Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–14.
ISSN-0037-8712 in press
doi:10.5269/bspm.66516

Solution Forms for Generalized Hyperbolic Cotangent Type Systems of $P-$difference Equations

Ahmed Ghezal and Imane Zemmouri

ABSTRACT: Due to the recent increasing interest in hyperbolic-cotangent types of scalar-or two-dimensional systems of difference equations and treatment of some particular states. This paper presents a natural extension of the $p-$dimensional of four-systems of this generalized type and treats general states. Which is an extension of Stevic's work (J. Inequal. Appl., 2021, 184 (2021)). We also show these systems are solvable by using appropriate variable transformations and obtaining systems of homogeneous linear difference equations with constant coefficients. Some numerical examples of these systems are presented.

Key Words: Hyperbolic-cotangent type systems, general solution, system of rational difference equations, product-type system.

Contents

1. Introduction

In the past few decades, that has been an increasing interest in studying difference equations and systems of difference equations, in particular, homogeneous linear with constant coefficients. These homogeneous linear difference equations and systems with constant coefficients are theoretically solvable (see., [2], [12], [13] − [14]) where there has been some recent applications in solvability of nonlinear ones, for example, see., [1], [9]− [10], [24]. Several non(linear) difference equations and systems have been offered in literature that indicate solvability, such as [3]− [11], [15], [17], [28]− [30]. Recently, Stevic et al. [21]− [27] considered a more general type of discret models which are analogous to a few trigonometric formulas, namely, the hyperbolic-cotangent type (motivated by [16], [18], [21], [23], [25], [26]), in particular, Stević [23] gave the solutions to the following hyperbolic-cotangent-type difference equation

$$x_n = \frac{x_{n-k}x_{n-l} + \alpha}{x_{n-k} + x_{n-l}}, n \geq 0, \tag{1.1}$$

this equation can be readily reduced to the state $\alpha = 1$, i.e.,

$$\frac{x_n}{\sqrt{\alpha}} = \frac{\frac{x_{n-k}}{\sqrt{\alpha}}\frac{x_{n-l}}{\sqrt{\alpha}} + 1}{\frac{x_{n-k}}{\sqrt{\alpha}} + \frac{x_{n-l}}{\sqrt{\alpha}}}, n \geq 0, \text{ when } \alpha > 0,$$

which looks similar to the following hyperbolic-cotangent sum formula

$$\coth(k + l) = \frac{\coth(k)\coth(l) + 1}{\coth(k) + \coth(l)},$$

2010 *Mathematics Subject Classification:* 39A05, 39A10, 39A20.
Submitted December 31, 2022. Published February 14, 2023

hence the name for this type, while Stević [22] gave the form of the solutions of the following two-dimentional systems of this type

$$x_n = \frac{u_{n-k}v_{n-l}+\alpha}{u_{n-k}+v_{n-l}}, y_n = \frac{w_{n-k}s_{n-l}+\alpha}{w_{n-k}+s_{n-l}}, n \geq 0,$$

where u_n, v_n, w_n and s_n are x_n or y_n. Now, due to the wonderful results that Stević et al. [27] obtained through generalized the scalar difference equation (1.1), i.e., the following difference equation

$$x_{n+k} = \frac{x_{n+l}x_n - \alpha\beta}{x_{n+l}+x_n-\alpha-\beta}, n \geq 0, \tag{1.2}$$

Among the interesting motives of the difference equation (1.2) is solvability in closed form, this allows them to be used to describe the long-term behavior of their solutions. It should also be noted that solvable scalar and systems of difference equations have abundant applications. Here we present the extensions of the difference equation in (1.2) and we show that are solvable in closed-form by giving its general solution. Now, we consider this type of the following $p-$dimensional systems of difference equations

$$x_{n+k}^{(i)} = \frac{x_{n+l}^{(i+s) \bmod p} x_n^{\left(i+s'\right) \bmod p} - \alpha\beta}{x_{n+l}^{(i+s) \bmod p} + x_n^{(i+s') \bmod p} - \alpha - \beta}, n \geq 0, i \in \{1, ..., p\}, \tag{1.3}$$

where $k \in \mathbb{N}$, $l \in \mathbb{N}_0$, $l < k$, $s, s' \in \{0, 1\}$, $\alpha, \beta \in \mathbb{R}$ and $x_j^{(i)} \in \mathbb{R}$, $j = \{0, ..., k-1\}$, $i \in \{1, ..., p\}$.

2. Main results

In this section, we present the solutions to the system (1.3) by considering various states separately.

2.1. First state $\alpha = \beta = 0$

In this state, we have a following hyperbolic-cotangent–type system of $p-$difference equations

$$x_{n+k}^{(i)} = \frac{x_{n+l}^{(i+s) \bmod p} x_n^{\left(i+s'\right) \bmod p}}{x_{n+l}^{(i+s) \bmod p} + x_n^{(i+s') \bmod p}}, n \geq 0, i \in \{1, ..., p\}, \tag{2.1}$$

which is an extension of the following works [21], [22], [25] and [26]. In order to determine the solution of this system, we use the change of variables $x_n^{(i)} = \left(y_n^{(i)}\right)^{-1}$, $i \in \{1, ..., p\}$, which allows us to obtain a following system of $p-$homogeneous linear difference equations with constant coefficients of order k,

$$y_{n+k}^{(i)} = y_{n+l}^{(i+s) \bmod p} + y_n^{\left(i+s'\right) \bmod p}, n \geq 0, i \in \{1, ..., p\}. \tag{2.2}$$

As it is known to researchers that the system (2.2) of linear difference equations with constant coefficients is solvable. For this, we get the first result that we summarize in the following theorem

Theorem 2.1. *Consider the system of rational difference equations (2.1). Then system (2.1) is solvable and we have*

a. *If $s = s' = 0$ then* $x_n^{(i)} = \left(\sum_{m=1}^{\tau}\sum_{r=0}^{R_m-1} K_{mr} n^r \lambda_m^n\right)^{-1}, i \in \{1, ..., p\}.$

b. *If $s = s' = 1$ then* $x_n^{(i)} = \left(\sum_{m=1}^{\widetilde{\tau}}\sum_{r=0}^{\widetilde{R}_m-1} \widetilde{K}_{mr} n^r \lambda_m^n\right)^{-1}, i \in \{1, ..., p\}.$

c. *If $s = 1 - s' = 0$ then* $x_n^{(i)} = \left(\sum_{m=1}^{\overline{\tau}}\sum_{r=0}^{\overline{R}_m-1} \overline{K}_{mr} n^r \lambda_m^n\right)^{-1}, i \in \{1, ..., p\}.$

d. *If* $s' = 1 - s = 0$ *then* $x_n^{(i)} = \left(\sum\limits_{m=1}^{\widehat{\tau}} \sum\limits_{r=0}^{\widehat{R}_m - 1} \widehat{K}_{mr} n^r \lambda_m^n \right)^{-1}, i \in \{1, ..., p\}$.

Proof. **a.** If $s = s' = 0$, the system (2.2), can be rewritten as

$$y_{n+k}^{(i)} = y_{n+l}^{(i)} + y_n^{(i)}, n \geq 0, i \in \{1, ..., p\},$$

hence, we get the general solution to the system (2.2),

$$y_n^{(i)} = \sum_{m=1}^{\tau} \sum_{r=0}^{R_m - 1} K_{mr} n^r \lambda_m^n, n \geq 0, i \in \{1, ..., p\},$$

where λ_m, $m \in \{1, ..., \tau\,(\tau \leq k)\}$ are the roots of the characteristic polynomial $P_k(\lambda) = \lambda^k - \lambda^l - 1$, $K_{mr} \in \mathbb{R}$, $r \in \{0, ..., R_m - 1\}$, $m \in \{1, ..., \tau\}$ and R_m, $m \in \{1, ..., \tau\}$ are the multiplicity of the characteristic roots λ_m, $m \in \{1, ..., \tau\}$, respectively.

b. If $s = s' = 1$, the system (2.2), can be rewritten as

$$\begin{aligned} y_{n+k}^{(i)} &= y_{n+l}^{(i+1) \bmod p} + y_n^{(i+1) \bmod p}, n \geq 0, i \in \{1, ..., p\} \\ &= y_{n+2l-k}^{(i+2) \bmod p} + 2y_{n+l-k}^{(i+2) \bmod p} + y_{n-k}^{(i+2) \bmod p} \\ &= y_{n+3l-2k}^{(i+3) \bmod p} + 3y_{n+2l-2k}^{(i+3) \bmod p} + 3y_{n+l-2k}^{(i+3) \bmod p} + y_{n-2k}^{(i+3) \bmod p} \\ &\vdots \\ &= \sum_{j=0}^{p} C_p^j y_{n+(p-j)l-(p-1)k}^{(i)}, \end{aligned}$$

where $C_p^j = \dfrac{p!}{j!\,(p-j)!}$, hence, we get the general solution to the system (2.2),

$$y_n^{(i)} = \sum_{m=1}^{\widetilde{\tau}} \sum_{r=0}^{\widetilde{R}_m - 1} \widetilde{K}_{mr} n^r \lambda_m^n, n \geq 0, i \in \{1, ..., p\},$$

where λ_m, $m \in \{1, ..., \widetilde{\tau}\,(\widetilde{\tau} \leq pk)\}$ are the roots of the characteristic polynomial $\widetilde{P}_k(\lambda) = \lambda^{pk} - \sum\limits_{j=0}^{p} C_p^j \lambda^{(p-j)l}$, $\widetilde{K}_{mr} \in \mathbb{R}$, $r \in \left\{0, ..., \widetilde{R}_m - 1\right\}$, $m \in \{1, ..., \tau\}$ and $\widetilde{R}_m$, $m \in \{1, ..., \widetilde{\tau}\}$ are the multiplicity of the characteristic roots λ_m, $m \in \{1, ..., \widetilde{\tau}\}$, respectively.

c. If $s = 1 - s' = 0$, the system (2.2), can be rewritten as

$$y_{n+k}^{(i)} = y_{n+l}^{(i)} + y_n^{(i+1) \bmod p}, n \geq 0, i \in \{1, ..., p\},$$

and we have

$$y_n^{(i+j) \bmod p} = y_{n+k}^{(i+j-1) \bmod p} - y_{n+l}^{(i+j-1) \bmod p}, n \geq 0, i, j \in \{1, ..., p\}. \quad (2.3)$$

Now, for $n \geq 0$, $i \in \{1, ..., p\}$, we have

$$\begin{aligned}
y_n^{(i+p-1) \bmod p} &= y_{n+k}^{(i+p-2) \bmod p} - y_{n+l}^{(i+p-2) \bmod p} \\
&= y_{n+2k}^{(i+p-3) \bmod p} - 2y_{n+k+l}^{(i+p-3) \bmod p} + y_{n+2l}^{(i+p-3) \bmod p} \\
&= y_{n+3k}^{(i+p-4) \bmod p} - 3y_{n+2k+l}^{(i+p-4) \bmod p} + 3y_{n+k+2l}^{(i+p-4) \bmod p} - y_{n+3l}^{(i+p-4) \bmod p} \\
&= y_{n+4k}^{(i+p-5) \bmod p} - 4y_{n+3k+l}^{(i+p-5) \bmod p} + 6y_{n+2k+2l}^{(i+p-5) \bmod p} - 4y_{n+k+3l}^{(i+p-5) \bmod p} + y_{n+4l}^{(i+p-5) \bmod p} \\
&\vdots \\
&= \sum_{j=0}^{p-1} (-1)^j C_{p-1}^j y_{n+(p-1-j)k+jl}^{(i)},
\end{aligned}$$

using (2.3) for $j = p$, we have

$$\begin{aligned}
y_n^{(i)} &= y_{n+k}^{(i+p-1) \bmod p} - y_{n+l}^{(i+p-1) \bmod p} \\
&= \sum_{j=0}^{p-1} (-1)^j C_{p-1}^j y_{n+(p-j)k+jl}^{(i)} - \sum_{j=1}^{p} (-1)^{j-1} C_{p-1}^{j-1} y_{n+(p-j)k+jl}^{(i)},
\end{aligned}$$

for $i \in \{1, ..., p\}$, this system can be rewritten as

$$y_{n+pk}^{(i)} + \sum_{j=1}^{p} (-1)^j C_p^j y_{n+(p-j)k+jl}^{(i)} - y_n^{(i)} = 0, i \in \{1, ..., p\},$$

hence, we get the general solution to the system (2.2),

$$y_n^{(i)} = \sum_{m=1}^{\overline{\tau}} \sum_{r=0}^{\overline{R}_m - 1} \overline{K}_{mr} n^r \lambda_m^n, n \geq 0, i \in \{1, ..., p\},$$

where λ_m, $m \in \{1, ..., \overline{\tau} (\overline{\tau} \leq pk)\}$ are the roots of the characteristic polynomial $\overline{P}_k(\lambda) = \sum_{j=0}^{p} (-1)^j C_p^j \lambda^{(p-j)k+jl} - 1$, $\overline{K}_{mr} \in \mathbb{R}$, $r \in \{0, ..., \overline{R}_m - 1\}$, $m \in \{1, ..., \overline{\tau}\}$ and $\overline{R}_m$, $m \in \{1, ..., \overline{\tau}\}$ are the multiplicity of the characteristic roots λ_m, $m \in \{1, ..., \overline{\tau}\}$, respectively.

d. If $s' = 1 - s = 0$, the system (2.2), can be rewritten as

$$y_{n+k}^{(i)} = y_{n+l}^{(i+1) \bmod p} + y_n^{(i)}, n \geq 0, i \in \{1, ..., p\},$$

and we have

$$y_{n+l}^{(i+j) \bmod p} = y_{n+k}^{(i+j-1) \bmod p} - y_n^{(i+j-1) \bmod p}, n \geq 0, i, j \in \{1, ..., p\}. \tag{2.4}$$

Now, for $n \geq 0$, $i \in \{1, ..., p\}$, we have

$$\begin{aligned}
y_{n+l}^{(i+p-1) \bmod p} &= y_{n+k}^{(i+p-2) \bmod p} - y_n^{(i+p-2) \bmod p} \\
&= y_{n+2k-l}^{(i+p-3) \bmod p} - 2y_{n+k-l}^{(i+p-3) \bmod p} + y_{n-l}^{(i+p-3) \bmod p} \\
&= y_{n+3k-2l}^{(i+p-4) \bmod p} - 3y_{n+2k-2l}^{(i+p-4) \bmod p} + 3y_{n+k-2l}^{(i+p-4) \bmod p} - y_{n-2l}^{(i+p-4) \bmod p} \\
&= y_{n+4k-3l}^{(i+p-5) \bmod p} - 4y_{n+3k-3l}^{(i+p-5) \bmod p} + 6y_{n+2k-3l}^{(i+p-5) \bmod p} - 4y_{n+k-3l}^{(i+p-5) \bmod p} + y_{n-3l}^{(i+p-5) \bmod p} \\
&\vdots \\
&= \sum_{j=0}^{p-1} (-1)^j C_{p-1}^j y_{n+(p-1-j)k-(p-2)l}^{(i)},
\end{aligned}$$

using (2.4) for $j=p$, we have

$$\begin{aligned} y_{n+l}^{(i)} &= y_{n+k}^{(i+p-1) \bmod p} - y_n^{(i+p-1) \bmod p} \\ &= \sum_{j=0}^{p-1} (-1)^j C_{p-1}^j y_{n+(p-j)k-(p-1)l}^{(i)} - \sum_{j=1}^{p} (-1)^{j-1} C_{p-1}^{j-1} y_{n+(p-j)k-(p-1)l}^{(i)}, \end{aligned}$$

for $i \in \{1, ..., p\}$, this system can be rewritten as

$$y_{n+pk-(p-1)l}^{(i)} + \sum_{j=1}^{p} (-1)^j C_p^j y_{n+(p-j)k-(p-1)l}^{(i)} - y_{n+l}^{(i)} = 0, i \in \{1, ..., p\},$$

hence, we get the general solution to the system (2.2),

$$y_n^{(i)} = \sum_{m=1}^{\widehat{\tau}} \sum_{r=0}^{\widehat{R}_m - 1} \widehat{K}_{mr} n^r \lambda_m^n, n \geq 0, i \in \{1, ..., p\},$$

where λ_m, $m \in \{1, ..., \widehat{\tau}\,(\widehat{\tau} \leq pk)\}$ are the roots of the characteristic polynomial $\widehat{P}_k(\lambda) = \sum_{j=0}^{p} (-1)^j C_p^j \lambda^{(p-j)k} - \lambda^{pl}$, $\widehat{K}_{mr} \in \mathbb{R}$, $r \in \left\{0, ..., \widehat{R}_m - 1\right\}$, $m \in \{1, ..., \widehat{\tau}\}$ and $\widehat{R}_m$, $m \in \{1, ..., \widehat{\tau}\}$ are the multiplicity of the characteristic roots λ_m, $m \in \{1, ..., \widehat{\tau}\}$, respectively.□

□

Remark 2.2. *When $s = s' = 0$ and $k = l + 1 = 2$, the system (2.2) becomes*

$$y_{n+2}^{(i)} = y_{n+1}^{(i)} + y_n^{(i)}, n \geq 0, i \in \{1, ..., p\}.$$

If the initial conditions $y_0^{(i)} = y_1^{(i)} = 1$, $i \in \{1, ..., p\}$ then $y_n^{(i)} = F_n$, $i \in \{1, ..., p\}$, where (F_n) is the Fibonacci sequence.

Example 2.3. *We consider interesting numerical example for the system (2.1) with $k = 4$, $l = 2$, $p = 4$ and the initial conditions*

$i\,/\,j$	0	1	2	3
1	$\frac{23}{3}$	$\frac{12}{5}$	$\frac{13}{2}$	$\frac{-12}{15}$
2	$\frac{-114}{3}$	13	16	$\frac{39}{4}$
3	2	$\frac{-12}{5}$	$\frac{15}{2}$	$\frac{17}{3}$
4	$\frac{-5}{2}$	3	6	5

Table 1. The initial conditions.

The plot of the system (2.1) is shown in Figure 1.

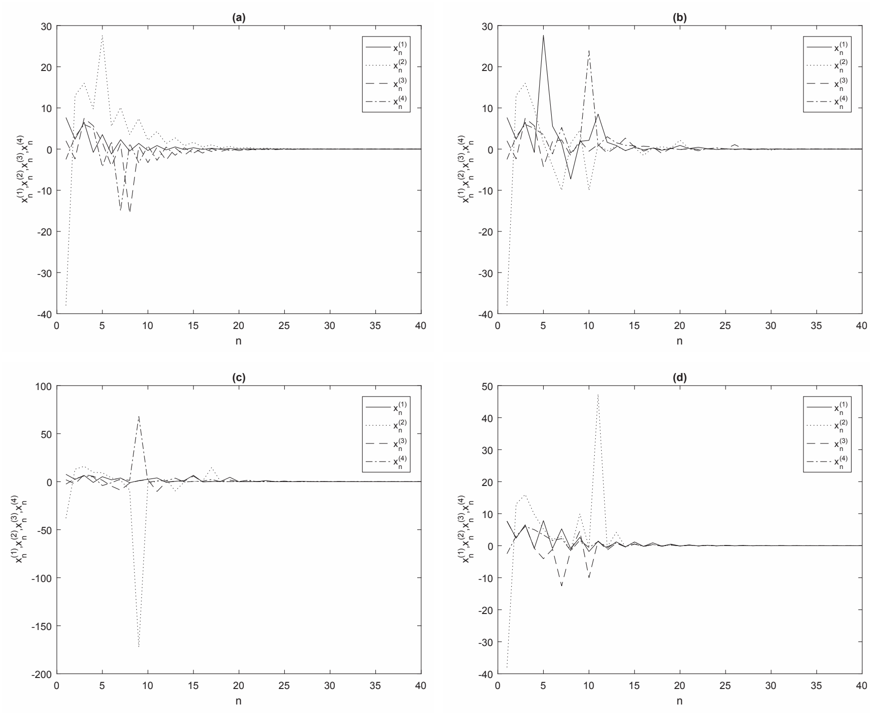

Figure 1: The plot of the system (2.1); when (a):$s = s' = 0$, (b):$s = s' = 1$, (c):$s' = 1 - s = 0$, (d):$s = 1 - s' = 0$ and we put the initial conditions in Table 1.

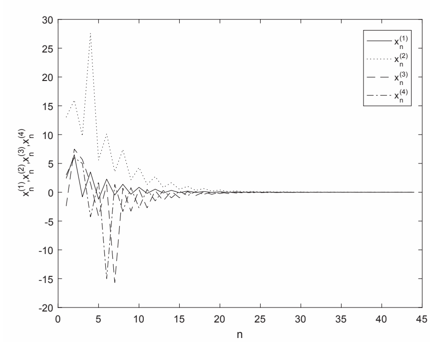

Figure 2: The plot of the solutions of system (2.1); when (a):$s = s' = 0$ and the initial conditions in Table 1.

2.2. Second state $\alpha \neq 0$ or $\beta \neq 0$

In this subsection, we also present the solutions to the system (1.3) by considering three-states separately.

2.2.1. State $\alpha+\beta = 0$. In this state, we have a following hyperbolic-cotangent–type system of $p-$difference equations

$$x_{n+k}^{(i)} = \frac{x_{n+l}^{(i+s) \bmod p} x_n^{(i+s') \bmod p} + \alpha^2}{x_{n+l}^{(i+s) \bmod p} + x_n^{(i+s') \bmod p}}, n \geq 0, i \in \{1, ..., p\}, \tag{2.5}$$

which is an extension of the following works [23] for one-dimensional and [21], [22], [25] and [26] for two-dimensional. It is easy to exhibit that (2.5) has two fundamental equilibrium points given by

$\left(\overline{x^{(1)}},...,\overline{x^{(p)}}\right) = \pm\alpha\underline{1}'_{(p)}$, where $\underline{1}_{(p)}$ is an unit vector. We can write system (2.5) after a simple calculation on the following form

$$x^{(i)}_{n+k} \pm \alpha = \frac{\left(x^{(i+s) \bmod p}_{n+l} \pm \alpha\right)\left(x^{(i+s') \bmod p}_{n} \pm \alpha\right)}{x^{(i+s) \bmod p}_{n+l} + x^{(i+s') \bmod p}_{n}}, n \geq 0, i \in \{1,...,p\},$$

thus, we get

$$\frac{x^{(i)}_{n+k} + \alpha}{x^{(i)}_{n+k} - \alpha} = \frac{x^{(i+s) \bmod p}_{n+l} + \alpha}{x^{(i+s) \bmod p}_{n+l} - \alpha}\frac{x^{(i+s') \bmod p}_{n} + \alpha}{x^{(i+s') \bmod p}_{n} - \alpha}, n \geq 0, i \in \{1,...,p\}.$$

In order to determine the solution of this system, we use the change of variables $y^{(i)}_n = \dfrac{x^{(i)}_n + \alpha}{x^{(i)}_n - \alpha}, i \in \{1,...,p\}$, which allows us to obtain a following product-type system of $p-$difference equations,

$$y^{(i)}_{n+k} = y^{(i+s) \bmod p}_{n+l} y^{(i+s') \bmod p}_{n}, n \geq 0, i \in \{1,...,p\}, \tag{2.6}$$

which is an extension of the following works [19] and [20]. As it is known to researchers that the product-type system (2.6) of nonlinear difference equations is solvable. For this, we get the second result that we summarize in the following theorem

Theorem 2.4. *Consider the system of difference equations* (2.5)*. Suppose that* $\left|x^{(i)}_j\right| > \alpha, j \in \{0,...,k-1\}$, $i \in \{1,...,p\}$, *then system* (2.5) *is solvable and we have*

a. *If* $s = s' = 0$ *then* $x^{(i)}_n = \alpha\dfrac{\exp\left(\sum\limits_{m=1}^{\tau}\sum\limits_{r=0}^{R_m-1} K_{mr}n^r\lambda^n_m\right)+1}{\exp\left(\sum\limits_{m=1}^{\tau}\sum\limits_{r=0}^{R_m-1} K_{mr}n^r\lambda^n_m\right)-1}, i \in \{1,...,p\}.$

b. *If* $s = s' = 1$ *then* $x^{(i)}_n = \alpha\dfrac{\exp\left(\sum\limits_{m=1}^{\widetilde{\tau}}\sum\limits_{r=0}^{\widetilde{R}_m-1} \widetilde{K}_{mr}n^r\lambda^n_m\right)+1}{\exp\left(\sum\limits_{m=1}^{\widetilde{\tau}}\sum\limits_{r=0}^{\widetilde{R}_m-1} \widetilde{K}_{mr}n^r\lambda^n_m\right)-1}, i \in \{1,...,p\}.$

c. *If* $s = 1 - s' = 0$ *then* $x^{(i)}_n = \alpha\dfrac{\exp\left(\sum\limits_{m=1}^{\overline{\tau}}\sum\limits_{r=0}^{\overline{R}_m-1} \overline{K}_{mr}n^r\lambda^n_m\right)+1}{\exp\left(\sum\limits_{m=1}^{\overline{\tau}}\sum\limits_{r=0}^{\overline{R}_m-1} \overline{K}_{mr}n^r\lambda^n_m\right)-1}, i \in \{1,...,p\}.$

d. *If* $s' = 1 - s = 0$ *then* $x^{(i)}_n = \alpha\dfrac{\exp\left(\sum\limits_{m=1}^{\widehat{\tau}}\sum\limits_{r=0}^{\widehat{R}_m-1} \widehat{K}_{mr}n^r\lambda^n_m\right)+1}{\exp\left(\sum\limits_{m=1}^{\widehat{\tau}}\sum\limits_{r=0}^{\widehat{R}_m-1} \widehat{K}_{mr}n^r\lambda^n_m\right)-1}, i \in \{1,...,p\}.$

Proof. The system (2.6) can be rewritten as

$$z^{(i)}_{n+k} = z^{(i+s) \bmod p}_{n+l} + z^{(i+s') \bmod p}_{n}, n \geq 0, i \in \{1,...,p\},$$

where $z^{(i)}_n = \log\left(y^{(i)}_n\right), i \in \{1,...,p\}$. This system is ultimately the same as system (2.1). Thus, the system (2.5) is solvable.□ □

Example 2.5. *We consider interesting numerical example for the system* (2.5) *with* $k = 4$, $l = 2$, $p = 4$, $\alpha = 0.25$ *and the initial conditions*

i/j	0	1	2	3
1	$\frac{23}{3}$	$\frac{12}{5}$	$\frac{13}{2}$	$\frac{12}{15}$
2	$\frac{114}{3}$	13	16	$\frac{39}{4}$
3	2	$\frac{12}{5}$	$\frac{15}{2}$	$\frac{17}{3}$
4	0.5	0.3	0.6	0.5

Table 2. The initial conditions.

The plot of the system (2.5) *is shown in Figure* 3.

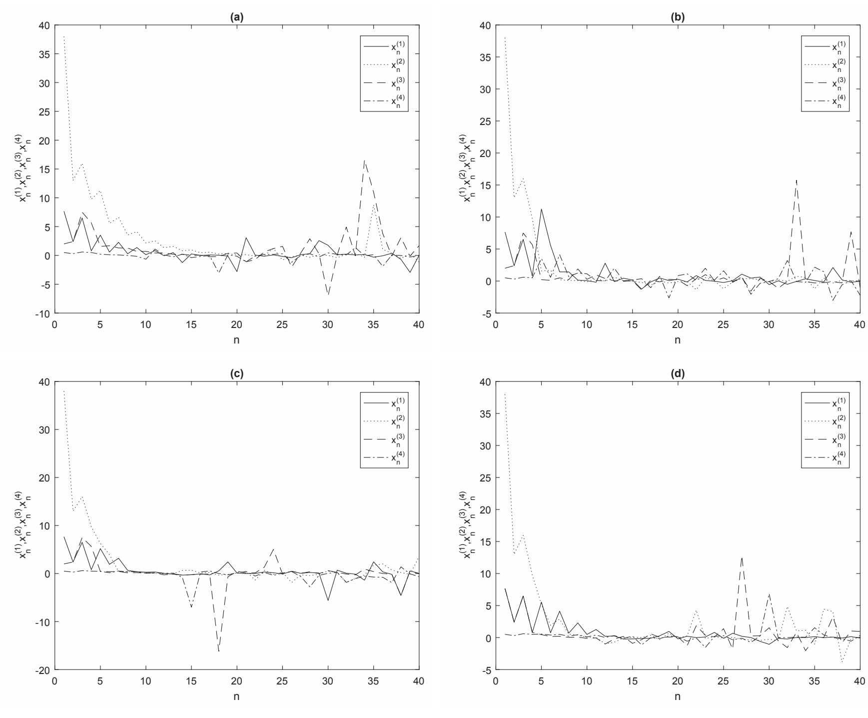

Figure 3: The plot of the system (2.5); when (a):$s = s' = 0$, (b):$s = s' = 1$,(c):$s' = 1 - s = 0$, (d):$s = 1 - s' = 0$ and we put the initial conditions in Table 2.

The plot of the solutions, when $(a): s = s' = 0$ *is shown in Figure* 4.

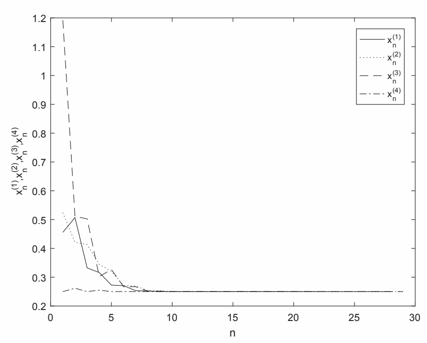

Figure 4: The plot of the solutions of system (2.5); when (a):$s = s' = 0$ and the initial conditions in Table 2.

2.2.2. State $\alpha = \beta$. In this state, we have a following hyperbolic-cotangent–type system of $p-$difference equations

$$x_{n+k}^{(i)} = \frac{x_{n+l}^{(i+s) \bmod p} x_n^{(i+s') \bmod p} - \alpha^2}{x_{n+l}^{(i+s) \bmod p} + x_n^{(i+s') \bmod p} - 2\alpha}, n \geq 0, i \in \{1, ..., p\}, \tag{2.7}$$

It is easy to exhibit that (2.7) has one fundamental equilibrium point given by $\left(\overline{x^{(1)}}, ..., \overline{x^{(p)}}\right) = \alpha \underline{1}'_{(p)}$. We can write system (2.7) after a simple calculation on the following form

$$x_{n+k}^{(i)} - \alpha = \frac{\left(x_{n+l}^{(i+s) \bmod p} - \alpha\right)\left(x_n^{(i+s') \bmod p} - \alpha\right)}{x_{n+l}^{(i+s) \bmod p} - \alpha + x_n^{(i+s') \bmod p} - \alpha}, n \geq 0, i \in \{1, ..., p\}.$$

In order to determine the solution of this system, we use the change of variables $y_n^{(i)} = x_n^{(i)} - \alpha, i \in \{1, ..., p\}$, which allows us to obtain a system of $p-$difference equations

$$y_{n+k}^{(i)} = \frac{y_{n+l}^{(i+s) \bmod p} y_n^{(i+s') \bmod p}}{y_{n+l}^{(i+s) \bmod p} + y_n^{(i+s') \bmod p}}, n \geq 0, i \in \{1, ..., p\},$$

similar to the system (2.6). For this, we get the third result that we summarize in the following theorem

Theorem 2.6. *Consider the system of difference equations (2.7). Then system (2.7) is solvable and we have*

- **a.** *If* $s = s' = 0$ *then* $x_n^{(i)} = \left(\sum_{m=1}^{\tau} \sum_{r=0}^{R_m - 1} K_{mr} n^r \lambda_m^n\right)^{-1} + \alpha, i \in \{1, ..., p\}.$
- **b.** *If* $s = s' = 1$ *then* $x_n^{(i)} = \left(\sum_{m=1}^{\widetilde{\tau}} \sum_{r=0}^{\widetilde{R}_m - 1} \widetilde{K}_{mr} n^r \lambda_m^n\right)^{-1} + \alpha, i \in \{1, ..., p\}.$
- **c.** *If* $s = 1 - s' = 0$ *then* $x_n^{(i)} = \left(\sum_{m=1}^{\overline{\tau}} \sum_{r=0}^{\overline{R}_m - 1} \overline{K}_{mr} n^r \lambda_m^n\right)^{-1} + \alpha, i \in \{1, ..., p\}.$
- **d.** *If* $s' = 1 - s = 0$ *then* $x_n^{(i)} = \left(\sum_{m=1}^{\widehat{\tau}} \sum_{r=0}^{\widehat{R}_m - 1} \widehat{K}_{mr} n^r \lambda_m^n\right)^{-1} + \alpha, i \in \{1, ..., p\}.$

Proof. The proof is similar to the proof of Theorem 2.1.□ □

Example 2.7. *We consider interesting numerical example for the system (2.7) with* $k = 4,\ l = 2,\ p = 4$ *and* $\alpha = 0.5$.

The plot of the system (2.7) *is shown in Figure* 5.

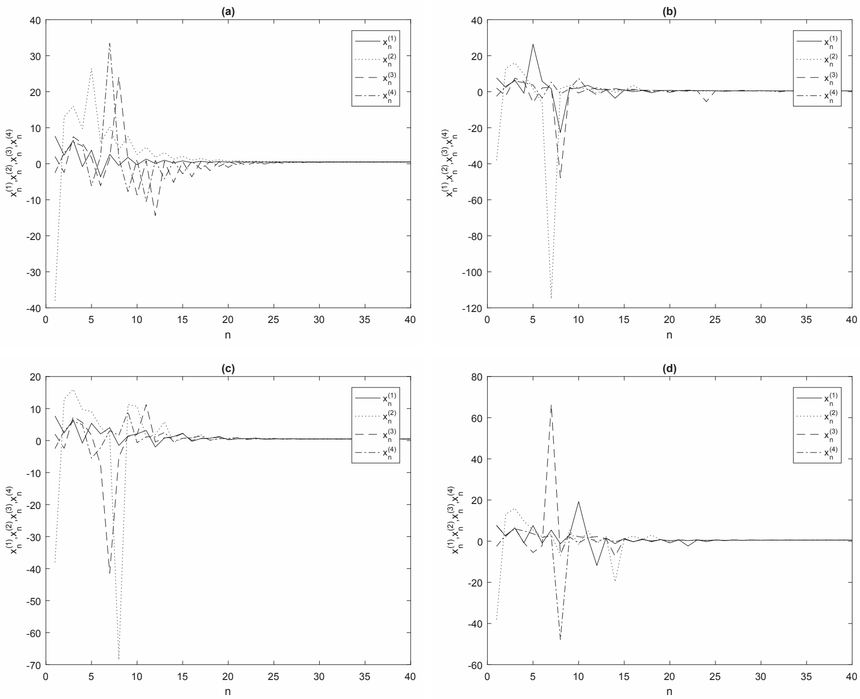

Figure 5: The plot of the system (2.7); when (a):$s = s' = 0$, (b):$s = s' = 1$, (c):$s' = 1 - s = 0$, (d):$s = 1 - s' = 0$ and we put the initial conditions in Table 1.

The plot of the solutions, when $(a) : s = s' = 0$ *is shown in Figure* 6.

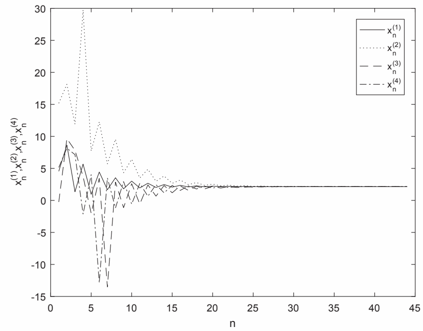

Figure 6: The plot of the solutions of system (2.7); when (a):$s = s' = 0$ and the initial conditions in Table 1.

2.2.3. State $\alpha \neq \beta$. It is easy to exhibit that (1.3) has two fundamental equilibrium points given by $\left(\overline{x^{(1)}}, ..., \overline{x^{(p)}}\right) = \alpha \underline{1}'_{(p)}$ and $\left(\overline{x^{(1)}}, ..., \overline{x^{(p)}}\right) = \beta \underline{1}'_{(p)}$. In this state, we can write system (1.3) after a simple calculation on the following form

$$x_{n+k}^{(i)}-\alpha=\frac{\left(x_{n+l}^{(i+s)\bmod p}-\alpha\right)\left(x_{n}^{(i+s')\bmod p}-\alpha\right)}{x_{n+l}^{(i+s)\bmod p}+x_{n}^{(i+s')\bmod p}-\alpha-\beta},n\geq 0,i\in\{1,...,p\},$$

$$x_{n+k}^{(i)}-\beta=\frac{\left(x_{n+l}^{(i+s)\bmod p}-\beta\right)\left(x_{n}^{(i+s')\bmod p}-\beta\right)}{x_{n+l}^{(i+s)\bmod p}+x_{n}^{(i+s')\bmod p}-\alpha-\beta},n\geq 0,i\in\{1,...,p\},$$

thus, we get

$$\frac{x_{n+k}^{(i)}-\beta}{x_{n+k}^{(i)}-\alpha}=\frac{x_{n+l}^{(i+s)\bmod p}-\beta}{x_{n+l}^{(i+s)\bmod p}-\alpha}\frac{x_{n}^{(i+s')\bmod p}-\beta}{x_{n}^{(i+s')\bmod p}-\alpha},n\geq 0,i\in\{1,...,p\}.$$

In order to determine the solution of this system, we use the change of variables $y_n^{(i)}=\dfrac{x_n^{(i)}-\beta}{x_n^{(i)}-\alpha}, i\in\{1,...,p\}$, which allows us to obtain a product-type system of $p-$difference equations similar to the system (2.6). For this, we get the fourth result that we summarize in the following theorem

Theorem 2.8. *Consider the system of difference equations* (1.3)*. Suppose that* $\left|x_j^{(i)}\right|>\max(\alpha,\beta)$, $j\in\{0,...,k-1\}$, $i\in\{1,...,p\}$*, then system* (1.3) *is solvable and we have*

a. *If* $s=s'=0$ *then* $x_n^{(i)}=\dfrac{\alpha\exp\left(\sum\limits_{m=1}^{\tau}\sum\limits_{r=0}^{R_m-1}K_{mr}n^r\lambda_m^n\right)-\beta}{\exp\left(\sum\limits_{m=1}^{\tau}\sum\limits_{r=0}^{R_m-1}K_{mr}n^r\lambda_m^n\right)-1}, i\in\{1,...,p\}.$

b. *If* $s=s'=1$ *then* $x_n^{(i)}=\dfrac{\alpha\exp\left(\sum\limits_{m=1}^{\widetilde{\tau}}\sum\limits_{r=0}^{\widetilde{R}_m-1}\widetilde{K}_{mr}n^r\lambda_m^n\right)-\beta}{\exp\left(\sum\limits_{m=1}^{\widetilde{\tau}}\sum\limits_{r=0}^{\widetilde{R}_m-1}\widetilde{K}_{mr}n^r\lambda_m^n\right)-1}, i\in\{1,...,p\}.$

c. *If* $s=1-s'=0$ *then* $x_n^{(i)}=\dfrac{\alpha\exp\left(\sum\limits_{m=1}^{\overline{\tau}}\sum\limits_{r=0}^{\overline{R}_m-1}\overline{K}_{mr}n^r\lambda_m^n\right)-\beta}{\exp\left(\sum\limits_{m=1}^{\overline{\tau}}\sum\limits_{r=0}^{\overline{R}_m-1}\overline{K}_{mr}n^r\lambda_m^n\right)-1}, i\in\{1,...,p\}.$

d. *If* $s'=1-s=0$ *then* $x_n^{(i)}=\dfrac{\alpha\exp\left(\sum\limits_{m=1}^{\widehat{\tau}}\sum\limits_{r=0}^{\widehat{R}_m-1}\widehat{K}_{mr}n^r\lambda_m^n\right)-\beta}{\exp\left(\sum\limits_{m=1}^{\widehat{\tau}}\sum\limits_{r=0}^{\widehat{R}_m-1}\widehat{K}_{mr}n^r\lambda_m^n\right)-1}, i\in\{1,...,p\}.$

Proof. The proof is similar to the proof of Theorem 2.4.□ □

Example 2.9. *We consider interesting numerical example for the system* (1.3) *with* $k=4,\ l=2,\ p=4,\ \alpha=0.5,\ \beta=0.1$ *and the initial conditions*

i/j	0	1	2	3
1	23	12	-13	15
2	-11	-3	6	4
3	-2	-12	15	-17
4	5	3	6	5

Table 3. The initial conditions.

The plot of the system (1.3) *is shown in Figure* 7.

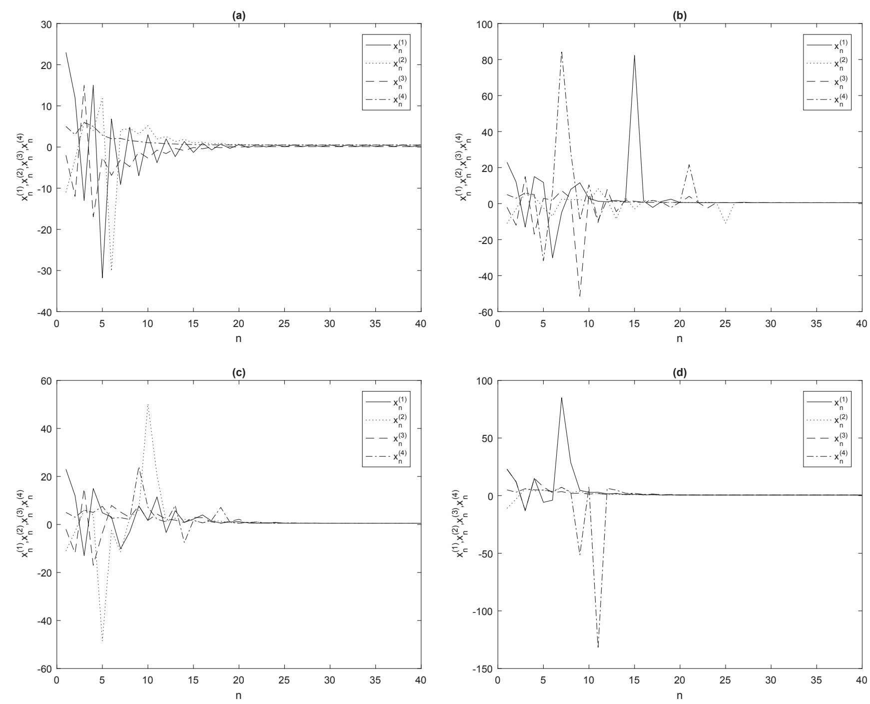

Figure 7: The plot of the system (1.3)); when (a):$s = s' = 0$, (b):$s = s' = 1$,(c):$s' = 1 - s = 0$, (d):$s = 1 - s' = 0$ and we put the initial conditions in Table 3.

The plot of the solutions, when $(a): s = s' = 0$ *is shown in Figure* 8.

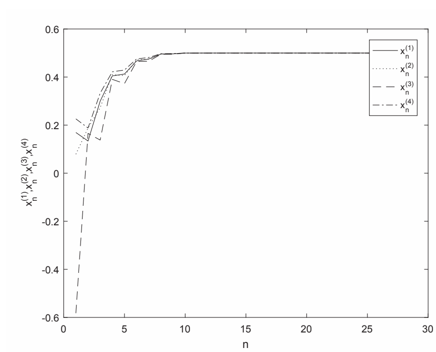

Figure 8: The plot of the solutions of system (1.3); when (a):$s = s' = 0$ and the initial conditions in Table 3.

Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. L. Berg., S. Stević. On some systems of difference equations. Applied Mathematics and Computation, 218, $1713 - 1718$ (2011).

2. D. Bernoulli. Observationes de seriebus quae formantur ex additione vel substractione quacunque terminorum se mutuo consequentium, ubi praesertim earundem insignis usus pro inveniendis radicum omnium aequationum algebraicarum ostenditur. Commentarii Acad Petropol. 1732, III(1728): $85 - 100$. (in Latin).

3. E. M., Elsayed. Solutions of rational difference system of order two. Mathematical and Computer Modelling, 55 (3 − 4), 378 − 384 (2012).

4. E. M., Elsayed. Solution for systems of difference equations of rational form of order two. Computational and Applied Mathematics, 33 (3), 751 − 765 (2014).

5. E. M., Elsayed. On a system of two nonlinear difference equations of order two. Proceedings of the Jangjeon Mathematical Society, 18, 353 − 368 (2015).

6. E. M. Elsayed.,B. S. Alofi. Dynamics and solutions structures of nonlinear system of difference equations. Mathematical Methods in the Applied Sciences, 1 − 18 (2022).

7. T. F. Ibrahim., A. Q. Khan. Forms of solutions for some two-dimensional systems of rational partial recursion equations. Mathematical Problems in Engineering, 2021 Article ID 9966197 (2021).

8. M. Kara., Y. Yazlik. Solvability of a system of nonlinear difference equations of higher order. Turkish Journal of Mathematics, 43(3), 1533 − 1565 (2019).

9. A. Ghezal., I. Zemmouri. On a solvable $p-$dimensional system of nonlinear difference equations. Journal of Mathematical and Computational Science, 12(2022), Article ID 195.

10. A. Ghezal., I. Zemmouri. Higher-order system of $p-$nonlinear difference equations solvable in closed-form with variable coefficients. Boletim da Sociedade Paranaense de Matemática, 41, 1 − 14 (2022).

11. A. Ghezal. Note on a rational system of $(4k+4)-$order difference equations: periodic solution and convergence. Journal of Applied Mathematics and Computing, (2022), https://doi.org/10.1007/s12190-022-01830-y.

12. Lagrange J-L., OEuvres T. Vol. II. Paris: Gauthier-Villars; 1868. (in French).

13. De Moivre A. Miscellanea Analytica de Seriebus et Quadraturis. Londini: J. Tonson & J.Watts; 1730. (in Latin).

14. De Moivre A. The Doctrine of Chances. 3rd ed. London: Strand Publishing; 1756.

15. A.Y. Özban. On the positive solutions of the system of rational difference equations $x_{n+1} = 1/y_{n-k}, y_{n+1} = y_n/x_{n-m}y_{n-m-k}$. Journal of Mathematical Analysis and Applications. 323 (1), 26 − 32 (2006).

16. G. Papaschinopoulos., C.J. Schinas and G. Stefanidou. On a $k-$order system of Lyness-type difference equations. Advances in Difference Equations, 2007, Article ID 31272 (2007).

17. D. Şimşek., Ogul, B. Solutions of the rational difference equations $x_{n+1} = x_{n-(2k+1)}/(1 + x_{n-k})$. MANAS Journal of Engineering, 5(3), 57 − 68 (2017).

18. S. Stević. On some solvable systems of difference equations. Applied Mathematics and Computation, 218(9), 5010−5018 (2012).

19. Stević, S., B. Irĭcanin and Z. Šmarda. On a product-type system of difference equations of second order solvable in closed form. Journal of Inequalities and Applications, 2015, Article ID 327 (2015).

20. Stević, S., B. Irĭcanin and Z. Šmarda. Two-dimensional product-type system of difference equations solvable in closed form. Advances in Difference Equations 2016, Article ID 253 (2016).

21. S. Stević. Sixteen practically solvable systems of difference equations. Advances in Difference Equations, 467, 1 − 32, (2019) .

22. S. Stević. Solvability of a general class of two-dimensional hyperbolic-cotangent-type systems of difference equations. Advances in Difference Equations, 294, 1 − 34 (2019).

23. S. Stević., B. Irĭcanin and W. Kosmala. More on a hyperbolic-cotangent class of difference equations. Mathematical Methods in the Applied Sciences, 42, 2974 − 2992 (2019).

24. S. Stević., D.T. Tollu. Solvability of eight classes of nonlinear systems of difference equations. Mathematical Methods in the Applied Sciences, 42, 4065 − 4112 (2019).

25. S. Stević. A note on general solutions to a hyperbolic-cotangent class of systems of difference equations. Advances in Difference Equations, 693, 1 − 12 (2020).

26. S. Stević. New class of practically solvable systems of difference equations of hyperbolic cotangent- type. Electronic Journal of Qualitative Theory of Differential Equations, 89, 1 − 25 (2020).

27. S. Stević., B. Irĭcanin., W. Kosmala and Z. Šmarda. Note on theoretical and practical solvability of a class of discrete equations generalizing the hyperbolic-cotangent class. Journal of Inequalities and Applications, 184, 1 − 12 (2021).

28. D.T. Tollu., Y. Yazlik and N.Taskara. On fourteen solvable systems of difference equations. Applied Mathematics and Computation, 233, 310 − 319 (2014).

29. N. Touafek., E. M. Elsayed. On the solutions of systems of rational difference equations. Mathematical and Computer Modelling, 55 (7 − 8), 1987 − 1997 (2012).

30. Y., Yazlik, Tollu, D.T., and N. Taskara. Behaviour of solutions for a system of two higher-order difference equations. Journal of Science and Arts, 45(4), 813 − 826 (2018).

Ahmed Ghezal,
Department of Mathematics and Computer Sciences,
University Center of Mila,
Algeria.
E-mail address: a.ghezal@centre-univ-mila.dz

and

Imane Zemmouri,
Department of Mathematics,
University of Annaba, Elhadjar 23,
Algeria.
E-mail address: imanezemmouri25@gmail.com