$ is a subgraph induced by all the nil elements of M^*. $\square$

It is clear from the preceding examples that $\Gamma^*_N(M)$ is a subgraph of $\Gamma_N(M)$

Theorem 3.2. *For both the graphs $\Gamma_N(\mathbb{Z}_{p^n})$ and $\Gamma_N(\mathbb{Z}_{p^n})$, any two distinct vertices m_1 and m_2 are not adjacent if and only if $m_1 + m_2 \equiv 0(mod\ p^{n-1})$ where $n \geq 2$.*

Proof. By definitions 2.1 and 2.2, two distinct vertices m_1 and m_2 are adjacent if $r^2(m_1 + m_2) = 0$ and $r(m_1 + m_2) \neq 0$ for some $r \in R$.

If $m_1 + m_2 \equiv 0(mod\ p^{n-1})$ then $(m_1 + m_2) = ap^{n-1}$ for some integer $1 \leq a \leq p$.

Consider m_1 and m_2 to be two adjacent vertices in any of the nil graphs. Then there exist some $r \in \mathbb{Z}_{p^n}$ such that $r^2(m_1 + m_2) = r^2(a.p^{n-1}) \equiv\ 0\ (mod\ p^n)$, which implies either $a = sp$ or $r = tp$ for some integer s and t.

Case 1: If $a = sp$, we have $r^2(m_1 + m_2) = r^2(a.p^{n-1}) = r^2(sp.p^{n-1}) = r^2sp^n \equiv 0(mod\ p^n)$ and also $r(m_1 + m_2) = 0$ as $r(spp^{n-1}) = 0$.

Case 2: If $r = tp$, we have $r^2(m_1+m_2) = (tp)^2(a.p^{n-1}) = 0$ and also $r(m_1+m_2) = 0$ as $tp(a.p^{n-1}) = 0$.

Both the cases lead us to a contradiction. Hence m_1 and m_2 are not adjacent in both the associated Nil graphs.

Conversely, let two distinct vertices m_1 and m_2 be non-adjacent in both the Nil graph, then we have whenever $r^2(m_1 + m_2) = 0$ then $r(m_1 + m_2) = 0$ for some $r \in R$.

Suppose $m_1 + m_2 = sp^{n-2}$ for some integer $1 \leq s \leq p$, then as $r \in \mathbb{Z}_{p^n}$ must satisfy $r(m_1 + m_2) = 0$ thus $r = tp^2$ for some integer t. Let $r_1 \in \mathbb{Z}_{p^n}$ such that $r_1 = tp \Rightarrow r_1(m_1 + m_2) = tp.sp^{n-2} = tsp^{n-1} \not\equiv 0$ and $r_1^2(m_1 + m_2) = (tp)^2.sp^{n-2} = t^2sp^n = 0$.

So, if $m_1 + m_2 = sp^{n-2}$ then there exists $r_1 \in \mathbb{Z}_{p^n}$ such that $r_1^2(m_1+m_2) = 0$ and $r_1(m_1+m_2) \neq 0$ which is a contradiction as m_1 and m_2 are non-adjacent.

Therefore $m_1 + m_2 \neq sp^i$ where $1 \leq i \leq n-2 \Rightarrow m_1 + m_2 = sp^{n-1}$ for some integer $1 \leq s \leq p$, i.e., $m_1 + m_2 \equiv 0(mod\ p^{n-1})$. □

Example 3.3. *Consider the two nil graphs $\Gamma(\mathbb{Z}_9)$ and $\Gamma_N^*(\mathbb{Z}_9)$ in* **Figure 3**, *we observe the following:*

In $\Gamma_N(\mathbb{Z}_9)$ and $\Gamma_N^*(\mathbb{Z}_9)$, we observed that 1 is not adjacent to 2, 5 and 8; 4 is not adjacent to 2, 5 and 8; and 7 is not adjacent to 2, 5 and 8 in both the nil graphs. The reason is that $1+2 = 1+5 = 1+8 = 4+2 = 4+5 = 4+8 = 7+2 = 7+5 = 7+8 \equiv 0(mod\ 3)$. Also 3 and 6 are not adjacent in $\Gamma_N(\mathbb{Z}_{3^2})$ as $3+6 \equiv 0(mod\ 3)$.

Theorem 3.4. *For prime p and positive integer n*

(i) $\Gamma_N^*(\mathbb{Z}_{p^n})$ is connected for all $p \geq 3$.

(ii) $\Gamma_N(\mathbb{Z}_{p^n})$ is connected for $p \geq 2$ and $n > 1$.

Proof. (i) Consider the two distinct vertices $m_1,\ m_2 \in V(\Gamma_N^*(\mathbb{Z}_{p^n}))$ such that m_1 and m_2 are not adjacent. To prove that $\Gamma_N^*(\mathbb{Z}_{p^n})$ is connected we require to show that there is always a path from m_1 to m_2. As m_1 and m_2 are not adjacent then $m_1 + m_2$ must be a multiple of p^{n-1}, *i.e.*, $m_1 + m_2 = ap^{n-1}$ where $1 \leq a \leq p \Rightarrow m_2 = ap^{n-1} - m_1$.

Let us consider $V^{'}(m_1) = \{(ap^{n-1} - m_1) \in V(\Gamma_N^*(\mathbb{Z}_{p^n}))\ :\ 1\ \leq a \leq p\}$

and $V^{'}(m_2) = \{(bp^{n-1} - m_2) \in V(\Gamma_N^*(\mathbb{Z}_{p^n}))\ :\ 1 \leq b \leq p\}$.

Let $nh(m_1) = \{m_1^{'}\ :\ \ m_1^{'}\ adj\ m_1\}$ and $nh(m_2) = \{m_2^{'}\ :\ \ m_2^{'}\ adj\ m_2\}$ be the neighbourhoods of m_1 and m_2 respectively.

Then for each $i = 1, 2$,

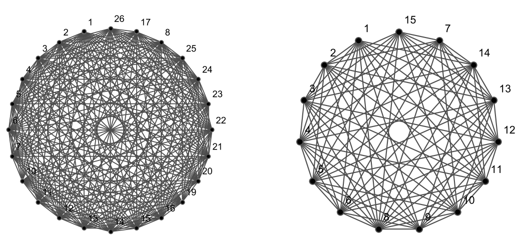
$|nh(m_i)| = |V(\Gamma_N^*(\mathbb{Z}_{p^n}))| - |V^{'}(m_i)| - 1$

$= (p^n - p) - p - 1 = p^n - 2p - 1 > \frac{|V(\Gamma_N^*(\mathbb{Z}_{p^n}))|}{2} = \frac{(p^n-p)}{2}$ since $p \geq 3$.

This implies that, $nh(m_1) \cap nh(m_2) \neq \phi$, *i.e.*, there exist at least one vertex $m^{'} \in nh(m_1) \cap nh(m_2)$ that is adjacent to both m_1 and m_2. Hence, we have a path $m_1 - m^{'} - m_2$ and this shows that $\Gamma_N^*(\mathbb{Z}_{p^n})$ is connected.

(ii) We already know that $\Gamma_N^*(\mathbb{Z}_{p^n}) \subset \Gamma_N(\mathbb{Z}_{p^n})$, and $\Gamma_N^*(\mathbb{Z}_{p^n})$ is connected for $p \geq 3$. So, we can show that the elements from the set $N^c(M^*)$ are adjacent to atleast one of the elements from the set $N(M^*)$ then $\Gamma_N(\mathbb{Z}_{p^n})$ will be connected for $p \geq 3$.

For any $m^{'} \in N^c(M^*)$ we can find at least one $m \in N(M^*)$ so that $m^{'} + m \not\equiv 0(mod\ p^{n-1})$. Thus the elements of $N^c(M^*)$ are adjacent to the elements of $N(M^*)$ in $\Gamma_N(\mathbb{Z}_{p^n})$ and is connected for $p \geq 3$. For $\Gamma_N(\mathbb{Z}_4)$ and $\Gamma_N(\mathbb{Z}_9)$ we can easily check that the two graphs are connected. Thus $\Gamma_N(\mathbb{Z}_{p^n})$ is connected for $p \geq 2$ and $n > 1$. □



(a) (b)

Figure 4: (a) $\Gamma_N(\mathbb{Z}_{27})$ and (b) $\Gamma_N^*(\mathbb{Z}_{16})$

Example 3.5. *Let us take examples of nil graphs, say $\Gamma_N(\mathbb{Z}_{27})$ and $\Gamma_N(\mathbb{Z}_{16})$ as shown in Figure* **Figure 4**.

In $\Gamma_N(\mathbb{Z}_{27})$(Figure 4(a)), let us take $m_1 = 1$ and $m_2 = 17$. We have $deg(1) = 20 \geq \frac{|V(\Gamma_N(\mathbb{Z}_{27}))|}{2} = 13$ and $deg(17) = 20 \geq \frac{|V(\Gamma_N(\mathbb{Z}_{27}))|}{2} = 13$ and $nh(1) \cap nh(17) = \{2, 3, 4, 5, 6, 7, 11, 12, 13, 14, 15, 16, 20, 21, 22, 23, 24, 25\} \neq \phi$. Hence, we can easily have a path from 1 to 17. Similarly we can show it for every pair of vertices of $\Gamma_N(\mathbb{Z}_{27})$ giving that $\Gamma_N(\mathbb{Z}_{27})$ is connected.

In $\Gamma^*_N(\mathbb{Z}_{16})$(Figure 4(b)), we have 8 as the only non-nilpotent element of $\mathbb{Z}_{16}$ and we observe that 8 is adjacent to all the other nilpotent elements in the Nil graph $\Gamma^*_N(\mathbb{Z}_{16})$. Hence $\Gamma^*_N(\mathbb{Z}_{16})$ is connected.

Theorem 3.6. *Let p be a prime then the Nil Graph $\Gamma^*_N(\mathbb{Z}_{p^n})$ is regular for all $p \geq 3$ and $n \geq 2$.*

Proof. In Theorem 3.4, we have seen that, for any vertex $v \in \Gamma^*_N(\mathbb{Z}_{p^n})$, the degree of v is $deg(v) = |nh(v)| = p^n - 2p - 1$. Thus for each p, $deg(v)$ is constant for all $v \in \Gamma^*_N(\mathbb{Z}_{p^n})$. Hence $\Gamma^*_N(\mathbb{Z}_{p^n})$ is a regular graph of degree $p^n - 2p - 1$. □

Theorem 3.7. *For Nil Graph $\Gamma^*_N(\mathbb{Z}_{2^n})$, where $n \geq 3$, the vertices having the highest degree are of the form 2^{n-2} and 3.2^{n-2} and all the other vertices have degree 1 less than the highest.*

Proof. We know a vertex m_1 in $\Gamma^*_N(\mathbb{Z}_{2^n})$ is not adjacent to vertices of the form of $qp^{(n-1)} - m_1$ for some $1 \leq q \leq p$ and there are p vertices of such form. The vertex 2^{n-2} is not adjacent to vertices of the form $q.2^{n-1} - 2^{n-2}$ where $1 \leq q \leq p \Rightarrow$ vertex 2^{n-2} is not adjacent to p number of vertices where $p = 2$, but for $q = 1$, $1.2^{n-1} - 2^{n-2} = 2^{n-2}(1.2 - 1) = 2^{n-2}$

$\Rightarrow 2^{n-2}$ is not adjacent to itself.

For $q = 2$, $2.2^{n-1} - 2^{n-2} = 2^{n-2}(2.2 - 1) = 2^{n-2}.3$

$\Rightarrow 2^{n-2}.3$ is not adjacent to $2^{n-2}.3$

For $q \geq 3$, $q.2^{n-1} - 2^{n-2} \notin V(\Gamma^*_N(\mathbb{Z}_{2^n}))$

Thus, besides themselves, vertices 2^{n-2} and $2^{n-2}.3$ are not adjacent to each other.

So, $deg(2^{n-2}) = deg(2^{n-2}.3) = N(M^*) - 1 - 1 = (2^n - 2) - 1 - 1 = 2^n - 4$. Whereas from *Theorem 3.4*, we see that any other vertex is not adjacent to itself and to p vertices and hence, $deg(m) = N(M^*) - p - 1 = (2^n - 2) - 2 - 1 = 2^n - 5$. □

Theorem 3.8. *For connected $\Gamma_N(\mathbb{Z}_{p^n})$ and $\Gamma^*_N(\mathbb{Z}_{p^n})$,*

$$Diam(\Gamma_N(\mathbb{Z}_{p^n})) = Diam(\Gamma^*_N(\mathbb{Z}_{p^n})) = 2.$$

Proof. In both graphs, $\Gamma_N(\mathbb{Z}_{p^n})$ and $\Gamma^*_N(\mathbb{Z}_{p^n})$, there exists at least one vertex u for each vertex v, such that $v + u \equiv 0(mod\ p^{n-1})$ and v and u are not adjacent. It is sufficient to show that there is always one vertex adjacent to both of non-adjacent vertices.

For $\Gamma^*_N(\mathbb{Z}_{p^n})$: Let m_1 and m_2 be any two non-adjacent vertices in $\Gamma^*_N(\mathbb{Z}_{p^n})$ and let $nh(m_1)$ and $nh(m_2)$ be the set of vertices of the neighbourhood of m_1 and m_2 respectively. In *Theorem 3.4* we see that $nh(m_1) \cap nh(m_2) \neq \phi$, *i.e.*, there exist at least one vertex $m^{'} \in nh(m_1) \cap nh(m_2)$ such that m_1 and m_2 both adjacent to $m^{'}$. Hence, the geodesic of any two non-adjacent vertices is 2.

For $\Gamma_N(\mathbb{Z}_{p^n})$: Clearly from *Theorem 3.4*, we see that for any two non-adjacent vertices say $m_1^{'}$ and $m_2^{'}$ in $\Gamma_N(\mathbb{Z}_{p^n})$ there is always at least one vertex m' which is a common vertex to both $m_1^{'}$ and $m_2^{'}$. Hence the geodesic of $m_1^{'}$ and $m_2^{'}$ is 2.

Thus, we have shown that the geodesic of both graphs $\Gamma^*_N(\mathbb{Z}_{p^n})$ and $\Gamma_N(\mathbb{Z}_{p^n})$ is either 1 or 2. This implies that the longest geodesic has a length of 2 and thus has a diameter of 2. □

Corollary 3.9. *Diameter of nil graph $\Gamma^*_N(\mathbb{Z}_{p^2}) = \infty$ for $p = 2$ and $p = 3$.*

Theorem 3.10. *Radius of $\Gamma_N(\mathbb{Z}_{2^n})$ is always 1.*

Proof. The only non-nilpotent element in $\Gamma_N(\mathbb{Z}_{2^n})$ is 2^{n-1} because non-nilpotent elements of the module $\mathbb{Z}_{p^n}$ are all multiples of p^{n-1}, and the only multiple of 2^{n-1} that is strictly less than p^n is 2^{n-1} itself.

By definition, $V(\Gamma_N(\mathbb{Z}_{2^n})) = M^* = N(M^*) \cup N^c(M^*)$, and because 2^{n-1} is the only non-nilpotent element, the remaining vertices are all nilpotent elements. Also in *Theorem 3.7* we have seen that a non-nilpotent element is always adjacent to all the nilpotent elements. That means $2^{n-1} \in V(\Gamma_N(\mathbb{Z}_{2^n}))$ is adjacent to all the other remaining vertices. Hence, the maximum distance from 2^{n-1} to any other vertex $m \in \Gamma_N(\mathbb{Z}_{2^n})$ is 1, *i.e.*, $e(2^{n-1}) = max\{d(2^{n-1}, m) \ : \ \forall m \in V(\Gamma_N(\mathbb{Z}_{2^n}))\} = 1$ For any two non adjacent vertices u and $v \in N(M^*)$ are connected through the path $u - 2^{n-1} - v$. So eccentricity $v \in N(M^*)$ is 1 *or* 2. Thus the minimum eccentricity of the graph $\Gamma_N(\mathbb{Z}_{2^n})$ is 1 and so $r(\Gamma_N(\mathbb{Z}_{2^n})) = 1$, where $n \geq 2$. □

Corollary 3.11. *The vertex 2^{n-1} has the highest degree in $\Gamma_N(\mathbb{Z}_{2^n})$.*

Corollary 3.12. *The graph centre of $\Gamma_N(\mathbb{Z}_{2^n})$ is the non-nilpotent element 2^{n-1}.*

Theorem 3.13. *The radius of all the connected graph $\Gamma_N(\mathbb{Z}_{p^n})$ is 2 for $p \geq 3$.*

Proof. The vertex set $V(\Gamma_N(\mathbb{Z}_{p^n}))$ consists of all the nilpotent element of the module $\mathbb{Z}_{p^n}$. For any arbitrary vertex $m_1 \in V(\Gamma_N(\mathbb{Z}_{p^n}))$ there exist at least one $m_2 \in V(\Gamma_N(\mathbb{Z}_{p^n}))$ of the form $m_2 = qp^{n-1} - m_1$ for some $1 \leq q \leq p$, then $e(m_1) = max\{d(m_1, m_2) \ : \ \forall m_2 \in V(\Gamma_N(\mathbb{Z}_{p^n}))\} = 2$.

We have $V(\Gamma_N(\mathbb{Z}_{p^n})) = N(M^*) \cup N^c(M^*)$.

As $\Gamma(\mathbb{Z}_{p^n})$ is an induced subgraph of $\Gamma_N(\mathbb{Z}_{p^n})$

$\Rightarrow$ any edge $e \in E(\Gamma_N(\mathbb{Z}_{p^n}))$

$\Rightarrow e \in E(\Gamma_N(\mathbb{Z}_{p^n}))$ and the maximum possible distance between two nilpotent elements m_1 and m_2 is 2 $\forall m_1, \ m_2 \in V(\Gamma_N(\mathbb{Z}_{p^n}))$.

Now we need to show two things, firstly the maximum possible distance between a nilpotent element and a non-nilpotent element is also 2. Secondly the maximum possible distance between two distinct non-nilpotent elements $m_1^{'}, \ m_2^{'} \in N^c(M^*)$ is also 2.

We know that a non-nilpotent element $m^{'} \in N^c(M^*)$ is always adjacent to all $m \in N(M^*)$, so the $d(m', m) = 1$ but we need to show that the maximum possible distance is 2.

Now we know that m is not adjacent to vertices of the form $qp^{n-1} - m$ and are adjacent to the remaining vertices which is not of the form $qp^{n-1} - m$ for $1 \leq q \leq p$.

Let $A = \{a \in N(M^*) \ : \ a = (qp^{n-1} - m), \ for \ 1 \leq q \leq p\}$

$B = \{b \in N(M^*) \ : \ b \neq (qp^{n-1} - m), \ for \ 1 \leq q \leq p\} - N^c(M^*)$

$|B| = |V(\Gamma_N(\mathbb{Z}_{p^n}))| - |A| - |N^c(M^*)| = (p^n - 1) - p - (p-1) > 0 \ \forall p \geq 3$

Hence there exists at least one $b \in B$ such that m is adjacent to b. Choosing the path $m - b - m^{'}$ we have the maximum possible distance between nilpotent element m and non-nilpotent element $m^{'}$ is 2.

From *Theorem 3.4* it is clear that for any two distinct non-nilpotent elements $m_1^{'}$ and $m_2^{'} \in N^c(M^*)$ there exists at least one $m_1 \in N(M^*)$ such that $m_1^{'} \ adj \ m_1$ and also $m_2^{'} \ adj \ m_1$. Hence, we always have a path $m_1^{'} - m_1 - m_2^{'}$ from $m_1^{'} \ to \ m_2^{'}$. Thus, the maximum possible distance between two non-nilpotent elements is also 2. We conclude that the eccentricity of any arbitrary $v \in V(\Gamma_N(\mathbb{Z}_{p^n}))$ is $e(v) = max\{d(v, u) \ : \ \forall u \in V(\Gamma_N(\mathbb{Z}_{p^n}))\} = 2$ where $v \in N(M^*) \cup N^c(M^*)$.

We have $min\{e(m_1) \ : \ \forall m_1 \in V(\Gamma_N(\mathbb{Z}_{p^n}))\} = 2 \Rightarrow r(\Gamma_N(\mathbb{Z}_{p^n})) = 2$. □

Theorem 3.14. *If $\Gamma_N^*(\mathbb{Z}_{p^n})$ is connected nil-graph then $r(\Gamma_N^*(\mathbb{Z}_{p^n})) = 2$.*

Proof. For connected graphs, we know that the eccentricity of a vertex m_1 is given by $e(m_1) = max\{d(m_1, m_2) \ : \ \forall m_2 \in V(\Gamma_N^*(\mathbb{Z}_{p^n}))\}$ where $d(m_1, m_2)$ is the length of the shortest path between m_1 to m_2, $m_1 \neq m_2$. Let m_1 be an arbitrary vertex in $\Gamma_N^*(\mathbb{Z}_{p^n})$. We know that the vertices m_1 and $qp^{(n-1)} - m_1 \in V(\Gamma_N^*(\mathbb{Z}_{p^n}))$ are not adjacent for alteast one integer q, $1 \leq q \leq p$ and let $m_2 \in V(\Gamma_N^*(\mathbb{Z}_{p^n}))$ be one such vertex. From *theorem 3.4*, we see that there exist at least one $m^{'} \in V(\Gamma_N^*(\mathbb{Z}_{p^n}))$ such that $m_1 - m^{'} - m_2$ is a path. Thus, we see that the $d(m_1, m_2) = 2$ and the eccentricity of any vertex $m_1 \in V(\Gamma_N^*(\mathbb{Z}_{p^n}))$ is always 2. Since m_1 is arbitrary we have $min\{e(m_1) \ : \ \forall m_1 \in \Gamma_N^*(\mathbb{Z}_{p^n})\} = 2$. Thus, the radius of the graph is also 2. □

Corollary 3.15. *Radius of nil graph $\Gamma_N^*(\mathbb{Z}_{p^2}) = \infty$ for $p = 2$ and $p = 3$.*

Corollary 3.16. *For $p \geq 3$ the graph centre of the graphs $\Gamma_N(\mathbb{Z}_{p^n})$ is the vertex set.*

This is because the eccentricity of each vertex is 2 which is equal to the radius of that graph $\Gamma_N(\mathbb{Z}_{p^n})$.

Corollary 3.17. *The graph center of all the connected Nil Graph $\Gamma_N^*(\mathbb{Z}_{p^n})$ is the vertex set.*

This is because the eccentricity of any vertex is always 2 which is equal to the radius of that graph $\Gamma_N^*(M)$.

Lemma 3.18. *(Dirac theorem) If G is a graph with n vertices, where $n \geq 3$ and $deg(v) \geq n/2$ for every vertex of G, then G is Hamiltonian.*

Theorem 3.19. *The graphs $\Gamma_N(\mathbb{Z}_{p^n})$ is Hamiltonian for all $p^n > 2^2$.*

Proof. As $\Gamma_N^*(\mathbb{Z}_{p^n})$ is an induced subgraph of $\Gamma_N(\mathbb{Z}_{p^n})$

$\Rightarrow$ any edge $e \in E(\Gamma_N^*(\mathbb{Z}_{p^n}))$, then $e \in E(\Gamma_N(\mathbb{Z}_{p^n}))$ and for any vertices $m_1 \in V(\Gamma_N^*(\mathbb{Z}_{p^n}))$, then $m_1 \in V(\Gamma_N(\mathbb{Z}_{p^n}))$.

From *Theorem 3.4*, we have whenever $m_1 \in V(\Gamma_N^*(\mathbb{Z}_{p^n}))$, $deg(m_1) \geq \frac{|V(\Gamma_N^*(\mathbb{Z}_{p^n}))|}{2}$ and also whenever $m_1 \in V(\Gamma_N(\mathbb{Z}_{p^n}))$, $\frac{|V(\Gamma_N(\mathbb{Z}_{p^n}))|}{2}$ since $\Gamma_N(\mathbb{Z}_{p^n})$ has additional non-nilpotent vertices but every non-nilpotent vertex is adjacent to all the other nilpotent vertices, *i.e.*, with every one additional non-nilpotent vertex in $\Gamma_N(\mathbb{Z}_{p^n})$ there is an increase in the degree of m_1 by one.

For $m^{'} \in N^c(M^*) \subset V(\Gamma_N(\mathbb{Z}_{p^n}))$, its degree is obviously greater than $\frac{|V(\Gamma_N(\mathbb{Z}_{p^n}))|}{2}$.

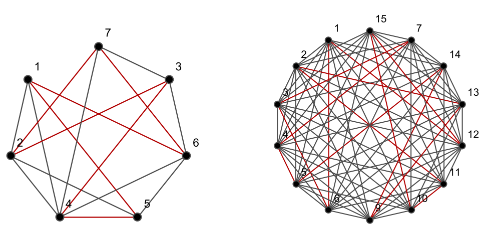
Hence by lemma 3.18, we conclude that $\Gamma_N(\mathbb{Z}_{p^n})$ is Hamiltonian for all $p^n > 2^2$. $\square$

Note: The graph $\Gamma_N(\mathbb{Z}_{2^2})$is not Hamiltonian as it is a path (see figure 1(a)).

Theorem 3.20. *Every connected graph $\Gamma_N^*(\mathbb{Z}_{p^n})$ is a Hamiltonian graph for $p \geq 2$ and $n > 1$.*

Proof. From *Theorem 3.4*, we see that for all connected graphs $\Gamma_N^*(\mathbb{Z}_{p^n})$, $deg(v) \geq \frac{|V(\Gamma_N^*(\mathbb{Z}_{p^n}))|}{2}$ for all $v \in \Gamma_N^*(\mathbb{Z}_{p^n})$. Now by *Lemma 3.18* the graph $\Gamma_N^*(\mathbb{Z}_{p^n}))$ is a Hamiltonian graph. $\square$

Example 3.21. *The Hamiltonian cycles of $\Gamma_N(\mathbb{Z}_8)$ and $\Gamma_N^*(\mathbb{Z}_{16})$ are highlighted in* **Figure 5**.



(a) (b)

Figure 5: Hamiltonian cycle in (a) $\Gamma_N(\mathbb{Z}_8)$ and (b) $\Gamma_N^*(\mathbb{Z}_{16})$

Theorem 3.22. *If p be an odd prime, then all the connected Nil Graphs of $\Gamma_N^*(\mathbb{Z}_{p^n})$ are Eulerian.*

Proof. From Theorem 3.6, we can see that the graph $\Gamma_N^*(\mathbb{Z}_{p^n})$, $p \geq 3$ is a connected regular graph of degree $(p^n - 2p - 1)$ which is even for every $p \geq 3$. Hence, $\Gamma_N^*(\mathbb{Z}_{p^n})$ is eulerian. $\square$

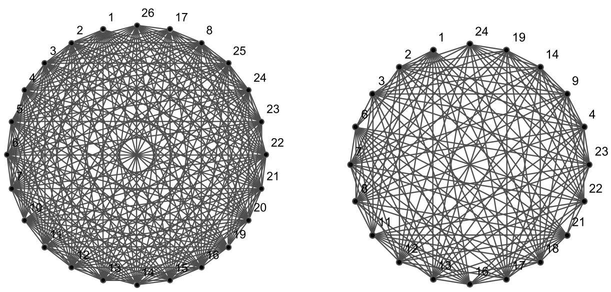
Theorem 3.23. *The graph $\Gamma_N(\mathbb{Z}_{p^n})$ is Eulerian for all odd prime p.*

Proof. We know that p is odd for all $p \geq 3$, and that module $\mathbb{Z}_{p^n}$ always contains $p-1$ non-nilpotent elements.

We can see from Theorem 3.22, that the graph $\Gamma_N^*(\mathbb{Z}p^n)$ is eulerian for all $p \geq 3$. Also Theorem 3.1 we see that that $\Gamma_N^*(\mathbb{Z}p^n)$ is an induced subgraph of $\Gamma_N(\mathbb{Z}p^n)$. Now, $\Gamma_N(\mathbb{Z}p^n)$ has an additional $p-1$ non-nilpotent elements and are adjacent to all the nil elements. Thus the degree of the vertex $m \in \Gamma_N(M)$ is even if $m \in N(M^*)$. Let $m' \in N^c(M^*)$, then $deg(m')$ = numbers of nilpotent elements in $\Gamma_N(M) = (p^n - 1) - (p-1) = even$.

Thus, all the vertices of graph $\Gamma_N(M)$ have an even degree. Hence, $\Gamma_N(M)$ is eulerian. □

Example 3.24. *The two properties given in theorem 3.22 and theorem 3.23 can be observed in the following graphs $\Gamma_N(\mathbb{Z}_{3^3})$ and $\Gamma_N^*(\mathbb{Z}_{5^2})$ shown in* **Figure 6** *and any nil graph $\Gamma_N(\mathbb{Z}_{p^n})$ and $\Gamma_N^*(\mathbb{Z}_{p^n})$.*



(a) (b)

Figure 6: (a) $\Gamma_N(\mathbb{Z}_{16})$ and (b) $\Gamma_N^*(\mathbb{Z}_{25})$

Corollary 3.25. *The nil graphs $\Gamma_N^*(\mathbb{Z}_{p^2})$ for $p = 2$ and $p = 3$ are neither Hamiltonian nor Eulerian.*

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