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On Power Integral Bases of Certain Pure Number Fields Defined By $x^{120} - m$

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abstract: Let $K = \mathbb{Q}(\alpha)$ be a pure number field with α a complex root of a monic irreducible polynomial $F(x) = x^{120} - m \in \mathbb{Z}[x]$, where $m \neq \pm 1$. In this paper, we study the monogenity of K. More precisely, we prove that if m is square free, $m \not\equiv 1 \pmod 4$, $m \not\equiv \pm 1 \pmod 9$, and $\overline{m} \notin \{\mp 1, 7, 18\} \pmod{25}$, then K is monogenic. On the other hand, if $m \equiv 1 \pmod 4$, $m \equiv 1 \pmod 9$, or $m \equiv 1 \pmod{25}$, then K is not monogenic. Our results are illustrated by some computational examples.

Key Words: Power integral basis, pure number fields, Theorem of Ore, prime ideal factorization, common index divisor.

Contents

1. Introduction

Let K be a number field generated by a complex root α of a monic irreducible polynomial $F(x) \in \mathbb{Z}[x]$ of degree n and $\mathbb{Z}_K$ its ring of integers which is a free $\mathbb{Z}$-module of rank $n = [K : \mathbb{Q}]$. If $\mathbb{Z}_K$ has a power integral basis $(1, \theta, ..., \theta^{n-1})$ for some $\theta \in \mathbb{Z}_K$; $\mathbb{Z}_K = \mathbb{Z}[\theta]$, then the field K is said to be monogenic. Otherwise, K is called not monogenic . Recall that for any $\theta \in \mathbb{Z}_K$, the abelian quotient group $\mathbb{Z}_K/\mathbb{Z}[\theta]$ is finite. Its order is called the index of $\mathbb{Z}[\theta]$, which we denote by $(\mathbb{Z}_K : \mathbb{Z}[\theta])$. The index of K is

$$i(K) = \gcd\{(\mathbb{Z}_K : Z[\theta]) \,|\, \theta \in \mathbb{Z}_K \,\text{and}\, K = \mathbb{Q}(\theta)\}.$$

A rational prime p dividing $i(K)$ is called a prime common index divisor of K. If $\mathbb{Z}_K$ has a power integral basis, then $i(K) = 1$. Thus, if there is a prime common index divisor of K, then K is not monogenic. The problem of giving an arithmetic characterization of monogenic number fields is called the problem of Hasse (see [13,14,22,21]). It is one of the most important problems in algebraic number theory. This problem is the subject of many studies and is of interest to several researchers. Let us recall some previous works regarding this problem. In [16], Gaál and Remete, calculated the elements of index 1 in pure quartic number fields $\mathbb{Q}(\sqrt[4]{m})$ for $1 < m < 10^7$ and $m \equiv 2, 3 \pmod 4$. In [15], Gaál and Győry gave an algorithm for solving index form equations in totally real quintic fields with Galois group S_5. In [6], Bilu, Gaál, and Győry studied the monogenity of totally real sextic number fields with Galois group S_6. In [2], Ahmad, Nakahara, and Husnine proved that if $m \equiv 2, 3 \pmod 4$ and $m \not\equiv \pm 1 \pmod 9$, then the pure sextic number field $\mathbb{Q}(\sqrt[6]{m})$ is monogenic. On the other hand, if $m \equiv 1 \pmod 4$ and $m \not\equiv \pm 1 \pmod 9$, then it is not monogenic (see [1]). Also, Hameed and Nakahara proved that if $m \equiv 1 \pmod 4$, then the octic number field $\mathbb{Q}(\sqrt[8]{m})$ is not monogenic, but if $m \equiv 2, 3 \pmod 4$, then it is monogenic (see [21]). In [9,10], El Fadil studied the monogenity of pure number fields of degree 12 and 24, respectively. In [11], El Fadil, Ben Yakkou, and Didi studied the monogenity of pure number fileds $\mathbb{Q}(\sqrt[42]{m})$. In [17], Gaál and Remete obtained, by applying the explicit form of the index equation, new deep results on monogenity

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of number fields $\mathbb{Q}(\sqrt[n]{m})$, where $3 \leq n \leq 9$ and m a square free rational integer. They also showed in [18] that if $m \equiv 2 \,\text{or}\, 3 \pmod 4$ is square free rational integer, then the octic field $K = \mathbb{Q}(i, \sqrt[4]{m})$ is not monogenic. Also in [25], Pethő and Pohst studied indices in multiquadratic number fields.

The aim of this paper is to study the monogenity of a pure number field K generated by a complex root α of a monic irreducible polynomial $F(x) = x^{120} - m$, with $m \neq \pm 1$ a rational integer. Our method is based to Newton polygon techniques applied on prime ideal factorization.

2. Main Results

Let K be a number field generated by a complex root α of a monic irreducible polynomial $F(x) = x^{120} - m \in \mathbb{Z}[x]$, where $m \neq \pm 1$ is a rational integer. The following theorem gives a necessary and sufficient conditions for $\mathbb{Z}_K = \mathbb{Z}[\alpha]$.

Theorem 2.1. *The ring $\mathbb{Z}[\alpha]$ is the ring of integers of K if and only if m is square free, $m \not\equiv 1 \pmod 4$, $m \not\equiv \pm 1 \pmod 9$, and $\overline{m} \notin \{\mp 1, 7, 18\} \pmod{25}$. In this case, K is monogenic and α generates a power integral of $\mathbb{Z}_K$.*

Remark 2.2. *Note that the significant Gassert's result ([19, Theorem 1.1]) yields only one way and cannot garantee the equivalence. However, Theorem 2.1 above gives the wanted equivalence in the context of pure number fields of degree* 120.

According to the above theorem, if m is not square free, $m \equiv 1 \pmod 4$, $m \equiv \pm 1 \pmod 9$, or $\overline{m}$ is contained in $\{\mp 1, 7, 18\} \pmod{25}$, then α does not generates a power integral basis of $\mathbb{Z}_K$. Henceforth, in these cases, Theorem 2.1 can not decide on the monogenity of K. The following theorem gives a partial answer. It produce infinite families of non-monogenic pure number fields K, that it $\mathbb{Z}_K$ has no power integral basis.

Theorem 2.3. *If one of the following conditions holds:*

1. $m \equiv 1 \mod 4$.
2. $m \equiv 1 \mod 9$.
3. $m \equiv 1 \mod 25$,

then K is not monogenic.

Remark 2.4. *Note that the condition m is square free is not required for the above theorem.*

As a consequence of the two previous theorems, the following result gives an important characterization of the monogenity of some special pure number fields of degree 120.

Corollary 2.5. *Let K be a pure number field generated by a root α of a monic irreducible polynomial $x^{120} - m^t$, with $m \neq \pm 1$ a square free rational integer and t a positive integer which is coprime to* 30*. Then*

1. *If $m \not\equiv 1 \pmod 4$, $m \not\equiv \pm 1 \pmod 9$, and $\overline{m} \notin \{\mp 1, 7, 18\} \pmod{25}$, then K is monogenic.*
2. *If $m \equiv 1 \mod 4$, $m \equiv 1 \mod 9$, or $m \equiv 1 \mod 25$, then K is not monogenic.*

3. Preliminaries

To prove our results, we based our method on prime ideal factorization. Let p be a rational prime. In 1878, Dedekind gave the explicit factorization of the principal ideal $p\mathbb{Z}_K$ when p does not divide the index $(\mathbb{Z}_K : \mathbb{Z}[\theta])$ for some primitive element $\theta \in \mathbb{Z}_K$ (see [8] and [23, Theorem 4.33]). He also gave a criterion known as Dedekind's criterion to test the divisibility of the index $(\mathbb{Z}_K : \mathbb{Z}[\theta])$ by p (see [7, Theorem 6.14], [8], and [23]). When p divides $i(K)$, then Dedekind's theorem cannot give the prime ideal factorization of $p\mathbb{Z}_K$. In 1928, Ore developed an algorithm to factorize $p\mathbb{Z}_K$. His method is based on Newton polygon techniques. The papers [12], [20], and [24] give a detailed survey on the theory and applications of Newton polygon techniques, including prime ideal factorization in number fields. Now, let us recall some fundamental notions on Newton polygon techniques. Let ν_p be the discrete valuation of $\mathbb{Q}_p(x)$ defined on $\mathbb{Z}_p[x]$ by

$$\nu_p\left(\sum_{i=0}^{r} a_i x^i\right) = \min\{\nu_p(a_i),\, 0 \le i \le r\}.$$

Let $\phi(x) \in \mathbb{Z}[x]$ be a monic polynomial whose reduction modulo p is irreducible. By successive euclidean divisions, any monic irreducible polynomial $F(x) \in \mathbb{Z}[x]$ admits a unique ϕ-adic development

$$F(x) = a_0(x) + a_1(x)\phi(x) + \cdots + a_n(x)\phi(x)^n,$$

with deg $(a_i(x)) <$ deg $(\phi(x))$. For every $0 \le i \le n$, let $u_i = \nu_p(a_i(x))$. The ϕ-Newton polygon of $F(x)$ is the lower boundary convex envelope of the set of points

$$\{(i, u_i) \,|\, 0 \le i \le n \text{ and } a_i(x) \ne 0\}$$

in the euclidean plane, which we denote by $N_\phi(F)$. The polygon $N_\phi(F)$ is the union of different adjacent sides $S_1, S_2, \dots, S_g$ with increasing slopes $\lambda_1 < \lambda_2 < \dots < \lambda_g$. We shall write $N_\phi(F) = S_1 + S_2 + \cdots + S_g$. The polygon determined by the sides of negative slopes of $N_\phi(F)$ is called the ϕ-principal Newton polygon of $F(x)$ and will be denoted by $N_\phi^+(F)$. Recall that the length of $N_\phi^+(F)$ is $l(N_\phi^+(F)) = \nu_{\overline{\phi}}(\overline{F(x)})$, the highest power of $\phi(x)$ dividing $F(x)$ modulo p. Let $\mathbb{F}_\phi$ be the finite residue field $\mathbb{Z}[x]/(p, \phi(x)) \simeq \mathbb{F}_p[x]/(\overline{\phi(x)})$. We attach to any abscissa $0 \le i \le l(N_\phi^+(F))$, the following residue coefficient:

$$c_i = \begin{cases} 0, & \text{if } (i, u_i) \text{ lies strictly above } N_\phi^+(F), \\ \dfrac{a_i(x)}{p^{u_i}} \pmod{(p, \phi(x))}, & \text{if } (i, u_i) \text{ lies on } N_\phi^+(F). \end{cases}$$

Let S be one of the sides of $N_\phi^+(F)$ and $\lambda = -\frac{h}{e}$ be its slope, where e and h are two positive coprime integers. The length of S, denoted $l(S)$ is the length of its projection to the horizontal axis. The degree of S is $d = d(S) = \frac{l(S)}{e}$; it is equal to the the number of segments into which the integral lattices divide S. More precisely, if (s, u_s) is the initial point of S, then the points with integer coordinates lying in S are exactly

$$(s, u_s),\ (s+e, u_s - h), \dots, (s + de, u_s - dh).$$

We attach to S the following residual polynomial defined by

$$R_{ł}(F)(y) = c_s + c_{s+e}y + \cdots + c_{s+(d-1)e}y^{d-1} + c_{s+de}y^d \in \mathbb{F}_\phi[y].$$

The ϕ-index of $F(x)$, denoted $ind_\phi(F)$, is deg(ϕ) times the number of points with natural integer coordinates that lie below or on the polygon $N_\phi^+(F)$, strictly above the horizontal axis and strictly beyond the vertical axis (see FIGURE 1). We say that the polynomial $F(x)$ is ϕ-regular with respect to p if for each side S of $N_\phi^+(F)$ of slope ł, its associated residual polynomial $R_{ł}(F)(y)$ is separable in $\mathbb{F}_\phi[y]$. The polynomial $F(x)$ is said to be p-regular if $F(x)$ is ϕ_i-regular for every $1 \le i \le t$, where $\overline{F(x)} = \prod_{i=1}^t \overline{\phi_i(x)}^{l_i}$ is the factorization of $\overline{F(x)}$ into a product of powers of distinct monic irreducible polynomials in $\mathbb{F}_p[x]$. For every $i = 1, \dots, t$, let $N_{\phi_i}^+(F) = S_{i1} + \cdots + S_{ir_i}$ and for every $j = 1, \dots, r_i$, let $R_{ł_{ij}}(F)(y) = \prod_{s=1}^{s_{ij}} \psi_{ijs}^{n_{ijs}}(y)$ be the factorization of $R_{ł_{ij}}(F)(y)$ in $\mathbb{F}_{\phi_i}[y]$. Now, we state the theorem of Ore, which plays a significant role in the proof of our results (see [12, Theorem 1.7 and Theorem 1.9], [20], and [24]):

Theorem 3.1. *(Ore's Theorem)*
Under the above notations, we have:

1.

$$\nu_p((\mathbb{Z}_K : \mathbb{Z}[\alpha])) \ge \sum_{i=1}^{t} ind_{\phi_i}(F).$$

The equality holds if $F(x)$ is p-regular.

2. If $F(x)$ is p-regular, then

$$p\mathbb{Z}_K = \prod_{i=1}^{t}\prod_{j=1}^{r_i}\prod_{s=1}^{s_{ij}} \mathfrak{p}_{ijs}^{e_{ij}},$$

where e_{ij} is the ramification index of the side S_{ij} and $f_{ijs} = deg(\phi_i) \times deg(\psi_{ijs})$ is the residue degree of $\mathfrak{p}_{ijs}$ over p.

Corollary 3.2. *Under the hypothesis of the above theorem, if for every $i = 1, \dots, t$, $l_i = 1$ or $N_{\phi_i}^{+}(F) = S_i$ has a single side of height 1, then p does not divide $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$.*

Example 3.3. *Consider the monic irreducible polynomial $F(x) = x^9 + 54x + 134$ which factors in $\mathbb{F}_3[x]$ as follow: $\overline{F(x)} = \overline{\phi(x)}^9$ where $\phi = x + 2$. The ϕ-adic development of $F(x)$ is*

$$F(x) = -486 + 2358\phi - 4608\phi^2 + 5376\phi^3 - 4032\phi^4 + 2016\phi^5 - 672\phi^6 + 144\phi^7 - 18\phi^8 + \phi^9$$

Thus, $N_\phi^{+}(F) = S_1 + S_2 + S_3$ with respect to ν_3 has three sides of degree 1 each joining the points $(0,5), (1,2), (3,1)$, and $(9,0)$ in the euclidean plane (see Figure 1) with respective slopes $\lambda_1 = -3$, $\lambda_2 = \frac{-1}{2}$, and $\lambda_3 = \frac{-1}{6}$. Thus, the residual polynomial $R_{\lambda_i}(F)(y)$ is irreducible polynomials in $\mathbb{F}_\phi[y] \simeq \mathbb{F}_3[y]$ as it is of degree 1 for $i = 1,2,3$. Thus, $F(x)$ is ϕ-regular. Hence it is 3-regular. Let $K = \mathbb{Q}(a)$ with α a root of $F(x)$. By Theorem 3.1, we have

$$\nu_2((\mathbb{Z}_K : \mathbb{Z}[\alpha])) = ind_\phi(F) = \deg(\phi) \times 4 = 4$$

and

$$3\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^6,$$

with residue degrees $f(\mathfrak{p}_k/3) = 1$ for $k = 1,2,3$.

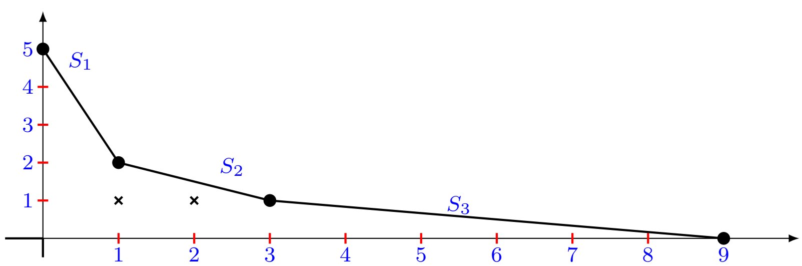

Figure 1: $N_\phi^{+}(F)$ with respect to ν_3.

Since it is difficult to find the ϕ-adic development of certain polynomials, we will use any adequate ϕ-admissible development of $F(x)$. This technique will allow us to comfortably apply Theorem 3.1. In what follows, we recall some useful facts concerning this technique. Let

$$F(x) = \sum_{j=0}^{n} A_j(x)\phi(x)^j, \ A_j(x) \in \mathbb{Z}_p[x] \tag{3.1}$$

be a ϕ-development of $F(x)$, not necessarily the ϕ-adic one. Take $\omega_j = \nu_p(A_j(x))$ for all $0 \le j \le n$. Let N be the principal Newton polygon of the set of points $\{(j,\omega_j),\ 0 \le j \le n, \omega_j \ne \infty\}$. To any $0 \le j \le n$, we attach the following residue coefficient:

$$c_j^{'} = \begin{cases} 0, & \text{if } (j,\omega_j) \text{ lies strictly above } N, \\ \dfrac{A_j(x)}{p^{\omega_j}} \pmod{(p, \phi(x))}, & \text{if } (j,\omega_j) \text{ lies on } N. \end{cases}$$

Likewise, for any side S of N with slope λ, we define the residual polynomial associated to S and denoted by $R_\lambda^{'}(F)(y)$ (similar to the residual polynomial $R_\lambda(F)(y)$ defined from the ϕ-adic development of $F(x)$).

We say that the ϕ-development (3.1) of $F(x)$ is admissible if $c_j^{'}\neq 0$ for each abscissa j of a vertex of N. Recall that $c_j^{'}\neq 0$ if and only if $\overline{\phi(x)}$ does not divide $\overline{\left(\frac{A_j(x)}{p^{\omega_j}}\right)}$. For more details, refer to [20]. The following lemma shows an important relationship between the ϕ-adic development and any ϕ-admissible development of a given polynomial $F(x)$.

Lemma 3.4. *([20, Lemma 1.12])*
If a ϕ-development of $F(x)$ is admissible, then $N_\phi^+(F)=N$ and $c_j^{'}=c_j$. In particular, for any segment S of N with slope λ, we have $R_\lambda^{'}(F)(y)=R_\lambda(F)(y)$ (up to multiply by a non-zero element of $\mathbb{F}_\phi$).

4. Proofs of main results

After recalling necessary preliminaries and results in the above section, we are now in the position to prove our main results. Let us begin by Theorem 2.1.

Proof of Theorem 2.1
Let $D(\alpha)$ be the discriminant of α and d_K the discriminant of K. By [23, Propositions 2.9 and 2.13], one has:

$$\begin{aligned} D(\alpha) &= D(1,\alpha,\dots,\alpha^{119})=(-1)^{\frac{120\times(120-1)}{2}}N_{K/\mathbb{Q}}(F^{'}(\alpha))=-N_{K/\mathbb{Q}}(120\cdot\alpha^{119})\\ &= 120^{120}N_{K/\mathbb{Q}}(\alpha)^{119}=(2^3\cdot 3\cdot 5)^{120}m^{119}=(\mathbb{Z}_K:\mathbb{Z}[\alpha])^2\cdot d_K. \end{aligned}$$

It follows that, $\mathbb{Z}[\alpha]$ is integrally closed if and only if p does not divide the index $(\mathbb{Z}_K:\mathbb{Z}[\alpha])$ for every rational prime p dividing $2\cdot 3\cdot 5\cdot m$. Let p be a rational prime dividing m, then $F(x)\equiv\phi^{120} \pmod p$, where $\phi=x$. The ϕ-principal Newton polygon of $F(x)$ with respect to ν_p, $N_\phi^+(F)=S$ has a single side with slope $\lambda=\frac{-\nu_p(b)}{120}$; it is the side joining the points $(0,\nu_p(b))$ and $(120,0)$. If $\nu_p(m)\geq 2$ (this means that m is not square free), then by using Theorem 3.1, we have

$$\nu_p(\mathbb{Z}_K:\mathbb{Z}[\alpha])\geq ind_\phi(F)=\frac{119(\nu_p(b)-1)+\gcd(\nu_p(b),120)-1}{2}\geq 2.$$

Consequently, p^2 divides the index $(\mathbb{Z}_K:\mathbb{Z}[\alpha])$ and α does not generate a power integral basis of $\mathbb{Z}_K$. If $\nu_p(m)=1$ for every prime divisor of m (i.e., m is square free), then $N_\phi^+(F)=S$ has a single side of height 1 with slope $\lambda=\frac{-1}{120}$. Thus, the residual polynomial $R_\lambda(F)(y)$ is irreducible over $\mathbb{F}_\phi\simeq\mathbb{F}_p$. By Theorem 3.1, we get $\nu_p((\mathbb{Z}_K:\mathbb{Z}[\alpha]))=ind_\phi(F)=0$, that is to say, p does not divide $(\mathbb{Z}_K:\mathbb{Z}[\alpha])$. Now, we deal with $p\in\{2,3,5\}$ when p does not divide m. Let us start by $p=2$. In this case, one has

$$\overline{F(x)}=\overline{(x^{15}-1)}^8 \text{ in } \mathbb{F}_2[x].$$

Since 2 does not divide 15, the polynomial $x^{15}-1$ is separable modulo 2. Let $\phi(x)$ be a monic irreducible factor of $x^{15}-1$ and write $x^{15}-1=\phi(x)U(x)+R(x)$ with $\overline{R(x)}=0$. Note that $\overline{\phi(x)}$ does not divide $\overline{U(x)}$, since $x^{15}-1$ is separable modulo 2. Write

$$\begin{aligned} F(x) &= x^{120}-m=(x^{15}-1+1)^8-m\\ &= (\phi(x)U(x)+R(x)+1)^8-m\\ &= (\phi(x)U(x))^8+\sum_{j=1}^{7}\binom{8}{j}(R(x)+1)^{8-j}U(x)^j\phi(x)^j+(R(x)+1)^8-m.\\ &= (\phi(x)U(x))^8+\sum_{j=1}^{7}\binom{8}{j}(R(x)+1)^{8-j}U(x)^j\phi(x)^j+\sum_{j=1}^{7}\binom{8}{j}(R(x))^j\\ &+ 1-m. \end{aligned}\tag{4.1}$$

Let $A_0(x)=\sum_{j=1}^{7}\binom{8}{j}(R(x))^j+1-m$. Note that $\nu_2(\sum_{j=1}^{7}\binom{8}{j}(R(x))^j)\geq 2$. If $m\not\equiv 1 \pmod 4$, then $\nu_2(A_0(x))=1$. Since $\overline{\phi(x)}$ does not divide $\overline{U(x)(R(x)+1)}$, the above ϕ-development (4.1) of $F(x)$ is admissible. Thus, by Lemma 3.4, for every irreducible factor $\phi(x)$ of $F(x)$ modulo 2, $N_\phi^+(F) = S$ has a single side of height 1 joining the points $(0,1)$ and $(8,0)$. By Corollary 3.2, 2 does not divide the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$. If $m \equiv 1 \pmod 4$, then $\nu_2(A_0(x)) \geq 2$. Let $A_0(x) = \sum_{j\geq 0} a_j(x)\phi(x)^j$ be the ϕ-adic development of $A_0(x)$ (that is $\deg(a_j(x)) < \deg(\phi(x))$). It follows that

$$F(x) = \sum_{j\geq 9} a_j(x)\phi(x^j) + (a_8(x) + U(x)^8)\phi(x)^8 + \sum_{j=1}^{7}(\binom{8}{j}(R(x)+1)^{8-j}U(x)^j + a_j(x))\phi(x)^j + a_0(x)$$

Since $\nu_2(A_0(x)) \geq 2$, $\nu_2(a_j(x)) \geq 2$, let

$$\sum_{j=1}^{7}(\binom{8}{j}(R(x)+1)^{8-j}U(x)^j + a_j(x))\phi(x)^j = \sum_{j\geq 1} b_j(x)\phi(x)^j,$$

with $\deg(b_j(x)) < \deg(\phi(x))$. We rewrite the ϕ-development (4.1) of $F(x)$ as follows:

$$F(x) = \sum_{j\geq 9}(a_j(x) + b_j(x))\phi(x)^j + (U(x)^8 + a_8(x) + b_8(x))\phi(x)^8 + \sum_{j=1}^{7} b_j(x)\phi(x)^j + a_0(x) \tag{4.2}$$

Note that the above development (4.2) is the unique ϕ-adic development of $F(x)$. For every $j = 1,\dots,7$, let $\nu_j = \nu_2(b_j(x))$ which is greater than 1 and $\nu_0 = \nu_2(a_0(x))$ which is greater than 2. Note also that $\nu_2\left(U(x)^8 + a_8(x) + b_8(x)\right) = 0$. So, $N_\phi^+(F)$ is the Newton polygon joining the points $(0,\nu_0)$, $(1,\nu_1),\dots,(7,\nu_7)$, and $(8,0)$ in the euclidean plane. It follows that $(1,1)$ is a point with natural integer coordinates that lie below or on the polygon $N_\phi^+(F)$, strictly above the horizontal axis and strictly beyond the vertical axis. So, by Theorem 3.1, 2 divides the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$. We conclude that when 2 does not divide m, then 2 does not divide the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$ if and only if $m \not\equiv 1 \pmod 4$. Now we deal with $p \in \{3,5\}$ and p does not divide m. Set $120 = p\cdot u$ with p does not divide u. In this case, $\overline{F(x)} = \overline{(x^u - m)}^p$ in $\mathbb{F}_p[x]$. Let $\phi(x)$ be a monic irreducible factor of $x^u - m$ modulo p, and set $x^u - m = \phi(x)U(x) + R(x)$ with $\overline{R(x)} = \overline{0}$ and $\overline{\phi(x)}$ does not divide $\overline{U(x)}$. Note that $\nu_p\left(\binom{p}{j}\right) = 1$ for every $j = 1,\dots,p-1$. Set $T(x) = \frac{1}{p^2}\sum_{j=0}^{p-1}\binom{p}{j}R(x)^{p-j}m^j \in \mathbb{Z}[x]$. By applying binomial theorem twice, we get

$$F(x) = x^{120} - m = (\phi(x)U(x))^p + \sum_{j=1}^{p-1}\binom{p}{j}(R(x)+m)^{p-j}U(x)^j\phi(x)^j + p^2T(x) + m^p - m. \tag{4.3}$$

If $\nu_p(m^p - m) = \nu_p(m^{p-1}-1) = 1$, then $\nu_p(p^2T(x) + m^p - m) = 1$. Since $\overline{\phi(x)}$ does not divide $(R(x)+m)^{p-j}U(x)^j$, the above ϕ-development (4.3) of $F(x)$ is admissible. By Lemma 3.4, $N_\phi^+(F) = S$ has a single side of height 1 joining the points $(0,1)$ and $(p,0)$. By Corollary 3.2, p does not divide the index $(\mathbb{Z}_K : \mathbb{Z}[\alpha])$. Suppose now that $\nu_p(m^p - m) \geq 2$. Let $T(x) = \sum_{j\geq 0} a_j(x)\phi(x)^j$ be the ϕ-adic development of $T(x)$ (i.e., $\deg(a_j(x)) < \deg(\phi(x))$). We rewrite the development (4.3) as follows:

$$F(x) = \sum_{j\geq p+1} a_j(x)\phi(x)^j + \sum_{j=1}^{p-1} A_j(x)\phi(x)^j + p^2a_0(x) + m^p - m, \tag{4.4}$$

where $A_j(x) = p\left(\dfrac{\binom{p}{j}(R(x)+m)^{p-j}}{p} + pa_j(x)\right)$ for every $j = 1,\dots,p-1$. It follows that $\omega_j = \nu_p(A_j(x)) = 1$ for every $j = 1,\dots,p-1$ and $\overline{\phi(x)}$ does not divide $\overline{\left(\frac{A_j(x)}{p^{\omega_j}}\right)}$. Moreover, $\omega_0 = \nu_p(A_0(x)) =$

$\nu_p(p^2a_0(x)+m^p-m)\geq 2$ and $\overline{\phi(x)}$ does not divide $\overline{\left(\frac{A_0(x)}{p^{\omega_0}}\right)}$, because $\deg(a_0(x))<\deg(\phi(x))$. Thus, the ϕ-developement (4.4) of $F(x)$ is admissible. By Lemma 3.4, for any monic irreducible factor of $F(x)$ modulo p, $N_\phi^+(F)=S_1+S_2$ has two sides of degree 1 each joining the points $(0,\omega_0)$, $(1,1)$, and $(p,0)$ in the euclidean plane with $\omega_0\geq 2$. Thus, $R_{\lambda_k}(F)(y)$ is irreducible over $\mathbb{F}_\phi$ as it is of degree 1 for every $k=1,2$. It follows that $F(x)$ is p-regular. By Theorem 3.1, $\nu_p(\mathbb{Z}_K:\mathbb{Z}[\alpha])=\sum ind_\phi(F)\geq 1$, where ϕ runs over all monic irreducible factors of $F(x)$ modulo p. We conclude that when $p\in\{3,5\}$ does not divide m, then p does not divide the index $(\mathbb{Z}_K:\mathbb{Z}[\alpha])$ if and only if $\nu_p(m^{p-1}-1)=1$, equivalently, if $m\not\equiv\pm1 \pmod 9$ and $m\equiv\pm1,7,18 \pmod{25}$. This completes the proof of the Theorem. $\square$

The following lemma gives a sufficient condition for a rational prime p to be a prime common index divisor of the field K. For more details, refer to [8] and [23, Theorems 4.33 and 4.34].

Lemma 4.1. *Let p be a rational prime and K a number field. For every positive integer m, let $P_p(m)$ be the number of distinct prime ideals of $\mathbb{Z}_K$ lying above p with residue degree m and $N_p(m)$ be the number of monic irreducible polynomials of $\mathbb{F}_p[x]$ of degree m. If for some positive integer m, $P_p(m)>N_p(m)$, then p is a prime common index divisor of K.*

Recall that the number of monic irreducible polynomials of degree m in $\mathbb{F}_p[x]$ is

$$N_p(m)=\frac{1}{m}\sum_{d|m}\mu(d)p^{\frac{m}{d}},$$

where μ is the Möubius function. This number was found by Gauss (see [23, Proposition 4.35]).

Proof of Theorem 2.3

In all cases, we prove that K is not monogenic by showing that $i(K)>1$.

1. Since $m\equiv 1 \pmod 4$, $\overline{F(x)}=\overline{(\phi_1(x)\phi_2(x)U(x))}^8$ in $\mathbb{F}_2[x]$, where $\phi_1(x)=x-1$, $\phi_2(x)=x^2+x+1$, $U(x)=1+x^3+x^6+x^9+x^{12}$. Write

$$\begin{array}{rcl} F(x) & = & x^{120}-m=(x^{15}-1+1)^8-m\\ & = & (\phi_1(x)\phi_2(x)U(x)+1)^8-m\\ & = & (\phi_1(x)\phi_2(x)U(x))^8+8(\phi_1(x)\phi_2(x)U(x))^7+28(\phi_1(x)\phi_2(x)U(x))^6\\ & + & 56(\phi_1(x)\phi_2U(x))^5+70(\phi_1(x)\phi_2(x)U(x))^4+56(\phi_1(x)\phi_2(x)U(x))^3\\ & + & 28(\phi_1(x)\phi_2(x)U(x))^2+8(\phi_1(x)\phi_2(x)U(x))+1-m. \end{array} \tag{4.5}$$

Since $\overline{\phi_1(x)}$ does not divide $\overline{\phi_2(x)U(x)}$ and $\overline{\phi_2(x)}$ does not divide $\overline{\phi_1(x)U(x)}$, the above ϕ_i-development is admissible for $i=1,2$. Let $\nu=\nu_2(1-m)$. We distinguish four cases.

• If $m\equiv 5 \pmod 8$; $\nu=2$, then by using (4.5) and Lemma 3.4, $N_{\phi_i}^+(F)=S$ has a single side of degree 2 with slope $\lambda=\frac{-1}{4}$ joining the points $(0,2)$ and $(8,0)$ for $i=1,2$ (see Figure 2). The residual polynomial provided by $\phi_1(x)$, $R_\lambda^1(F)(y)=1+y+y^2\in\mathbb{F}_{\phi_1}[y]$ which is irreducible, and the residual polynomial provided by $\phi_2(x)$ is

$$R_\lambda^2(F)(y) \quad = \quad 1+(\phi_1(j)U(j))^4y+(\phi_1(j)U(j))^8y^2\in\mathbb{F}_{\phi_2}[y],$$

where $\phi_2(j)=1+j+j^2=0$ in $\mathbb{F}_{\phi_2}[y]$. It follows that

$$\begin{array}{rcl} R_\lambda^2(F)(y) & = & 1+(1+j)y+jy^2\\ & = & (y+1)(jy+1) \end{array}$$

Thus, $F(x)$ is 2-regular. By Theorem 3.1, we see that

$$2\mathbb{Z}_K=(\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3)^4\mathfrak{a},$$

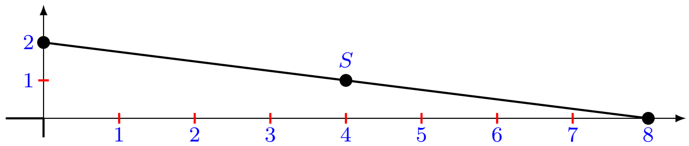

Figure 2: $N^+_{\phi_i}(F), i = 1, 2$ when $m \equiv 5 \pmod 8$

where is $\mathfrak{a}$ is a non-zero ideal of $\mathbb{Z}_K$ (provided by the factors of $U(x)$ modulo 2) and $\mathfrak{p}_k$ is a prime ideal of residue degree $f(\mathfrak{p}_k/2) = 2$ for $k = 1, 2, 3$. So, $P_2(2) \geq 3 > N_2(2) = 1$ (note that there is a unique monic irreducible polynomial over $\mathbb{F}_2$, namely $x^2 + x + 1$). By Lemma 4.1, 2 divides $i(K)$, and so K is not monogenic.

• If $m \equiv 9 \pmod{16}$; $\nu = 3$, then by using (4.5) and Lemma 3.4, $N^+_{\phi_i}(F) = S_{i1} + S_{i2}$ has two sides joining the points $(0, 3), (4, 1)$, and $(8, 0)$ in the euclidean plane, with respective degrees $d(S_{i1}) = 2$ and $d(S_{i2}) = 1$, with respective slopes $\lambda_{i1} = \frac{-1}{2}$ and $\lambda_{i2} = \frac{-1}{4}$ for $i = 1, 2$ (see Figure 3). Thus, $R_{\lambda_{11}}(F)(y) = 1 + y + y^2$ and $R_{\lambda_{12}}(F)(y) = 1 + y$ which are irreducible in $\mathbb{F}_{\phi_2}[y] \simeq \mathbb{F}_2[y]$. On the other hand,

$$\begin{aligned} R_{\lambda_{21}}(F)(y) &= 1 + ((\phi_1 U)(j))^2 y + ((\phi_1 U)(j))^4 \\ &= (y + 1)((1 + j)y + 1) \end{aligned}$$

which is separable over $\mathbb{F}_{\phi_2}[y]$. Also, $R_{\lambda_{22}}(F)(y) = 1 + y$. We conclude that $F(x)$ is 2-regular. By applying Theorem 3.1, we obtain that

$$2\mathbb{Z}_K = (\mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3)^2\mathfrak{p}_4^4\mathfrak{q}_1^4\mathfrak{a},$$

where $\mathfrak{a}$ is a non-zero ideal of $\mathbb{Z}_K$, $\mathfrak{p}_k$ is a prime ideal of $\mathbb{Z}_K$ of residue degree $f(\mathfrak{p}_k/2) = 2$ for $k = 1, 2, 3, 4$ and $\mathfrak{q}$ is a prime ideal with $f(\mathfrak{q}/2) = 1$. So, there are four prime ideals of residue degree 2 each lying above 2. Since there is a unique monic irreducible polynomial of degree 2 in $\mathbb{F}_2[x]$, by Lemma 4.1, 2 divides $i(K)$. Consequently K is not monogenic.

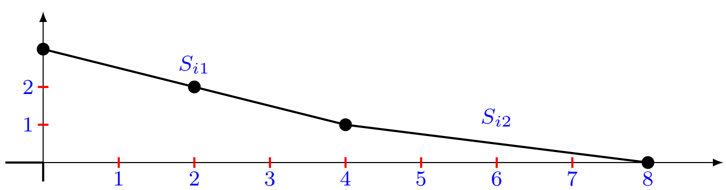

Figure 3: $N^+_{\phi_i}(F)$, $i = 1, 2$ when $m \equiv 9 \pmod{16}$

• If $m \equiv 17 \pmod{32}$; $\nu = 4$, then by using (4.5) and Lemma 3.4, $N^+_{\phi_i}(F) = S_{i1} + S_{i2} + S_{i3}$ with respective degrees $d(S_{i1}) = 2$ and $d(S_{i2}) = d(S_{i3}) = 1$ (see Figure 4). Similarly to the above case, we get that $F(x)$ is 2-redular. By applying Theorem 3.1, we see that

$$2\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4^2\mathfrak{p}_5^4\mathfrak{q}_1^2\mathfrak{q}_2^4\mathfrak{a},$$

where $\mathfrak{a}$ is a non-zero ideal of $\mathbb{Z}_K$, $\mathfrak{p}_k$ is a prime ideal of residue degree $f(\mathfrak{p}_k/2) = 2$ for $k = 1, \ldots, 5$, and $\mathfrak{q}_k$ is a prime ideal of $\mathbb{Z}_K$ of residue degree 1 for $k = 1, 2$. Then, for $p = 2$, we have $P_2(2) \geq 5 > 1 = N_2(2)$, by Lemma 4.1, 2 divides $i(K)$. So, K is not monogenic.

• If $m \equiv 1 \pmod{32}$, then $N^+_{\phi_i}(F) = S_{i1} + S_{i2} + S_{i3} + S_{i4}$ has four sides of degree 1 each, for $i = 1, 2$ (see Figure 5). Their attached residual polynomial $R_{\lambda_{ik}}(F)(y)$ is irreducible in $\mathbb{F}_{\phi_i}[y]$ for every $i = 1, 2$ and $k = 1, \ldots, 4$ as it is of degree 1. So, the polynomial $F(x)$ is 2-regular. By Theorem 3.1, we see that

$$2\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3^2\mathfrak{p}_4^8\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3^2\mathfrak{q}_4^8\mathfrak{a},$$

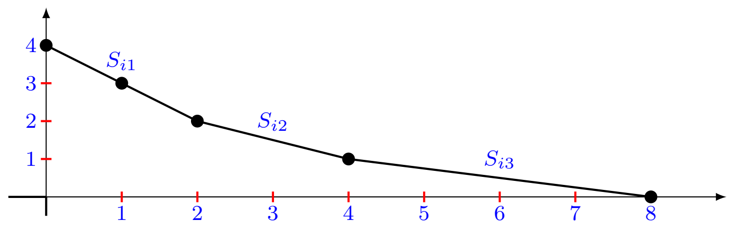

Figure 4: $N^+_{\phi_i}(F)$, $i = 1, 2$ when $m \equiv 17 \pmod{16}$

where $\mathfrak{a}$ is a non-zero ideal of $\mathbb{Z}_K$ and $\mathfrak{p}_k$ is a prime ideal of residue degree $f(\mathfrak{p}_k/2) = 2$ for $k = 1, 2, 3, 4$ and $\mathfrak{q}_k$ is a prime ideal of $\mathbb{Z}_K$ of residue degree 1 for $k = 1, 2, 3, 4$. It follows that there are at least four prime ideals of residue degree 1 each lying above 2. As there are only two monic irreducible polynomial of degree 1 in $\mathbb{F}_2[x]$, namely x and $x + 1$, by Lemma 4.1, 2 divides $i(K)$. Consequently, K is not monogenic. We conclude that if $m \equiv 1 \pmod 4$, then K is not monogenic.

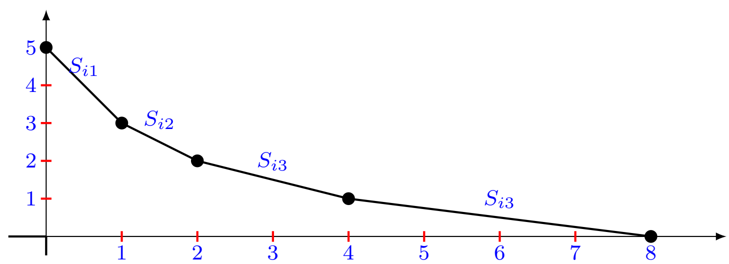

Figure 5: $N^+_{\phi_i}(F)$, $i = 1, 2$ when $m \equiv 1 \pmod{32}$

2. If $m \equiv 1 \pmod 9$; $\mu = \nu_3(m^2 - m) = \nu_3(1 - m) \geq 2$, then $\overline{F(x)} = \overline{(x^{40} - 1)}^3 = \overline{(\phi_1(x)\phi_2(x)V(x))}^3$ in $\mathbb{F}_3[x]$, where $\phi_1(x) = x - 1$, $\phi_2(x) = x + 1$ such that $\overline{\phi_i(x)}$ does not divide $\overline{V(x)}$. By Proof of Theorem 2.1 when we have determined $N^+_\phi(F)$ for $p \in \{3, 5\}$ and p does not divide m, $N^+_{\phi_i}(F) = S_{i1} + S_{i2}$ has two sides of degree 1 each joining the points $(0, \omega_i)$, $(1, 1)$, and $(3, 0)$, with $\omega_i \geq 2$ (see Figure 6). Thus, $R_{\lambda_{ik}}(F)(y)$ is irreducible in $\mathbb{F}_{\phi_k}[y]$ for every $k = 1, 2$, $i = 1, 2$. By Theorem 3.1, we have

$$3\mathbb{Z}_K = \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3^2\mathfrak{p}_4^2\mathfrak{a},$$

where $\mathfrak{a}$ is a proper ideal of $\mathbb{Z}_K$ and $\mathfrak{p}_j$ is a prime ideal with residue degree $f(\mathfrak{p}_j/3) = 1$ for $j = 1, 2, 3, 4$. Then, for $p = 3$, we have $P_3 \geq 4 > N_3(1) = 3$. By Lemma 4.1, 3 divides $i(K)$. Consequently, K is not monogenic.

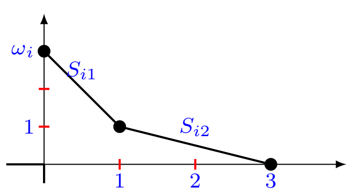

Figure 6: $N^+_{\phi_i}(F)$, $i = 1, 2$ when $m \equiv 1 \pmod 9$

3. If $m \equiv 1 \pmod{25}$; $\omega = \nu_5(m^4 - 1) \geq 2$, then $\overline{F(x)} = \overline{(x^{24} - 1)}^5 \overline{\left(\prod_{i=1}^4 \phi_i(x)V(x)\right)}^5$, where $\phi_i(x) = x - i$ for $i = 1, 2, 3, 4$, such that $\overline{\phi_k(x)}$ does not divide $\overline{V(x)}$.By Proof of Theorem 2.1, for $p = 5$, $i = 1, 2, 3, 4$, $\mathbb{N}^+_{\phi_i}(F) = S_{i1} + S_{i2}$ has two sides with respective slopes $\lambda_{i1} \leq -1$ and $\lambda_{i2} = \frac{-1}{4}$ joining

the points $(0,\omega_i)$, $(1,1)$, and $(0,5)$. Thus, the residual polynomial $R_{\mathfrak{l}_{ik}}(F)(y) \in \mathbb{F}_{\phi_i}[y] \simeq \mathbb{F}_5[y]$ is irreducible as it is of degree 1 for $i = 1,2,3,4$, $k = 1,2$. By Theorem 3.1, we see that

$$5\mathbb{Z}_K = \prod_{i=1}^{4} \mathfrak{p}_{i1}\mathfrak{p}_{i2}^4\mathfrak{a},$$

where $\mathfrak{a}$ is a non-zero ideal of $\mathbb{Z}_K$ and $\mathfrak{p}_{ik}$ is a prime ideal of $\mathbb{Z}_K$ of residue degree $f(\mathfrak{p}_{kj}/5) = \deg(\phi_i(x)) \times \deg(R_{\mathfrak{l}_{ik}}(F)(y)) = 1 \times 1 = 1$ for every $i = 1,\ldots,4$ and $k = 1,2$. Thus, there are at least 8 prime ideals of $\mathbb{Z}_K$ of residue degree 1 each lying above 5. As there are only 5 monic irreducible polynomials of degree 1 in $\mathbb{F}_5[x]$, namely x, $x-1$, $x-2$, $x-3$ and $x-4$, by Lemma 4.1, 5 divides $i(K)$. Hence, K is not monogenic.

□

Proof of Corollary 2.5
As $\gcd(t,30) = 1$, let (x,y) be the positive solution of the Diophantine equation $xt - 120y = 1$ with $0 \le y < t$. Let $\eta = \frac{\alpha^x}{m^y}$. Then $\eta^{120} = \frac{\alpha^{120x}}{m^{120y}} = \frac{m^{tx}}{m^{120y}} = m^{tx-120y} = m$. Since $G(x) = x^{120} - m \in \mathbb{Z}[x]$ is an Eisenstein polynomial with respect to any prime divisor of m, then it is irreducible over $\mathbb{Q}$. On the other hand, as $\eta \in K$ and $[K:\mathbb{Q}] = \deg(G(x))$, $K = \mathbb{Q}(\eta)$. Up to replace $F(x)$ by $G(x)$, then by a direct application of Theorems 2.1 and 2.3, we conclude the corollary. □

5. Examples

Let $F(x) \in \mathbb{Z}[x]$ be a monic irreducible polynomial and K the number field generated by a root α of $F(x)$.

1. Let $F(x) = x^{120} - 2022 \in \mathbb{Z}[x]$. As $F(x)$ is a 3-Eisenstein polynomial, then it is irreducible over $\mathbb{Q}$. Since $2022 \equiv 2 \pmod 4$, $2022 \equiv 6 \pmod 9$, and $2022 \equiv 22 \pmod{25}$, by Theorem 2.1, K is monogenic and α generates a power integral basis of $\mathbb{Z}_K$.

2. Let $F(x) = x^{120} - 178 \in \mathbb{Z}[x]$. As $F(x)$ is a 2-Eisenstein polynomial, then it is irreducible over $\mathbb{Q}$. Since $178 \equiv 2 \pmod 4$, $178 \equiv 7 \pmod 9$ and $178 \equiv 3 \pmod{25}$, by Theorem 2.1, K is monogenic.

3. Let $F(x) = x^{120} - 106 \in \mathbb{Z}[x]$. As $F(x)$ is a 2-Eisenstein polynomial, then it is irreducible over $\mathbb{Q}$. Since $106 \equiv 1 \pmod 4$, by Theorem 2.3, K is not monogenic.

4. Let $F(x) = x^{120} - 82 \in \mathbb{Z}[x]$. As $F(x)$ is a 2-Eisenstein polynomial, then it is irreducible over $\mathbb{Q}$. Since $82 \equiv 1 \pmod 9$, by Theorem 2.3, K is not monogenic.

5. Let $F(x) = x^{120} - 626 \in \mathbb{Z}[x]$. As $F(x)$ is a 2-Eisenstein polynomial, then it is irreducible over $\mathbb{Q}$. Since $626 \equiv 1 \pmod{25}$, by Theorem 2.3, K cannot be monogenic.

6. If $F(x) = x^{120} - 6^7$, then K is monogenic and $\theta = \frac{\alpha^7}{6}$ generates a power integral basis of $\mathbb{Z}_K$.

Final comments

It is important to note that the fundamental method which allows to test whether a number field is monogenic or not is to solve the index form equation which is very complicated for higher number field degrees, see e.g [13] and [14]. In this work, we have based our method of Newton polygon techniques applied on prime ideal factorization which is an efficient tool to investigate the monogenity of such pure number fields.

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