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Maximum Number of Limit Cycles of a Second Order Differential System

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ABSTRACT: In this paper, we study the limit cycles of a perturbed differential system in $\mathbb{R}^2$, given by

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x-\epsilon(1+\sin^n(\theta)\cos^m(\theta))H(x,y), \end{cases}$$

where $\epsilon > 0$ is a small parameter, m and n are non-negative integers, $\tan(\theta) = y/x$, and $H(x, y)$ is a real polynomial of degree $l \geq 1$. Using Averaging theory of first order we provide an upper bound for the maximum number of limit cycles. Also, we provide some examples to confirm and illustrate our results.

Key Words: Limit cycles, averaging theory, polynomial differential systems.

Contents

1. Introduction

The limit cycle is one of the most used notion in the qualitative theory of planar differential system recently with the arising of its maximum number determination problem. The study of limit cycles was initiated by Poincaré [8] in 1881 where he defined this notion as a periodic orbit isolated in the set of all periodic orbits of a differential system. He defined also the notion of a center of a real planar differential system which it is an isolated critical point in the neighborhood of periodic orbits. In this paper, we will produce limit cycles by perturbing the periodic orbits of a center using Averaging theory, which it is a powerful tool for studying the existence, number and behavior of limit cycles. There are many results concerning the maximum number of limit cycles bifurcating from the linear center of planar polynomial differential systems using averaging theory (see for instance [5], [6], [2]).

The Mathieu differential equation ([7])

$$\ddot{x} + b(1 + \cos t)x = 0, \tag{1.1}$$

where b is a real constant, is the simplest mathematical model of an excited system depending on a parameter, it describes the dynamics of a system with harmonic parametric excitation and a nonlinear term corresponding to a restoring force. In [4] T. Chen & J. Llibre studied the limit cycles of a kind of generalization of the equation (1.1). They considered the second order differential system

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x-\epsilon(1+\cos^m(\theta))Q(x,y), \end{cases}$$

where $\epsilon > 0$ sufficiently small, m is an arbitrary non negative integer, $Q(x, y)$ is a polynomial of degree $n \geq 1$ and $\theta = \arctan(y/x)$. They studied the maximum number of limit cycles which can bifurcate from

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the linear center $\dot{x} = y,\ \dot{y} = -x$ of the previous system with $\epsilon = 0$. They obtained an upper bound for this system with respect to the parity of m and n. In this work, using the averaging theory of first order, we study the maximum number of limit cycles of the following differential system

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x - \epsilon(1 + \sin^n(\theta)\cos^m(\theta))H(x, y), \end{cases} \tag{1.2}$$

where $\epsilon > 0$ sufficiently small, m and n are arbitrary non negative integers, $H(x, y)$ is a polynomial of degree $l \geq 1$ and $\theta = \arctan(y/x)$.

2. Preliminaries

In this section, we give some tools that we shall use for proving the main result.
The first tool is the averaging averaging which is the basic method that we will use in our work.
Consider the differential system at the following initial value

$$\dot{x} = \varepsilon F(t, x(t)) + \varepsilon^2 R(t, x(t), \varepsilon), \quad x(0) = x_0, \tag{2.1}$$

with $x \in D \subset \mathbb{R}^n$, D is a bounded domain and $t \geq 0$. Suppose that $F(t, x)$ and $R(t, x, \varepsilon)$ are T-periodic in t. The averaged system associated with the system (2.1) is defined by

$$\dot{y}(t) = \varepsilon f(y(t)), \quad y(0) = x_0, \tag{2.2}$$

where the averaged function is defined by

$$f(y) = \frac{1}{T}\int_0^T F(s, y)\, ds. \tag{2.3}$$

The following theorem gives us the conditions for which the singular points of the averaged system (2.2) provide $T-$ periodic orbits of the system (2.1). For a proof for this theorem, see ([3], [9]).

Theorem 2.1. *Consider the system (2.1) and suppose that the vector functions F, R, D_xF, D_x^2F and D_xR are continuous and bounded by a constant M (independent of ε) in $[0, \infty) \times D$ with $-\varepsilon_0 < \varepsilon < \varepsilon_0$. Then In addition, suppose that F and R are T-periodic in t with T independent of ε.*

1. *If $p \in D$ is a singular point of the averaged system (2.2) such that*

$$\det(D_x f(p)) \neq 0, \tag{2.4}$$

then for $|\varepsilon| > 0$ small enough, there exists a T-periodic solution $x_\varepsilon(t)$ of the system (2.1) such that $x_\varepsilon(0) \to p$ when $\varepsilon \to 0$.

2. *If the singular point $y = p$ of the averaged system (2.2) is hyperbolic then for $|\varepsilon| > 0$ small enough, the corresponding periodic solution of the system (2.1) is unique, hyperbolic and has the same kind of stability of p.*

In order to study the simple zeros of the averaged function we shall apply the Descartes Theorem.

Theorem 2.2 (Descartes Theorem)**.** *Let*

$$p(r) = a_{i_1} r^{i_1} + a_{i_2} r^{i_2} + ... + a_{i_n} r^{i_n}$$

be a polynomial with real coefficients, with $0 \leq i_1 < i_2 < ... < i_n$ and $a_{i_j} \neq 0$ real constants for $j \in 1, 2, ..., n$. When $a_{i_j} a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of the signs is m, then $p(r)$ has at most m positive real zeros. Furthermore, we can choose the coefficients of $p(r)$ in such a way that $p(r)$ has exactly $n - 1$ positive real zeros.

For a proof of the previous Theorem see [1].

Before we get to compute the averaged function, we introduce some of the needed formulas

$$\int_0^{2\pi} \cos(\theta)^p sin(\theta)^{2q} d\theta = \frac{(2q-1)!!}{(2q+p)(2q+p-2)...(p+2)} \int_0^{2\pi} \cos(\theta)^p d\theta,$$

where $p \in \mathbb{R}\backslash\{-2,-4,...\}, q \in \mathbb{N}$.

$$\int_0^{2\pi} \cos(\theta)^p \sin(\theta)^{2q+1} d\theta = 0, \; p \in \mathbb{R}\backslash\{-1,-3,...\}, \quad q \in \mathbb{N}.$$

$$\int_0^{2\pi} \cos(\theta)^{2l} d\theta = \frac{(2l-1)!!}{2^l l!} 2\pi, \quad l > 0.$$

$$\int_0^{2\pi} \cos(\theta)^{2l+1} d\theta = 0, \quad l \geq 0.$$

For more information on these integrals see [10].

3. Main Result

Our main result is the following one.

Theorem 3.1. *Using the averaging theory of first order, the maximum number of limit cycles of the differential system (1.2) bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ is as follows*

1. ***If l is even***
   - *(1.a) we have at most $l-1$ limit cycles, if m is even and n is odd or m is odd and n is even.*
   - *(1.b) we have at most $(l-2)/2$ limit cycles, if m is even and n is even or m is odd and n is odd.*
2. ***If l is odd***
   - *(2.a) we have at most l limit cycles, if m is even and n is odd or m is odd and n is even.*
   - *(2.b) we have at most $(l-1)/2$ limit cycles, if m is even and n is even or m is odd and n is odd.*

Theorem 3.1 is proved in the following subsection.

3.1. Proof of Theorem 3.1

Assume that the polynomial $H(x,y) = \sum_{i+j=0}^{l} a_{ij} x^i y^j$. By applying change of the variables (x,y) into the polar coordinates (r,θ) defined by $x = r\cos(\theta)$, $y = r\sin(\theta)$, with $r > 0$, the system (1.2) becomes

$$\begin{cases} \dot{r} &=& -\epsilon \displaystyle\sum_{i+j=0}^{l} R_{ij}(\theta) r^{i+j}, \\ \dot{\theta} &=& -1 - \epsilon \displaystyle\sum_{i+j=0}^{l} \Theta_{ij}(\theta) r^{i+j-1}, \end{cases}$$

where

$$\begin{cases} R_{ij}(\theta) &=& a_{ij}(\cos^i(\theta)\sin^{j+1}(\theta) + \cos^{i+m}(\theta)\sin^{j+n+1}(\theta)), \\ \Theta_{ij}(\theta) &=& a_{ij}(\cos^{i+1}(\theta)\sin^{j}(\theta) + \cos^{i+m+1}(\theta)\sin^{j+n}(\theta)). \end{cases}$$

From the previous differential system we take θ as the new independent variable as follows

$$\frac{dr}{d\theta} = \epsilon \sum_{i+j=0}^{l} R_{ij}(\theta) r^{i+j} + O(\epsilon^2) = \epsilon F(r,\theta) + O(\epsilon^2). \qquad (3.1)$$

Now, we compute the averaged function f, associated with the previous equation, which is given by

$$f(r) = \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)\,d\theta. \tag{3.2}$$

We distinguish two cases for the parity of l, each case has four subcases for the parity of m and n.

Case (1) . Suppose that l is even, we have two subcases for studying $f(r)$.

Subcase (1.1) If m is even and n is odd, we have

$$\begin{aligned}
f(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} a_{ij}r^{i+j}(\cos^i\theta)\sin^{j+1}(\theta) + \cos^{i+m}(\theta)\sin^{j+n+1}(\theta))d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\right. \\
&\quad \left. + \sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^{i+m}(\theta)\sin^{j+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p-1=0}^{l} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{i+2p=0}^{l} a_{i,2p}r^{i+2p}\cos^{i+m}(\theta)\sin^{2p+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p=2}^{l+1} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{i+2p=1}^{l+1} a_{i,2p}r^{i+2p}\cos^{i+m}(\theta)\sin^{2p+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{2q+2p=1}^{l+1} a_{2q,2p}r^{2q+2p}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{2q+2p=2}^{l} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{2q+2p=2}^{l} a_{2q,2p}r^{2q+2p}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{q+p=1}^{l/2} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{q+p=1}^{l/2} a_{2q,2p}r^{2q+2p}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)\right] d\theta
\end{aligned}$$

$$= \frac{1}{2\pi} \sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1} \int_0^{2\pi} \cos^2 q(\theta) \sin^{2p}(\theta) d\theta$$

$$+ \frac{1}{2\pi} \sum_{q+p=1}^{l/2} a_{2q,2p} r^{2q+2p} \int_0^{2\pi} \cos^{2q+m}(\theta) \sin^{2p+n+1}(\theta) d\theta$$

$$= \frac{1}{2\pi} \sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1} \frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)} \frac{(2q-1)!!}{2^q q!} 2\pi$$

$$+ \frac{1}{2\pi} \sum_{q+p=1}^{l/2} a_{2q,2p} r^{2q+2p}$$

$$\times \frac{(2p+n)!!}{(2p+n+2q+m+1)(2p+n+2q+m-1)...(2q+m+2)} \frac{(2q+m-1)!!}{2^{q+m/2}(q+m/2)!} 2\pi$$

$$= \sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1} \frac{(2p-1)!!(2q-1)!!}{2^{p+q} q!(p+q)(p+q+1)...(q+1)}$$

$$+ \sum_{q+p=1}^{l/2} a_{2q,2p} r^{2q+2p}$$

$$\times \frac{(2p+n)!!(2q+m-1)!!}{2^{q+m/2}(q+m/2)!(2p+n+2q+m+1)(2p+n+2q+m-1)...(2q+m+2)}$$

$$= \sum_{k=1}^{l} A_k r^k.$$

Subcase (1.2) If m and n are even, we have

$$f(r) = \frac{1}{2\pi} \int_0^{2\pi} F(r,\theta) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} a_{ij} r^{i+j} (\cos^i(\theta) \sin^{j+1}(\theta) + \cos^{i+m}(\theta) \sin^{j+n+1}(\theta)) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{i+j=0}^{l} a_{ij} r^{i+j} \cos^i(\theta) \sin^{j+1}(\theta) + \sum_{i+j=0}^{l} a_{ij} r^{i+j} \cos^{i+m}(\theta) \sin^{j+n+1}(\theta) \right] d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{i+2p-1=0}^{l} a_{i,2p-1} r^{i+2p-1} \cos^i(\theta) \sin^{2p}(\theta) + \sum_{i+2p-1=0}^{l} a_{i,2p-1} r^{i+2p-1} \cos^{i+m}(\theta) \sin^{2p+n}(\theta) \right] d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[\sum_{i+2p=2}^{l+1} a_{i,2p-1} r^{i+2p-1} \cos^i(\theta) \sin^{2p}(\theta) + \sum_{i+2p=2}^{l+1} a_{i,2p-1} r^{i+2p-1} \cos^{i+m}(\theta) \sin^{2p+n}(\theta) \right] d\theta$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1} r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right.\\
&\quad \left. + \sum_{2q+2p=1}^{l+1} a_{2q,2p-1} r^{2q+2p-1}\cos^{2q+m}(\theta)\sin^{2p+n}(\theta)\right] d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{2q+2p=2}^{l} a_{2q,2p-1} r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right.\\
&\quad \left. + \sum_{2q+2p=2}^{l} a_{2q,2p-1} r^{2q+2p-1}\cos^{2q+m}(\theta)\sin^{2p+n}(\theta)\right] d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\cos^{2q}(\theta)\sin^{2p}(\theta)\right.\\
&\quad \left. + \sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\cos^{2q+m}(\theta)\sin^{2p+n}(\theta)\right] d\theta\\
&= \frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\left[\int_0^{2\pi}\cos^{2q}(\theta)\sin^{2p}(\theta)\right.\\
&\quad \left. + \int_0^{2\pi}\cos^{2q+m}(\theta)\sin^{2p+n}(\theta)\right] d\theta\\
&= \frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\left[\frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)}\frac{(2q-1)!!}{2^q q!}2\pi\right.\\
&\quad \left. + \frac{(2p+n-1)!!}{(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+2)}\frac{(2q+m-1)!!}{2^{q+m/2}(q+m/2)!}2\pi\right]\\
&= \sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\left[\frac{(2p-1)!!(2q-1)!!}{2^{p+q}q!(p+q)(p+q+1)...(q+1)}\right.\\
&\quad \left. + \frac{(2p+n-1)!!(2q+m-1)!!}{2^{q+m/2}(q+m/2)!(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+2)}\right]\\
&= \sum_{k=1}^{l} B_k r^{2k-1}.
\end{aligned}
$$

Subcase (1.3) If m and n are odd, we have

$$
\begin{aligned}
f(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi} a_{ij} r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta) + \cos^{i+m}(\theta)\sin^{j+n+1}(\theta))d\theta\\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+j=0}^{l} a_{ij} r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\right.\\
&\quad \left. + \sum_{i+j=0}^{l} a_{ij} r^{i+j}\cos^{i+m}(\theta)\sin^{j+n+1}(\theta)\right] d\theta
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{i+2p-1=0}^{l} a_{i,2p-1} r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{i+2p=0}^{l} a_{i,2p} r^{i+2p}\cos^{i+m}(\theta)\sin^{2p+n+1}(\theta)\Bigg]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{i+2p=2}^{l+1} a_{i,2p-1} r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{2q-1+2p=0}^{l} a_{2q-1,2p} r^{2q+2p-1}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)\Bigg]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1} r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{2q+2p=2}^{l+1} a_{2q-1,2p} r^{2q+2p-1}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)\Bigg]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{2q+2p=2}^{l} a_{2q,2p-1} r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{2q+2p=2}^{l} a_{2q-1,2p} r^{2q+2p-1}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)\Bigg]d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta) \\
&\quad + \sum_{q+p=1}^{l/2} a_{2q-1,2p} r^{2q+2p-1}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)\Bigg]d\theta \\
&= \frac{1}{2\pi}\sum_{q+p=1}^{l/2} r^{2q+2p-1}\Bigg[a_{2q,2p-1}\int_0^{2\pi}\cos^2 q(\theta)\sin^{2p}(\theta)d\theta \\
&\quad a_{2q-1,p}\int_0^{2\pi}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)d\theta\Bigg] \\
&= \frac{1}{2\pi}\sum_{q+p=1}^{l/2} r^{2q+2p-1}\Bigg[a_{2q,2p-1}\frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)}\frac{(2q-1)!!}{2^q q!}2\pi \\
&\quad + a_{2q-1,2p}\frac{(2p+n)!!}{(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+1)} \\
&\times \frac{(2q+m-2)!!}{2^{q+(m-1)/2}(q+(m-1)/2)!}2\pi\Bigg] \\
&= \sum_{q+p=1}^{l/2} r^{2q+2p-1}\Bigg[a_{2q,2p-1}\frac{(2p-1)!!(2q-1)!!}{2^{p+q}q!(p+q)(p+q+1)...(q+1)} \\
&\quad + a_{2q-1,2p} \\
&\times \frac{(2p+n)!!(2q+m-2)!!}{2^{q+(m-1)/2}(q+(m-1)/2)!(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+1)}\Bigg] \\
&= \sum_{k=1}^{l/2} C_k r^{2k-1}.
\end{aligned}
$$

Subcase (1.4) If m is odd and n is even, we have

$$
\begin{aligned}
f(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} a_{ij}r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta)+\cos^{i+m}(\theta)\sin^{j+n+1}(\theta))d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\right. \\
&\quad \left. + \sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^{i+m}(\theta)\sin^{j+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p-1=0}^{l} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{i+2p-1=0}^{l} a_{i,2p}r^{i+2p}\cos^{i+m}(\theta)\sin^{2p+n}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p=2}^{l+1} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{i+2p=1}^{l+1} a_{i,2p}r^{i+2p}\cos^{i+m}(\theta)\sin^{2p+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{2q+2p=2}^{l} a_{2q-1,2p-1}r^{2q+2p-2}\cos^{2q-1+m}(\theta)\sin^{2p+n}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{2q+2p=2}^{l} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{2q+2p=2}^{l} a_{2q-1,2p-1}r^{2q+2p-2}\cos^{2q+m-1}(\theta)\sin^{2p+n}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{q+p=1}^{l/2} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{q+p=1}^{l/2} a_{2q-1,2p-1}r^{2q+2p-2}\cos^{2q+m-1}(\theta)\sin^{2p+n}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q,2p-1}r^{2q+2p-1}\int_0^{2\pi}\cos^2 q(\theta)\sin^{2p}(\theta)d\theta \\
&\quad + \frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q-1,2p-1}r^{2q+2p-2}\int_0^{2\pi}\cos^{2q+m-1}(\theta)\sin^{2p+n}(\theta)d\theta
\end{aligned}
$$

$$= \frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1} \frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)} \frac{(2q-1)!!}{2^q q!} 2\pi$$

$$+ \frac{1}{2\pi}\sum_{q+p=1}^{l/2} a_{2q-1,2p-1} r^{2q+2p-2} \times$$

$$\frac{(2p+n-1)!!}{(2p+n+2q+m-1)(2p+n+2q+m-1-2)...(2q+m-1+2)} \frac{(2q+m-2)!!}{2^{q+m/2}(q+m/2)!} 2\pi$$

$$= \sum_{q+p=1}^{l/2} a_{2q,2p-1} r^{2q+2p-1} \frac{(2p-1)!!(2q-1)!!}{2^{p+q} q!(p+q)(p+q+1)...(q+1)} + \sum_{q+p=1}^{l/2} a_{2q-1,2p-1} r^{2q+2p-2} \times$$

$$\frac{(2p+n-1)!!(2q+m-2)!!}{2^{q+(m-1)/2}(q+(m-1)/2)!(2p+n+2q+m-1)(2p+n+2q+m-3)...(2q+m+1)}$$

$$= \sum_{k=1}^{l} D_k r^k.$$

Case (2) . If l is odd, then we have 4 subcases.

Subcase (2.1) If m is even and n is odd, we have

$$\begin{aligned}
f(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta) d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} a_{ij} r^{i+j} (\cos^i(\theta)\sin^{j+1}(\theta) + \cos^{i+m}(\theta)\sin^{j+n+1}(\theta)) d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{i+j=0}^{l} a_{ij} r^{i+j} \cos^i(\theta)\sin^{j+1}(\theta) \right. \\
&\quad \left. + \sum_{i+j=0}^{l} a_{ij} r^{i+j} \cos^{i+m}(\theta)\sin^{j+n+1}(\theta) \right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{i+2p-1=0}^{l} a_{i,2p-1} r^{i+2p-1} \cos^i(\theta)\sin^{2p}(\theta) \right. \\
&\quad \left. + \sum_{i+2p=0}^{l} a_{i,2p} r^{i+2p} \cos^{i+m}(\theta)\sin^{2p+n+1}(\theta) \right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \left[\sum_{i+2p=2}^{l+1} a_{i,2p-1} r^{i+2p-1} \cos^i(\theta)\sin^{2p}(\theta) \right. \\
&\quad \left. + \sum_{i+2p=1}^{l+1} a_{i,2p} r^{i+2p} \cos^{i+m}(\theta)\sin^{2p+n+1}(\theta) \right] d\theta
\end{aligned}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)$$
$$+\sum_{2q+2p=1}^{l+1} a_{2q,2p}r^{2q+2p}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)\Bigg]d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)$$
$$+\sum_{2q+2p=2}^{l+1} a_{2q,2p}r^{2q+2p}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)\Bigg]d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)$$
$$+\sum_{q+p=1}^{(l+1)/2} a_{2q,2p}r^{2q+2p}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)\Bigg]d\theta$$
$$= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}r^{2q+2p-1}\int_0^{2\pi}\cos^2 q(\theta)\sin^{2p}(\theta)d\theta$$
$$+\frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} a_{2q,2p}r^{2q+2p}\int_0^{2\pi}\cos^{2q+m}(\theta)\sin^{2p+n+1}(\theta)d\theta$$
$$= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}r^{2q+2p-1}\frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)}\frac{(2q-1)!!}{2^q q!}2\pi$$
$$+\frac{1}{2\pi}\sum_{q+p=1}^{(l+2)/2} a_{2q,2p}r^{2q+2p}\times$$
$$\frac{(2p+n)!!}{(2p+n+1+2q+m)(2p+n+1+2q+m-2)...(2q+m+2)}\frac{(2q+m-1)!!}{2^{q+m/2}(q+m/2)!}2\pi$$
$$= \sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}r^{2q+2p-1}\frac{(2p-1)!!(2q-1)!!}{2^{p+q}q!(p+q)(p+q+1)...(q+1)}$$
$$+\sum_{q+p=1}^{(l+1)/2} a_{2q,2p}r^{2q+2p}$$
$$\times\frac{(2p+n)!!(2q+m-1)!!}{2^{q+m/2}(q+m/2)!(2p+n+2q+m+1)(2p+n+2q+m-1)...(2q+m+2)}$$
$$= \sum_{k=1}^{l+1} E_k r^k.$$

Subcase (2.2) If m and n are even, we have

$$f(r) = \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} a_{ij}r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta)+\cos^{i+m}(\theta)\sin^{j+n+1}(\theta))d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\right.$$

$$\left.+\sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^{i+m}(\theta)\sin^{j+n+1}(\theta)\right]d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p-1=0}^{l} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right.$$

$$\left.+\sum_{i+2p-1=0}^{l} a_{i,2p-1}r^{i+2p-1}\cos^{i+m}(\theta)\sin^{2p+n}(\theta)\right]d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p=2}^{l+1} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right.$$

$$\left.+\sum_{i+2p=1}^{l+1} a_{i,2p-1}r^{i+2p-1}\cos^{i+m}(\theta)\sin^{2p+n}(\theta)\right]d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right.$$

$$\left.+\sum_{2q+2p=2}^{l+1} a_{2q,2p-1}r^{2q+2p-1}\cos^{2q+m}(\theta)\sin^{2p+n}(\theta)\right]d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right.$$

$$\left.+\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}r^{2q+2p-1}\cos^{2q+m}(\theta)\sin^{2p+n}(\theta)\right]d\theta$$

$$= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}r^{2q+2p-1}\int_0^{2\pi}\cos^2 q(\theta)\sin^{2p}(\theta)+\cos^{2q+m}(\theta)\sin^{2p+n}(\theta)d\theta$$

$$= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}r^{2q+2p-1}\left[\frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)}\frac{(2q-1)!!}{2^q q!}2\pi\right.$$

$$\left.\frac{(2p+n-1)!!}{(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+2)}\frac{(2q+m-1)!!}{2^{q+m/2}(q+m/2)!}2\pi\right]$$

$$= \sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}r^{2q+2p-1}\frac{(2p-1)!!(2q-1)!!}{2^{p+q}q!(p+q)(p+q+1)...(q+1)}$$

$$+\sum_{q+p=1}^{(l+1)/2} a_{2q,2p}r^{2q+2p}$$

$$\times\frac{(2p+n-1)!!(2q+m-1)!!}{2^{q+m/2}(q+m/2)!(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+2)}$$

$$= \sum_{k=1}^{(l+1)/2} F_k r^{2k-1}.$$

Subcase (2.3) If m and n are odd, we have

$$
\begin{aligned}
f(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} a_{ij}r^{i+j}(\cos^i(\theta)\sin^{j+1}(\theta)+\cos^{i+m}(\theta)\sin^{j+n+1}(\theta))d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\right. \\
&\quad \left. + \sum_{i+j=0}^{l} a_{ij}r^{i+j}\cos^{i+m}(\theta)\sin^{j+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p-1=0}^{l} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{2q-1+j=0}^{l} a_{2q-1,j}r^{2q-1+j}\cos^{2q-1+m}(\theta)\sin^{j+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{i+2p=2}^{l+1} a_{i,2p-1}r^{i+2p-1}\cos^i(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{2q+j=2}^{l+1} a_{2q-1,j}r^{2q+j-1}\cos^{2q+m-1}(\theta)\sin^{j+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{2q+2p=2}^{l+1} a_{2q-1,2p}r^{2q+2p-1}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right. \\
&\quad \left. + \sum_{q+p=1}^{(l+1)/2} a_{2q-1,2p}r^{2q+2p-1}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)\right] d\theta \\
&= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} r^{2q+2p-1}\left[a_{2q,2p-1}\int_0^{2\pi}\cos^2 q(\theta)\sin^{2p}(\theta)d\theta\right. \\
&\quad \left. + a_{2q-1,p}\int_0^{2\pi}\cos^{2q+m-1}(\theta)\sin^{2p+n+1}(\theta)d\theta\right] \\
&= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} r^{2q+2p-1}\left[a_{2q,2p-1}\frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)}\frac{(2q-1)!!}{2^q q!}2\pi + a_{2q-1,2p}\times\right. \\
&\quad \frac{(2p+n)!!}{(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+1)} \\
&\quad \left. \times\frac{(2q+m-2)!!}{2^{q+(m-1)/2}(q+(m-1)/2)!}2\pi\right]
\end{aligned}
$$

$$= \sum_{q+p=1}^{(l+1)/2} r^{2q+2p-1} \Bigg[a_{2q,2p-1} \frac{(2p-1)!!(2q-1)!!}{2^{p+q} q!(p+q)(p+q+1)...(q+1)}$$
$$+ a_{2q-1,2p}$$
$$\times \frac{(2p+n)!!(2q+m-2)!!}{2^{q+(m-1)/2}(q+(m-1)/2)!(2p+n+2q+m)(2p+n+2q+m-2)...(2q+m+1)} \Bigg]$$
$$= \sum_{k=1}^{(l+1)/2} G_k r^{2k-1}.$$

Subcase (2.4) If m is odd and n is even, we have

$$f(r) = \frac{1}{2\pi} \int_0^{2\pi} F(r,\theta) d\theta$$
$$= \frac{1}{2\pi} \int_0^{2\pi} a_{ij} r^{i+j} (\cos^i(\theta) \sin^{j+1}(\theta) + \cos^{i+m}(\theta) \sin^{j+n+1}(\theta)) d\theta$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \Bigg[\sum_{i+j=0}^{l} a_{ij} r^{i+j} \cos^i(\theta) \sin^{j+1}(\theta)$$
$$+ \sum_{i+j=0}^{l} a_{ij} r^{i+j} \cos^{i+m}(\theta) \sin^{j+n+1}(\theta) \Bigg] d\theta$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \Bigg[\sum_{i+2p-1=0}^{l} a_{i,2p-1} r^{i+2p-1} \cos^i(\theta) \sin^{2p}(\theta)$$
$$+ \sum_{i+2p-1=0}^{l} a_{i,2p-1} r^{i+2p-1} \cos^{i+m}(\theta) \sin^{2p+n}(\theta) \Bigg] d\theta$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \Bigg[\sum_{i+2p=2}^{l+1} a_{i,2p-1} r^{i+2p-1} \cos^i(\theta) \sin^{2p}(\theta)$$
$$+ \sum_{2q-1+2p-1=0}^{l} a_{2q-1,2p-1} r^{2q+2p-2} \cos^{2q+m-1}(\theta) \sin^{2p+n}(\theta) \Bigg] d\theta$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \Bigg[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1} r^{2q+2p-1} \cos^2 q(\theta) \sin^{2p}(\theta)$$
$$+ \sum_{2q+2p=2}^{l+1} a_{2q-1,2p-1} r^{2q+2p-2} \cos^{2q+m-1}(\theta) \sin^{2p+n}(\theta) \Bigg] d\theta$$
$$= \frac{1}{2\pi} \int_0^{2\pi} \Bigg[\sum_{2q+2p=2}^{l+1} a_{2q,2p-1} r^{2q+2p-1} \cos^2 q(\theta) \sin^{2p}(\theta)$$
$$+ \sum_{2q+2p=2}^{l+1} a_{2q-1,2p-1} r^{2q+2p-2} \cos^{2q+m-1}(\theta) \sin^{2p+n}(\theta) \Bigg] d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left[\sum_{q+p=1}^{(l+1)/2} a_{2q,2p-1}r^{2q+2p-1}\cos^2 q(\theta)\sin^{2p}(\theta)\right.$$
$$\left.+\sum_{q+p=1}^{(l+1)/2} a_{2q-1,2p}r^{2q+2p-1}\cos^{2q+m-1}(\theta)\sin^{2p+n}(\theta)\right]d\theta$$
$$= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} r^{2q+2p-1}\left[a_{2q,2p-1}\int_0^{2\pi}\cos^2 q(\theta)\sin^{2p}(\theta)d\theta\right.$$
$$\left.+a_{2q-1,p-1}\int_0^{2\pi}\cos^{2q+m-1}(\theta)\sin^{2p+n}(\theta)d\theta\right]$$
$$= \frac{1}{2\pi}\sum_{q+p=1}^{(l+1)/2} r^{2q+2p-1}\left[a_{2q,2p-1}\frac{(2p-1)!!}{(2p+2q)(2p+2q+2)...(2q+2)}\frac{(2q-1)!!}{2^q q!}2\pi\right.$$
$$+a_{2q-1,2p-1}\frac{(2p+n-1)!!}{(2p+n+2q+m-1)(2p+n+2q+m-3)...(2q+m+1)}$$
$$\left.\times\frac{(2q+m-2)!!}{2^{q+(m-1)/2}(q+(m-1)/2)!}2\pi\right]$$
$$= \sum_{q+p=1}^{(l+1)/2} r^{2q+2p-1}\left[a_{2q,2p-1}\frac{(2p-1)!!(2q-1)!!}{2^{p+q}q!(p+q)(p+q+1)...(q+1)}+a_{2q-1,2p-1}\right.$$
$$\left.\times\frac{(2p+n-1)!!(2q+m-2)!!}{2^{q+(m-1)/2}(q+(m-1)/2)!(2p+n+2q+m-1)(2p+n+2q+m-3)...(2q+m+1)}\right]$$
$$= \sum_{k=1}^{l+1} I_k r^k.$$

Finally, we shall discuss the number of zeros of the averaged function $f(r)$ in the cases above using the Descartes Theorem.

From the subcases (1.1), (1.4), (2.1) and (2.4) we obtain that the function $f(r)$ is generated by a linear combination of a set $\zeta_1 = \{r, r^2, ..., r^p\}$ with $p \in \{l, l+1\}$. Using Descartes Theorem, it results that $f(r)$ can have at most $l-1$ solutions if l is even, m and n do not have the same parity. Also it can have l solutions if l is odd, m and n have not the same parity. Consequently, by Theorem 2.1, for $\epsilon > 0$ small enough, the differential system (1.2) can have at most $l-1$ or l limit cycles. From the subcases (1.2), (1.3), (2.2) and (2.3) we obtain that the averaged function $f(r)$ is generated by a linear combination of a set $\zeta_2 = \{r, r^3, ..., r^p\}$ with $p \in \{l-1, l\}$. Using Descartes Theorem, it results that $f(r)$ can have at most $(l-2)/2$ solutions if l is even, and, m and n have the same parity. Also it can have $(l-1)/2$ solutions if l is odd, and, m and n have the same parity. Similarly, by using the Theorem 2.1 and for $\epsilon > 0$ small enough, the system (1.2) can have at most $(l-2)/2$ or $(l-1)/2$ limit cycles.

4. Applications

In this section, we provide some numerical examples.

Example 4.1. Consider the differential system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x-\epsilon(1+\sin(\theta)\cos^2(\theta))(y-x^2). \end{cases} \tag{4.1}$$

Transforming system (4.1) into polar coordinates ($x = r\cos(\theta)$ and $y = r\sin(\theta)$) with

$$\begin{cases} \dot{r} = \frac{x\dot{x}+y\dot{y}}{r}, \\ \dot{\theta} = \frac{x\dot{y}-y\dot{x}}{r^2}, \end{cases}$$

gives us

$$\begin{cases} \dot{r} &= -r\epsilon\sin(\theta)(1+\sin(\theta)\cos^2(\theta))(\sin(\theta)-r\cos^2(\theta)), \\ \dot{\theta} &= -1-\epsilon\cos(\theta)(1+\sin(\theta)\cos^2(\theta))(\sin(\theta)-r\cos^2(\theta)). \end{cases}$$

Taking θ as a new independent time variable

$$\frac{dr}{d\theta} = \frac{dr}{dt}\frac{dt}{d\theta},$$

we obtain this equation

$$\begin{aligned} \frac{dr}{d\theta} &= r\epsilon\sin(\theta)(1+\sin(\theta)\cos^2(\theta))(\sin(\theta)-r\cos^2(\theta)) \\ &= \epsilon F(r,\theta)+O(\epsilon^2), \end{aligned}$$

where

$$F(r,\theta) = r\sin(\theta)(1+\sin(\theta)\cos^2(\theta))(\sin(\theta)-r\cos^2(\theta)).$$

Now, we calculate the averaged function $f(r)$

$$\begin{aligned} f(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} r\sin(\theta)(1+\sin(\theta)\cos^2(\theta))(\sin(\theta)-r\cos^2(\theta))d\theta \\ &= \frac{1}{2\pi}\left[\int_0^{2\pi} r\sin^2(\theta)d\theta - \int_0^{2\pi} r^2\sin(\theta)\cos^2(\theta)d\theta \right. \\ &\quad \left. + \int_0^{2\pi} r\sin^3(\theta)\cos^2(\theta)d\theta - \int_0^{2\pi} r^2\sin^2(\theta)\cos^4(\theta)d\theta\right] \\ &= \frac{1}{2\pi}\left[\pi r - \frac{\pi}{8}r^2\right] \\ &= \frac{r}{2}\left[1-\frac{r}{8}\right]. \end{aligned}$$

The equation $f(r)=0$ has only one positive zero, $r*=8$. We have

$$\frac{df(r)}{dr} = \frac{1}{2}(1-\frac{r}{4}),$$

and

$$\frac{df(8)}{dr} = -\frac{1}{2} \neq 0.$$

So, since $n=1$, $m=2$ and $l=2$, from Theorem 3.1, the system (4.1) can have at most one limit cycle.

Example 4.2. Consider the differential system

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x-\epsilon(1+\sin^2(\theta)\cos^2(\theta))(y-yx^2-x^4). \end{cases} \tag{4.2}$$

Transforming system (4.2) into polar coordinates gives us

$$\begin{cases} \dot{r} &= -r\epsilon\sin(\theta)(1+\sin^2(\theta)\cos^2(\theta))(\sin(\theta)-r^2\sin(\theta)\cos^2(\theta)-r^3\cos^4(\theta)), \\ \dot{\theta} &= -1-\epsilon\cos(\theta)(1+\sin^2(\theta)\cos^2(\theta))(\sin(\theta)-r^2\sin(\theta)\cos^2(\theta)-r^3\cos^4(\theta)). \end{cases}$$

We take θ as an independent time variable, we find

$$\begin{aligned} \frac{dr}{d\theta} &= r\epsilon\sin(\theta)(1+\sin^2(\theta)\cos^2(\theta))(\sin(\theta)-r^2\sin(\theta)\cos^2(\theta)-r^3\cos^4(\theta)) \\ &= \epsilon F(r,\theta)+O(\epsilon^2), \end{aligned}$$

where

$$F(r,\theta) = r\sin(\theta)(1+\sin^2(\theta)\cos^2(\theta))(\sin(\theta) - r^2\sin(\theta)\cos^2(\theta) - r^3\cos^4(\theta)).$$

Calculating the averaged function $f(r)$, we obtain

$$\begin{aligned}
f(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} r\sin(\theta)(1+\sin^2(\theta)\cos^2(\theta))(\sin(\theta) - r^2\sin(\theta)\cos^2(\theta) - r^3\cos^4(\theta))d\theta \\
&= \frac{1}{2\pi}\left[\int_0^{2\pi} r\sin^2(\theta)d\theta - \int_0^{2\pi} r^3\sin^2(\theta)\cos^2(\theta)d\theta\right. \\
&\quad - \int_0^{2\pi} r^4\sin(\theta)\cos^4(\theta)d\theta + \int_0^{2\pi} r\sin^4(\theta)\cos^2(\theta)d\theta \\
&\quad \left. - \int_0^{2\pi} r^3\sin^4(\theta)\cos^4(\theta)d\theta - \int_0^{2\pi} r^4\sin^3(\theta)\cos^6(\theta)d\theta\right] \\
&= \frac{1}{2\pi}\left[\pi r - \frac{\pi}{4}r^3 + \frac{\pi}{8}r - \frac{3\pi}{64}r^3\right] \\
&= \frac{r}{16}\left[9 - \frac{19}{8}r^2\right].
\end{aligned}$$

We get only one positive zero for the equation $f(r) = 0$, which is $r* = \sqrt{\frac{72}{19}}$. Furthermore, we have

$$\frac{df(r)}{dr} = \frac{1}{2}(\frac{9}{8} - \frac{57}{64}r^2),$$

and

$$\frac{df(\sqrt{\frac{72}{19}})}{dr} = -\frac{9}{8} \neq 0.$$

Since $n = 2,\ m = 2$ and $l = 4$, from Theorem 3.1, the system (4.2) can have at most one limit cycle.

Example 4.3. Consider the differential system

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x - \epsilon(1+\sin^2(\theta)\cos(\theta))(y - yx^2). \end{cases} \tag{4.3}$$

Writing system (4.3) into polar coordinates gives us

$$\begin{cases} \dot{r} &= -r\epsilon\sin(\theta)(1+\sin^2(\theta)\cos(\theta))(\sin(\theta) - r^2\sin(\theta)\cos^2(\theta)), \\ \dot{\theta} &= -1 - \epsilon\cos(\theta)(1+\sin^2(\theta)\cos(\theta))(\sin(\theta) - r^2\sin(\theta)\cos^2(\theta)). \end{cases}$$

We take θ as an independent time variable, we obtain

$$\begin{aligned}
\frac{dr}{d\theta} &= r\epsilon\sin(\theta)(1+\sin^2(\theta)\cos(\theta))(\sin(\theta) - r^2\sin(\theta)\cos^2(\theta)) \\
&= \epsilon F(r,\theta) + O(\epsilon^2),
\end{aligned}$$

where

$$F(r,\theta) = r\sin(\theta)(1+\sin^2(\theta)\cos(\theta))(\sin(\theta) - r^2\sin(\theta)\cos^2(\theta)).$$

Finally, The averaged function $f(r)$ is calculated as follows

$$\begin{aligned}
f(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} r\sin(\theta)(1+\sin^2(\theta)\cos(\theta))(\sin(\theta)-r^2\sin(\theta)\cos^2(\theta))d\theta \\
&= \frac{1}{2\pi}\left[\int_0^{2\pi} r\sin^2(\theta)d\theta - \int_0^{2\pi} r^3\sin^2(\theta)\cos^2(\theta)d\theta \right.\\
&\quad \left. + \int_0^{2\pi} r\sin^4(\theta)\cos(\theta)d\theta - \int_0^{2\pi} r^3\sin^4(\theta)\cos^3(\theta)d\theta\right] \\
&= \frac{1}{2\pi}\left[\pi r - \frac{\pi}{4}r^3\right] \\
&= \frac{r}{2}\left[1-\frac{r^2}{4}\right].
\end{aligned}$$

$f(r)=0$ has only one positive zero, $r* = 2$. We get from

$$\frac{df(r)}{dr} = \frac{1}{2}(1-\frac{3}{4}r^2),$$

that

$$\frac{df(2\sqrt{2})}{dr} = -1 \neq 0.$$

Thus, since $n=2, \quad m=1$ and $l=3$, from Theorem 3.1, the system (4.3) can have at most one limit cycle.

Example 4.4. Consider the differential system

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x-\epsilon(1+\sin(\theta)\cos(\theta))(y-y^2x). \end{cases} \tag{4.4}$$

In polar coordinates the previous system becomes

$$\begin{cases} \dot{r} &= -r\epsilon\sin(\theta)(1+\sin(\theta)\cos(\theta))(\sin(\theta)-r^2\sin^2(\theta)\cos(\theta)), \\ \dot{\theta} &= -1-\epsilon\cos(\theta)(1+\sin(\theta)\cos(\theta))(\sin(\theta)-r^2\sin^2(\theta)\cos(\theta)). \end{cases}$$

taking θ as the independent time variable, we get

$$\begin{aligned}
\frac{dr}{d\theta} &= r\epsilon\sin(\theta)(1+\sin(\theta)\cos(\theta))(\sin(\theta)-r^2\sin^2(\theta)\cos(\theta)) \\
&= \epsilon F(r,\theta) + O(\epsilon^2),
\end{aligned}$$

where

$$F(r,\theta) = r\sin(\theta)(1+\sin(\theta)\cos(\theta))(\sin(\theta)-r^2\sin^2(\theta)\cos(\theta)).$$

We calculate the averaged function $f(r)$

$$\begin{aligned} f(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(r,\theta)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} r\sin(\theta)(1+\sin(\theta)\cos(\theta))(\sin(\theta)-r^2\sin^2(\theta)\cos(\theta))d\theta \\ &= \frac{1}{2\pi}\left[\int_0^{2\pi} r\sin^2(\theta)d\theta - \int_0^{2\pi} r^3\sin^3(\theta)\cos(\theta)d\theta \right. \\ &\quad \left. + \int_0^{2\pi} r\sin^3(\theta)\cos(\theta)d\theta - \int_0^{2\pi} r^3\sin^4(\theta)\cos^2(\theta)d\theta\right] \\ &= \frac{1}{2\pi}\left[\pi r - \frac{\pi}{8}r^3\right] \\ &= \frac{r}{2}\left[1-\frac{r^2}{8}\right]. \end{aligned}$$

We find that $f(r)=0$ has only one positive zero, $r* = 2\sqrt{2}$. So

$$\frac{df(r)}{dr} = \frac{1}{2}(1-\frac{3}{8}r^2),$$

and

$$\frac{df(2)}{dr} = -1 \neq 0.$$

Consequently, since $n=1,\ m=1$ and $l=3$, from Theorem 3.1, the system (4.4) can have at most one limit cycle.

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