Solution Forms for Generalized Hyperbolic Cotangent Type Systems of $P$–difference Equations

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ABSTRACT: Due to the recent increasing interest in hyperbolic-cotangent types of scalar-or two-dimensional systems of difference equations and treatment of some particular states. This paper presents a natural extension of the $p$–dimensional of four-systems of this generalized type and treats general states. Which is an extension of Stevic’s work (J. Inequal. Appl., 2021, 184 (2021)). We also show these systems are solvable by using appropriate variable transformations and obtaining systems of homogeneous linear difference equations with constant coefficients. Some numerical examples of these systems are presented.

Key Words: Hyperbolic-cotangent type systems, general solution, system of rational difference equations, product-type system.

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1. Introduction

In the past few decades, that has been an increasing interest in studying difference equations and systems of difference equations, in particular, homogeneous linear with constant coefficients. These homogeneous linear difference equations and systems with constant coefficients are theoretically solvable (see., [2], [12], [13] – [14]) where there has been some recent applications in solvability of nonlinear ones, for example, see., [1], [9]– [10], [24]. Several non(linear) difference equations and systems have been offered in literature that indicate solvability, such as [3] – [11], [15], [17], [28] – [30]. Recently, Stevic et al. [21] – [27] considered a more general type of discret models which are analogous to a few trigonometric formulas, namely, the hyperbolic-cotangent type (motivated by [16], [18], [21], [23], [25], [26]), in particular, Stević [23] gave the solutions to the following hyperbolic-cotangent-type difference equation

$$x_n = \frac{x_{n-k}x_{n-l} + \alpha}{x_{n-k} + x_{n-l}}, n \geq 0,$$

(1.1)

this equation can be readily reduced to the state $\alpha = 1$, i.e.,

$$x_n = \frac{x_{n-k}x_{n-l}}{\sqrt{\alpha} + \sqrt{\alpha}} + 1, n \geq 0, \text{ when } \alpha > 0,$$

which looks similar to the following hyperbolic-cotangent sum formula

$$\coth (k + l) = \frac{\coth (k) \coth (l) + 1}{\coth (k) + \coth (l)},$$

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hence the name for this type, while Stević [22] gave the form of the solutions of the following two-dimensional systems of this type

\[ x_n = \frac{u_{n-k}v_{n-l} + \alpha}{u_{n-k} + v_{n-l}}, \quad y_n = \frac{w_{n-k}s_{n-l} + \alpha}{w_{n-k} + s_{n-l}}, \quad n \geq 0, \]

where \( u_n, v_n, w_n \) and \( s_n \) are \( x_n \) or \( y_n \). Now, due to the wonderful results that Stević et al. [27] obtained through generalized the scalar difference equation (1.1), i.e., the following difference equation

\[ x_{n+k} = \frac{x_{n+l}x_n - \alpha \beta}{x_{n+l} + x_n - \alpha - \beta}, \quad n \geq 0, \]

(1.2)

Among the interesting motives of the difference equation (1.2) is solvability in closed form, this allows them to be used to describe the long-term behavior of their solutions. It should also be noted that solvable scalar and systems of difference equations have abundant applications. Here we present the extensions of the difference equation in (1.2) and we show that are solvable in closed-form by giving its general solution. Now, we consider this type of the following \( p \)-dimensional systems of difference equations

\[ x_{n+k}^{(i)} = \frac{x_{n+l}^{(i+s) \mod p} x_n^{(i+s') \mod p} - \alpha \beta}{x_{n+l}^{(i+s) \mod p} + x_n^{(i+s') \mod p} - \alpha - \beta}, \quad n \geq 0, i \in \{1, ..., p\}, \]

(1.3)

where \( k \in \mathbb{N}, l \in \mathbb{N}_0, l < k, s, s' \in \{0, 1\}, \alpha, \beta \in \mathbb{R} \) and \( x_j^{(i)} \in \mathbb{R}, j = \{0, ..., k-1\}, i \in \{1, ..., p\} \).

2. Main results

In this section, we present the solutions to the system (1.3) by considering various states separately.

2.1. First state \( \alpha = \beta = 0 \)

In this state, we have a following hyperbolic-cotangent-type system of \( p \)-difference equations

\[ x_{n+k}^{(i)} = \frac{x_{n+l}^{(i+s) \mod p} x_n^{(i+s') \mod p}}{x_{n+l}^{(i+s) \mod p} + x_n^{(i+s') \mod p}}, \quad n \geq 0, i \in \{1, ..., p\}, \]

(2.1)

which is an extension of the following works [21], [22], [25] and [26]. In order to determine the solution of this system, we use the change of variables \( x_n^{(i)} = \left( y_n^{(i)} \right)^{-1}, i \in \{1, ..., p\} \), which allows us to obtain a following system of \( p \)-homogeneous linear difference equations with constant coefficients of order \( k \),

\[ y_{n+k}^{(i)} = y_{n+l}^{(i+s) \mod p} + y_n^{(i+s') \mod p}, \quad n \geq 0, i \in \{1, ..., p\}. \]

(2.2)

As it is known to researchers that the system (2.2) of linear difference equations with constant coefficients is solvable. For this, we get the first result that we summarize in the following theorem

**Theorem 2.1.** Consider the system of rational difference equations (2.1). Then system (2.1) is solvable and we have

a. If \( s = s' = 0 \) then \( x_n^{(i)} = \left( \sum_{m=1}^{\tau} R_{m-1} \sum_{r=0}^{m-1} K_{m} n^r \lambda_m^r \right)^{-1} \), \( i \in \{1, ..., p\} \).

b. If \( s = s' = 1 \) then \( x_n^{(i)} = \left( \sum_{m=1}^{\tau} \bar{R}_{m-1} \sum_{r=0}^{m-1} \bar{K}_{m} n^r \lambda_m^r \right)^{-1} \), \( i \in \{1, ..., p\} \).

c. If \( s = 1 - s' = 0 \) then \( x_n^{(i)} = \left( \sum_{m=1}^{\tau} \overline{R}_{m-1} \sum_{r=0}^{m-1} \overline{K}_{m} n^r \lambda_m^r \right)^{-1} \), \( i \in \{1, ..., p\} \).
d. If \( s' = 1 - s = 0 \) then \( x_{n}^{(i)} = \left( \sum_{m=1}^{\tilde{r}} \sum_{r=0}^{R_{m} - 1} \tilde{K}_{mr} n^r \lambda_{m}^n \right)^{-1} , i \in \{1, ..., p\} \).

**Proof.**

a. If \( s = s' = 0 \), the system (2.2), can be rewritten as

\[
y_{n}^{(i)} = y_{n+l}^{(i)} + y_{n}^{(i)} , n \geq 0 , i \in \{1, ..., p\} ,
\]

hence, we get the general solution to the system (2.2),

\[
y_{n}^{(i)} = \sum_{m=1}^{r} \sum_{r=0}^{R_{m} - 1} K_{mr} n^r \lambda_{m}^n , n \geq 0 , i \in \{1, ..., p\} ,
\]

where \( \lambda_{m} , m \in \{1, ..., \tau (\tau \leq k)\} \) are the roots of the characteristic polynomial \( P_{k}(\lambda) = \lambda^{k} - \lambda^{l} - 1 \), \( K_{mr} \in \mathbb{R} \), \( r \in \{0, ..., R_{m} - 1\} \), \( m \in \{1, ..., \tau\} \) and \( R_{m} , m \in \{1, ..., \tau\} \) are the multiplicity of the characteristic roots \( \lambda_{m} , m \in \{1, ..., \tau\} \), respectively.

b. If \( s = s' = 1 \), the system (2.2), can be rewritten as

\[
y_{n}^{(i)} = y_{n+l}^{(i) \mod p} + y_{n}^{(i+1) \mod p} , n \geq 0 , i \in \{1, ..., p\}
\]

\[
y_{n}^{(i)} = y_{n+2l}^{(i+2) \mod p} + 2y_{n+l}^{(i+2) \mod p} + y_{n-k}^{(i+2) \mod p}
\]

\[
y_{n+3l-2k}^{(i+3) \mod p} + 3y_{n+2l-2k}^{(i+3) \mod p} + 3y_{n+l-2k}^{(i+3) \mod p} + y_{n-2k}
\]

\[
\vdots
\]

\[
y_{n}^{(i)} = \sum_{j=0}^{p} C_{p}^{j} y_{n+(p-j)l-(p-1)k}^{(i) \mod p}
\]

where \( C_{p}^{j} = \frac{p!}{j!(p-j)!} \), hence, we get the general solution to the system (2.2),

\[
y_{n}^{(i)} = \sum_{m=1}^{\tilde{r}} \sum_{r=0}^{R_{m} - 1} \tilde{K}_{mr} n^r \lambda_{m}^n , n \geq 0 , i \in \{1, ..., p\} ,
\]

where \( \lambda_{m} , m \in \{1, ..., \tilde{r} (\tilde{r} \leq pk)\} \) are the roots of the characteristic polynomial \( \tilde{P}_{k}(\lambda) = \lambda^{pk} - \sum_{j=0}^{p} C_{p}^{j} \lambda^{(p-j)l} \), \( \tilde{K}_{mr} \in \mathbb{R} \), \( r \in \{0, ..., \tilde{R}_{m} - 1\} \), \( m \in \{1, ..., \tau\} \) and \( \tilde{R}_{m} , m \in \{1, ..., \tilde{r}\} \) are the multiplicity of the characteristic roots \( \lambda_{m} , m \in \{1, ..., \tilde{r}\} \), respectively.

c. If \( s = 1 - s' = 0 \), the system (2.2), can be rewritten as

\[
y_{n}^{(i)} = y_{n+l}^{(i) \mod p} + y_{n}^{(i+1) \mod p} , n \geq 0 , i \in \{1, ..., p\}
\]

and we have

\[
y_{n}^{(i+j) \mod p} = y_{n+k}^{(i+j-1) \mod p} - y_{n+l}^{(i+j-1) \mod p} , n \geq 0 , i, j \in \{1, ..., p\} .
\]
Now, for \( n \geq 0, i \in \{1, \ldots, p\} \), we have

\[
y^{(i+p-1) \mod p} = y^{(i+p-2) \mod p} - y^{(i+p-1) \mod p}
\]

\[
y_n = y^{(i+p-1) \mod p} - 2y_{n+k+1}^{(i+p-3) \mod p} + y_{n+l}^{(i+p-3) \mod p}
\]

\[
y_n = y^{(i+p-4) \mod p} - 3y_{n+2k+l}^{(i+p-4) \mod p} + 3y_{n+k+2l}^{(i+p-4) \mod p} - y_{n+3l}^{(i+p-4) \mod p}
\]

\[
y_n = y^{(i+p-5) \mod p} - 4y_{n+3k+l}^{(i+p-5) \mod p} + 6y_{n+2k+2l}^{(i+p-5) \mod p} - 4y_{n+k+3l}^{(i+p-5) \mod p} + y_{n+4l}^{(i+p-5) \mod p}
\]

\[
\vdots
\]

\[
= \sum_{j=0}^{p-1} (-1)^j C_{p-1}^j y^{(i)}_{n+(p-1-j)k+jl},
\]

using (2.3) for \( j = p \), we have

\[
y^{(i)}_n = y^{(i+p-1) \mod p} - y^{(i+p-1) \mod p}
\]

\[
y_n = \sum_{j=0}^{p-1} (-1)^j C_{p-1}^j y^{(i)}_{n+(p-j)k+jl} - \sum_{j=1}^{p} (-1)^j C_{p-1}^{j-1} y^{(i)}_{n+(p-j)k+jl},
\]

for \( i \in \{1, \ldots, p\} \), this system can be rewritten as

\[
y_n = \sum_{m=1}^{\tau} \sum_{r=0}^{R_{n-1}} \lambda_m^{n} n^r, n \geq 0, i \in \{1, \ldots, p\},
\]

where \( \lambda_m, m \in \{1, \ldots, \tau (\tau \leq pk)\} \) are the roots of the characteristic polynomial \( P_k(\lambda) = \sum_{j=0}^{p} (-1)^j C_{p}^j (\lambda^{(p-j)k+jl} - 1), \lambda_{mr} \in \mathbb{R}, r \in \{0, \ldots, R_m - 1\}, m \in \{1, \ldots, \tau\} \) and \( R_m, m \in \{1, \ldots, \tau\} \) are the multiplicity of the characteristic roots \( \lambda_m, m \in \{1, \ldots, \tau\} \), respectively.

**d.** If \( s' = 1 - s = 0 \), the system (2.2), can be rewritten as

\[
y^{(i)}_{n+k} = y^{(i+1) \mod p} + y^{(i)}_n, n \geq 0, i \in \{1, \ldots, p\},
\]

and we have

\[
y^{(i+j) \mod p}_{n+k} = y^{(i+j-1) \mod p} + y^{(i+j-1) \mod p}_n, n \geq 0, i, j \in \{1, \ldots, p\}.
\]

Now, for \( n \geq 0, i \in \{1, \ldots, p\} \), we have

\[
y^{(i+p-1) \mod p}_{n+l} = y^{(i+p-2) \mod p}_n - y^{(i+p-2) \mod p}
\]

\[
y_n = y^{(i+p-3) \mod p} - 2y_{n+k+1}^{(i+p-3) \mod p} + y_{n-l}^{(i+p-3) \mod p}
\]

\[
y_n = y^{(i+p-4) \mod p} - 3y_{n+2k+l}^{(i+p-4) \mod p} + 3y_{n+k+2l}^{(i+p-4) \mod p} - y_{n-2l}^{(i+p-4) \mod p}
\]

\[
y_n = y^{(i+p-5) \mod p} - 4y_{n+3k+l}^{(i+p-5) \mod p} + 6y_{n+2k+2l}^{(i+p-5) \mod p} - 4y_{n+k+3l}^{(i+p-5) \mod p} + y_{n-3l}^{(i+p-5) \mod p}
\]

\[
\vdots
\]

\[
= \sum_{j=0}^{p-1} (-1)^j C_{p-1}^j y^{(i)}_{n+(p-1-j)k-(p-2)l},
\]
using (2.4) for \( j = p \), we have

\[
y^{(i)}_{n+l} = y^{(i)}_{n+k} - \sum_{j=0}^{p-1} (-1)^j C_p^{(i)} y^{(i)}_{n+((p-j)k-(p-1))l} - \sum_{j=1}^{p} (-1)^{j-1} C_p^{j-1} y^{(i)}_{n+((p-j)k-(p-1))l},
\]

for \( i \in \{1, \ldots, p\} \), this system can be rewritten as

\[
y^{(i)}_{n+pk-(p-1)l} + \sum_{j=1}^{p} (-1)^j C_p^{j} y^{(i)}_{n+((p-j)k-(p-1))l} - y^{(i)}_{n+l} = 0, i \in \{1, \ldots, p\},
\]

hence, we get the general solution to the system (2.2),

\[
y^{(i)}_n = \sum_{m=1}^{\hat{m}} \sum_{r=0}^{\hat{r}_m-1} \hat{K}_{mr} n^r \lambda^m_{m,n}, n \geq 0, i \in \{1, \ldots, p\},
\]

where \( \lambda_m, m \in \{1, \ldots, \hat{\tau} (\hat{\tau} \leq pk) \} \) are the roots of the characteristic polynomial \( \hat{P}_k(\lambda) = \sum_{j=0}^{p} (-1)^j C_p^{j} \lambda^{(p-j)k} - \lambda^p, \hat{K}_{mr} \in \mathbb{R}, r \in \{0, \ldots, \hat{r}_m - 1\}, m \in \{1, \ldots, \hat{\tau}\} \) and \( \hat{r}_m, m \in \{1, \ldots, \hat{\tau}\} \) are the multiplicity of the characteristic roots \( \lambda_m, m \in \{1, \ldots, \hat{\tau}\} \), respectively. □

**Remark 2.2.** When \( s = s' = 0 \) and \( k = l + 1 = 2 \), the system (2.2) becomes

\[
y^{(i)}_{n+2} = y^{(i)}_{n+1} + y^{(i)}_n, n \geq 0, i \in \{1, \ldots, p\}.
\]

If the initial conditions \( y^{(i)}_0 = y^{(i)}_1 = 1, i \in \{1, \ldots, p\} \) then \( y^{(i)}_n = F_n, i \in \{1, \ldots, p\} \), where \( (F_n) \) is the Fibonacci sequence.

**Example 2.3.** We consider interesting numerical example for the system (2.1) with \( k = 4, l = 2, p = 4 \) and the initial conditions

<table>
<thead>
<tr>
<th>( i/j )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>23</td>
<td>12</td>
<td>13</td>
<td>-12</td>
</tr>
<tr>
<td>2</td>
<td>-3</td>
<td>5</td>
<td>2</td>
<td>15</td>
</tr>
<tr>
<td>3</td>
<td>-2</td>
<td>13</td>
<td>16</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>-5</td>
<td>3</td>
<td>6</td>
<td>5</td>
</tr>
</tbody>
</table>

Table 1. The initial conditions.

The plot of the system (2.1) is shown in Figure 1.
Figure 1: The plot of the system (2.1); when (a): $s = s' = 0$, (b): $s = s' = 1$, (c): $s' = 1 - s = 0$, (d): $s = 1 - s' = 0$ and we put the initial conditions in Table 1.

Figure 2: The plot of the solutions of system (2.1); when (a): $s = s' = 0$ and the initial conditions in Table 1.

2.2. Second state $\alpha \neq 0$ or $\beta \neq 0$

In this subsection, we also present the solutions to the system (1.3) by considering three-states separately.

2.2.1. State $\alpha + \beta = 0$. In this state, we have a following hyperbolic-cotangent–type system of $p$–difference equations

$$x^{(i)}_{n+k} = \frac{x^{(i+s) \mod p}_{n+l} x^{(i+s') \mod p}_{n} + \alpha^2}{x^{(i+s) \mod p}_{n+l} + x^{(i+s') \mod p}_{n}}, n \geq 0, i \in \{1, \ldots, p\},$$

which is an extension of the following works [23] for one-dimensional and [21], [22], [25] and [26] for two-dimensional. It is easy to exhibit that (2.5) has two fundamental equilibrium points given by
\( (x^{(1)}, \ldots, x^{(p)}) = \pm \alpha \mathbf{1}_{(p)} \), where \( \mathbf{1}_{(p)} \) is an unit vector. We can write system (2.5) after a simple calculation on the following form

\[
x^{(i)}_{n+k} \pm \alpha = \frac{x^{(i+s)}_{n+l} \pm \alpha}{x^{(i+s)}_{n+l} \mp x^{(i+s')}_{n+l}}, \quad n \geq 0, i \in \{1, \ldots, p\},
\]

thus, we get

\[
x^{(i)}_{n+k} + \alpha = \frac{x^{(i+s)}_{n+l} + \alpha x^{(i+s')}_{n+l} + \alpha}{x^{(i+s)}_{n+l} - \alpha x^{(i+s')}_{n+l} - \alpha}, \quad n \geq 0, i \in \{1, \ldots, p\}.
\]

In order to determine the solution of this system, we use the change of variables \( y^{(i)}_n = \frac{x^{(i)}_n}{x^{(i)}_n - \alpha}, i \in \{1, \ldots, p\} \), which allows us to obtain a following product-type system of \( p \)-difference equations,

\[
y^{(i)}_{n+k} = y^{(i+s)}_{n+l} y^{(i+s')}_{n+l}, \quad n \geq 0, i \in \{1, \ldots, p\},
\]

which is an extension of the following works [19] and [20]. As it is known to researchers that the product-type system (2.6) of nonlinear difference equations is solvable. For this, we get the second result that we summarize in the following theorem

**Theorem 2.4.** Consider the system of difference equations (2.5). Suppose that \( |x^{(i)}_j| > \alpha, j \in \{0, \ldots, k-1\} \), \( i \in \{1, \ldots, p\} \), then system (2.5) is solvable and we have

**a.** If \( s = s' = 0 \) then \( x^{(i)}_n = \alpha \frac{\exp \left( \sum_{m=1}^{r} \sum_{r=0}^{R_{m-1}} K_{m r n} \lambda_{m}^n \right) + 1}{\exp \left( \sum_{m=1}^{\hat{r}} \sum_{r=0}^{\hat{R}_{m-1}} \hat{K}_{m r n} \lambda_{m}^n \right) - 1}, i \in \{1, \ldots, p\} \).

**b.** If \( s = s' = 1 \) then \( x^{(i)}_n = \alpha \frac{\exp \left( \sum_{m=1}^{\hat{r}} \sum_{r=0}^{\hat{R}_{m-1}} \hat{K}_{m r n} \lambda_{m}^n \right) + 1}{\exp \left( \sum_{m=1}^{\hat{r}} \sum_{r=0}^{\hat{R}_{m-1}} \hat{K}_{m r n} \lambda_{m}^n \right) - 1}, i \in \{1, \ldots, p\} \).

**c.** If \( s = 1 - s' = 0 \) then \( x^{(i)}_n = \alpha \frac{\exp \left( \sum_{m=1}^{\hat{r}} \sum_{r=0}^{\hat{R}_{m-1}} \hat{K}_{m r n} \lambda_{m}^n \right) + 1}{\exp \left( \sum_{m=1}^{\hat{r}} \sum_{r=0}^{\hat{R}_{m-1}} \hat{K}_{m r n} \lambda_{m}^n \right) - 1}, i \in \{1, \ldots, p\} \).

**d.** If \( s' = 1 - s = 0 \) then \( x^{(i)}_n = \alpha \frac{\exp \left( \sum_{m=1}^{\hat{r}} \sum_{r=0}^{\hat{R}_{m-1}} \hat{K}_{m r n} \lambda_{m}^n \right) + 1}{\exp \left( \sum_{m=1}^{\hat{r}} \sum_{r=0}^{\hat{R}_{m-1}} \hat{K}_{m r n} \lambda_{m}^n \right) - 1}, i \in \{1, \ldots, p\} \).

**Proof.** The system (2.6) can be rewritten as

\[
z^{(i)}_{n+k} = z^{(i+s)}_{n+l} + z^{(i+s')}_{n+l}, \quad n \geq 0, i \in \{1, \ldots, p\},
\]

where \( z^{(i)}_n = \log \left( y^{(i)}_n \right), i \in \{1, \ldots, p\} \). This system is ultimately the same as system (2.1). Thus, the system (2.5) is solvable. \( \square \)
Example 2.5. We consider interesting numerical example for the system (2.5) with \( k = 4, \ l = 2, \ p = 4, \ \alpha = 0.25 \) and the initial conditions

\[
\begin{array}{cccc}
  i/j & 0 & 1 & 2 & 3 \\
  1 & 23 & 12 & 13 & 12 \\
  2 & 11 & 4 & 5 & 2 & 15 \\
  3 & 3 & 12 & 16 & 4 \\
  4 & 2 & 5 & 2 & 3 \\
\end{array}
\]

Table 2. The initial conditions.

The plot of the system (2.5) is shown in Figure 3.

Figure 3: The plot of the system (2.5); when (a):\( s = s' = 0 \), (b):\( s = s' = 1 \),(c):\( s' = 1 - s = 0 \), (d):\( s = 1 - s' = 0 \) and we put the initial conditions in Table 2.

The plot of the solutions, when (a) \( s = s' = 0 \) is shown in Figure 4.
In order to determine the solution of this system, we use the change of variables \( \alpha \) and \( \beta \).

2.2.2. State \( \alpha = \beta \). In this state, we have a following hyperbolic-cotangent–type system of \( p \)–difference equations

\[
x_{n+k}^{(i)} = \frac{x_n^{(i+s)} \mod p \cdot x_n^{(i+s')} \mod p - \alpha^2}{x_{n+l}^{(i+s)} \mod p + x_n^{(i+s')} \mod p - 2\alpha}, \quad n \geq 0, \ i \in \{1, ..., p\},
\]

(2.7)

It is easy to exhibit that (2.7) has one fundamental equilibrium point given by \( (\overline{x^{(1)}}, ..., \overline{x^{(p)}}) = \alpha \overline{y^{(p)}} \).

We can write system (2.7) after a simple calculation on the following form

\[
x_{n+k}^{(i)} - \alpha = \frac{x_n^{(i+s)} \mod p - \alpha \cdot x_n^{(i+s')} \mod p - \alpha}{x_{n+l}^{(i+s)} \mod p - \alpha + x_n^{(i+s')} \mod p - \alpha}, \quad n \geq 0, \ i \in \{1, ..., p\}.
\]

In order to determine the solution of this system, we use the change of variables \( y_n^{(i)} = x_n^{(i)} - \alpha, \ i \in \{1, ..., p\} \), which allows us to obtain a system of \( p \)–difference equations

\[
y_{n+k}^{(i)} = \frac{y_{n+l}^{(i+s)} \mod p \cdot y_n^{(i+s')} \mod p}{y_{n+l}^{(i+s)} \mod p + y_n^{(i+s')} \mod p}, \quad n \geq 0, \ i \in \{1, ..., p\},
\]

similar to the system (2.6). For this, we get the third result that we summarize in the following theorem

**Theorem 2.6.** Consider the system of difference equations (2.7). Then system (2.7) is solvable and we have

a. If \( s = s' = 0 \) then \( x_n^{(i)} = \left( \sum_{m=1}^{\tau} \sum_{r=0}^{R_m-1} K_{mr} n^r \lambda_m^n \right)^{-1} + \alpha, i \in \{1, ..., p\} \).

b. If \( s = s' = 1 \) then \( x_n^{(i)} = \left( \sum_{m=1}^{\tau} \sum_{r=0}^{R_m-1} \hat{K}_{mr} n^r \lambda_m^n \right)^{-1} + \alpha, i \in \{1, ..., p\} \).

c. If \( s = 1 - s' = 0 \) then \( x_n^{(i)} = \left( \sum_{m=1}^{\tau} \sum_{r=0}^{R_m-1} \overline{K}_{mr} n^r \lambda_m^n \right)^{-1} + \alpha, i \in \{1, ..., p\} \).

d. If \( s' = 1 - s = 0 \) then \( x_n^{(i)} = \left( \sum_{m=1}^{\tau} \sum_{r=0}^{R_m-1} \overline{K}_{mr} n^r \lambda_m^n \right)^{-1} + \alpha, i \in \{1, ..., p\} \).

**Proof.** The proof is similar to the proof of Theorem 2.1. \( \square \)

**Example 2.7.** We consider interesting numerical example for the system (2.7) with \( k = 4, \ l = 2, \ p = 4 \) and \( \alpha = 0.5 \).
The plot of the system (2.7) is shown in Figure 5.

![Figure 5](image1)

Figure 5: The plot of the system (2.7); when (a): \( s = s' = 0 \), (b): \( s = s' = 1 \), (c): \( s' = 1 - s = 0 \), (d): \( s = 1 - s' = 0 \) and we put the initial conditions in Table 1.

The plot of the solutions, when (a): \( s = s' = 0 \) is shown in Figure 6.

![Figure 6](image2)

Figure 6: The plot of the solutions of system (2.7); when (a): \( s = s' = 0 \) and the initial conditions in Table 1.

2.2.3. State \( \alpha \neq \beta \). It is easy to exhibit that (1.3) has two fundamental equilibrium points given by \( (\overline{x}^{(1)}, ..., \overline{x}^{(p)}) = \alpha \overline{1}_{(p)} \) and \( (\overline{x}^{(1)}, ..., \overline{x}^{(p)}) = \beta \overline{1}_{(p)} \). In this state, we can write system (1.3) after a simple
In order to determine the solution of this system, we use the change of variables calculation on the following form

\[
x_{n+k}^{(i)} - \alpha = \frac{(x_{n+l}^{(i+s) \mod p} - \alpha) (x_{n}^{(i+s') \mod p} - \alpha)}{x_{n+l}^{(i+s) \mod p} + x_{n}^{(i+s') \mod p} - \alpha - \beta}, \quad n \geq 0, \ i \in \{1,...,p\},
\]

\[
x_{n+k}^{(i)} - \beta = \frac{(x_{n+l}^{(i+s) \mod p} - \beta) (x_{n}^{(i+s') \mod p} - \beta)}{x_{n+l}^{(i+s) \mod p} + x_{n}^{(i+s') \mod p} - \alpha - \beta}, \quad n \geq 0, \ i \in \{1,...,p\},
\]

thus, we get

\[
\frac{x_{n+k}^{(i)} - \beta}{x_{n+k}^{(i)} - \alpha} = \frac{x_{n+l}^{(i+s) \mod p} - \beta x_{n}^{(i+s') \mod p} - \beta}{x_{n+l}^{(i+s) \mod p} - \alpha x_{n}^{(i+s') \mod p} - \alpha}, \quad n \geq 0, \ i \in \{1,...,p\}.
\]

In order to determine the solution of this system, we use the change of variables \(y_{n}^{(i)} = \frac{x_{n}^{(i)} - \beta}{x_{n}^{(i)} - \alpha}, \ i \in \{1,...,p\}\), which allows us to obtain a product-type system of \(p\)-difference equations similar to the system (2.6). For this, we get the fourth result that we summarize in the following theorem

**Theorem 2.8.** Consider the system of difference equations (1.3). Suppose that \(|x_{j}^{(i)}| > \max(\alpha, \beta), j \in \{0,...,k-1\}, i \in \{1,...,p\}\), then system (1.3) is solvable and we have

a. If \(s = s' = 0\) then \(x_{n}^{(i)} = \frac{\alpha \exp \left( \sum_{m=1}^{\tau} \sum_{r=0}^{R_{m}-1} K_{mr} n^{r} \lambda_{m}^{n} \right) - \beta}{\exp \left( \sum_{m=1}^{\tau} \sum_{r=0}^{R_{m}-1} K_{mr} n^{r} \lambda_{m}^{n} \right) - 1}, i \in \{1,...,p\}\).

b. If \(s = s' = 1\) then \(x_{n}^{(i)} = \frac{\alpha \exp \left( \sum_{m=1}^{\tau} \sum_{r=0}^{R_{m}-1} \tilde{K}_{mr} n^{r} \lambda_{m}^{n} \right) - \beta}{\exp \left( \sum_{m=1}^{\tau} \sum_{r=0}^{R_{m}-1} \tilde{K}_{mr} n^{r} \lambda_{m}^{n} \right) - 1}, i \in \{1,...,p\}\).

c. If \(s = 1 - s' = 0\) then \(x_{n}^{(i)} = \frac{\alpha \exp \left( \sum_{m=1}^{\tau} \sum_{r=0}^{R_{m}-1} \tilde{K}_{mr} n^{r} \lambda_{m}^{n} \right) - \beta}{\exp \left( \sum_{m=1}^{\tau} \sum_{r=0}^{R_{m}-1} \tilde{K}_{mr} n^{r} \lambda_{m}^{n} \right) - 1}, i \in \{1,...,p\}\).

d. If \(s' = 1 - s = 0\) then \(x_{n}^{(i)} = \frac{\alpha \exp \left( \sum_{m=1}^{\tau} \sum_{r=0}^{R_{m}-1} \tilde{K}_{mr} n^{r} \lambda_{m}^{n} \right) - \beta}{\exp \left( \sum_{m=1}^{\tau} \sum_{r=0}^{R_{m}-1} \tilde{K}_{mr} n^{r} \lambda_{m}^{n} \right) - 1}, i \in \{1,...,p\}\).

**Proof.** The proof is similar to the proof of Theorem 2.4. \(\square\)

**Example 2.9.** We consider interesting numerical example for the system (1.3) with \(k = 4, l = 2, p = 4, \alpha = 0.5, \beta = 0.1\) and the initial conditions

\[
\begin{array}{cccc}
i / j & 0 & 1 & 2 & 3 \\
1 & 23 & 12 & -13 & 15 \\
2 & -11 & -3 & 6 & 4 \\
3 & -2 & -12 & 15 & -17 \\
4 & 5 & 3 & 6 & 5 \\
\end{array}
\]

Table 3. The initial conditions.
The plot of the system (1.3) is shown in Figure 7.

Figure 7: The plot of the system (1.3)); when (a): $s = s' = 0$, (b): $s = s' = 1$, (c): $s' = 1 - s = 0$, (d): $s = 1 - s' = 0$ and we put the initial conditions in Table 3.

The plot of the solutions, when (a) : $s = s' = 0$ is shown in Figure 8.

Figure 8: The plot of the solutions of system (1.3); when (a): $s = s' = 0$ and the initial conditions in Table 3.

Conflict of interest
The authors declare that they have no conflicts of interest.

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