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Some Results on Lattice Involving Generalized Permuting Tri-Derivations

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ABSTRACT: In this paper we introduce the notion of generalized permuting tri-derivations, g-derivations, and g-tri-derivations on lattices, and we study and generalize some properties discussed in [14] and [29]. We also give some properties characterizing the g-derivations, the g-tri-derivations, and the generalized permuting tri-derivations and their trace.

Key Words: Lattice, generalized permuting tri-derivations, permuting tri-derivations.

Contents

1. Introduction

In 1940, the concept of lattice theory was introduced by Birkhoff [1]. After many researchers have studied this concept in different points as partially ordered set (Poset) given by Hoffmann and in [11] Balbes and Dwinger gave the concept on distributive lattices. Some authors have studied analytic and algebraic properties of lattices [3,4,13].

The derivation is consequent topic to study, Posner [2] defined the derivation on ring and many researchers studied the derivation theory in rings, near rings and on BCI-algebras [6,8,19,31]. Multi-derivations (e.g. bi-derivations, tri-derivations, or n-derivations in general) have been studied in prime and semi-prime rings [9,22,23,24]. The concept of generalized derivation in rings is introduced by Braser [10] and Hvala [5]. This concept has been studied by many researchers, for example Agraç and Albas [25] on prime rings, Ozturk and Sapancy [26] on symmetric bi-derivation in prime rings and Jana et all [32] on KUS-algebras. In the lattice context, the notion of lattice derivation was defined and developed by Szasz in [19] and used by Ferrari [21] and Xin [20] to study further properties.

Xin et all [20] introduced the concept of derivation for a lattice and discussed some related properties. The concept of generalized derivation on a lattice is introduced and some related properties are discussed in [20], and for generalized derivation on a lattice by Alshehri [27].

On the application side, lattices have played an extremely important role in many disciplines, in information theory [15], information retrieval [16], information access control [12], and in [18]. Durfee applied tools from the geometry of numbers to solve various problems in cryptanalysis. They used algebraic techniques to cryptanalyze several public key cryptosystems and used tools from the theory of integer lattices to obtain some results.

Recently, in [28] Çeven defined symmetric bi-derivations and their trace for a lattice and proved some results, and in [30] he extended his definitions and theorems to the n-derivation of lattices.

Öztürk et all [29] introduced the idea of permuting tri-derivations in lattices and studied some related properties. In [14] Çeven also studied the notion of generalized symmetric bi-derivation on lattices and

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investigated various properties. Our research was mainly inspired by the work in [14] and [29].
Our work is divided into two parts. The first part deals with the notions of g-derivations and g-tri-derivations on lattices, as well as some results, while the second part deals with the notion of generalized permuting tri-derivations on lattices, as well as its fundamental properties.
In the first part, we defined new notions called g-derivations and g-tri-derivations on lattices, therefore we enriched this part with some important examples and theorems that characterize the g-derivations and g-tri-derivations.
In the second part, we present a new notion called generalized permuting tri-derivations on a lattice, as well as examples that demonstrate the existence of this class of applications. We also present some properties that characterize in detail the generalized permuting tri-derivation and its trace, as well as we have generalized some results from the references [14] and [29].

2. Some preliminaries

In this section, we will provide some definitions and results that will be useful in the following.

Lemma 2.1. [1, Lemma 1, p: 8] *Let L be a nonempty set endowed with the operations $\wedge$ and $\vee$. By a lattice $(L, \vee, \wedge)$ we mean a set L which satisfies the following conditions:*

1. $x \wedge x = x$; $x \vee x = x$,

2. $x \wedge y = y \wedge x$; $x \vee y = y \vee x$,

3. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$; $(x \vee y) \vee z = x \vee (y \vee z)$,

4. $(x \wedge y) \vee x = x$; $(x \vee y) \wedge x = x$,

for all $x, y, z \in L$.

Lemma 2.2. [1, Lemma 1, p: 8] *Let $(L, \wedge, \vee)$ be a lattice. A binary relation $\leq$ is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.*

Lemma 2.3. [20, Lemma 2.6, p: 308] *Let $(L, \wedge, \vee)$ be a lattice. Define the binary relation $\leq$ as in Definition 2.2. Then $(L, \leq)$ is a poset and for any $x, y \in L$, $x \wedge y$ is the greatest lower bound of $\{x, y\}$ and $x \vee y$ is the least upper bound of $\{x, y\}$.*

Definition 2.4. [1, Definition, p: 12] *A lattice L is distributive if the identity* (1) *or* (2) *holds:*

1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$;

2. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$;

In any lattice, the conditions (1) *and* (2) *are equivalent.*

Definition 2.5. [1, Definition, p: 25] *An ideal is a non-void subset I of a lattice L with the properties:*

1. $x \leq y, \quad y \in I \Rightarrow x \in I$;

2. $x, y \in I \Rightarrow x \vee y \in I$.

If I_1 and I_2 are ideals of lattice L, so $I_1 \cap I_2$ is as well.

Definition 2.6. [29, Definition 6, p: 416] *Let L be a lattice. A mapping $D\colon L \times L \times L \longrightarrow L$ is called permuting if it satisfies the following condition:*

$$D(x, y, z) = D(x, z, y) = D(y, x, z) = D(y, z, x) = D(z, x, y) = D(z, y, x)$$

for all $x, y, z \in L$.

Definition 2.7. [29, Definition 6, p: 416] *A mapping $d\colon L \longrightarrow L$ defined by $d(x) = D(x, x, x)$ is called the trace of D, where D is a permuting mapping.*

Definition 2.8. [29, Definition 7, p: 416] *Let L be a lattice. The map $D\colon L^3 \to L$ will be called a permuting Tri-derivations if D is a derivation according to all components; that is,*

$$\begin{aligned} D(x \wedge w, y, z) &= (D(x, y, z) \wedge w) \vee (x \wedge D(w, y, z)) \\ D(x, y \wedge w, z) &= (D(x, y, z) \wedge w) \vee (y \wedge D(x, w, z)) \\ D(x, y, z \wedge w) &= (D(x, y, z) \wedge w) \vee (z \wedge D(x, y, w)) \end{aligned}$$

for all $x, y, z, w \in L$.

Proposition 2.9. [29] Let L be a lattice and D be a permuting tri-derivations on L with the trace d. Then the following assertions hold:

(i) $D(x, y, z) \leq x,\ D(x, y, z) \leq y$ and $D(x, y, z) \leq z$,

(ii) $D(x, y, z) \leq (x \wedge y) \wedge z$,

(iii) $d(x) \leq x$,

(iv) $d^2(x) = d(x)$,

for all $x, y, z \in L$.

Corollary 2.10. [29, Corollary 1, p: 418] *Let L be a lattice and D be a permuting tri-derivations on L. If 1 is the greatest element of L and 0 is the smallest element of L, then $D(x, y, z) = 0$ if least one of the component is 0, and $D(1, x, y) \leq x$ and $D(1, x, y) \leq y$, for all $x, y \in L$.*

3. Some results on Lattices involving g-derivations

This section introduces a new notion called g-derivations and g-tri-derivations for a lattice, followed by an example that demonstrates the existence of this type of application.

Definition 3.1. *Let L_1 and L_2 be two lattices and $f, g : L_1 \to L_2$ be two maps. We say f is g-derivation from L_1 to L_2, if $f(x \wedge y) = (f(x) \wedge g(y)) \vee (g(x) \wedge f(y))$, for all $x, y \in L_1$.*

From the above definition, for each $x \in L_1$, we have $f(x) = f(x) \wedge g(x)$. This implies $f(x) \leq g(x)$, for each $x \in L_1$.

Whenever $L_1 = L_2 = L$ and $g(x) = x$, then our definition is the old definition of derivation on a lattice L.

Example 3.2. Let R be a reduced SA-ring (i.e., a ring for which the sum of two annihilator ideals is an annihilator ideal). It is important to know that in this case R is an IN-ring, i.e., for each two ideals I, J of R $Ann(I \cap J) = Ann(I) + Ann(J)$, see [7]. Now consider two lattices $Ann(R)$ (i.e., the lattice of annihilator ideals of R) and $Id(R)$ (i.e., the lattice of ideals of R), see [17]. Put $f : Id(R) \to Ann(R)$, by $f(I) = Ann(I)$ and $J : Id(R) \to Ann(R)$, by $g(I) = R$, for each ideal I of R. Then f is a g-derivation. For, $f(I \wedge J) = Ann(I \cap J) = Ann(I) + Ann(J)$. On the other hand, $(f(I) \wedge g(J)) \vee (g(I) \wedge f(J)) = (Ann(I) \cap R) \vee (R \cap Ann(J)) = Ann(I) \vee Ann(J) = Ann(Ann(Ann(I) + Ann(J))) = Ann(I) + Ann(J)$. Moreover, this is a g-derivation which is not an f-derivation. Consider two ideals $I = eR$ and $J = (1-e)R$, for some idempotent $e \in R$. Then $f(I \wedge J) = f(o) = R$, but $f(I) \wedge f(J) = 0$.

We may have a map $f : L_1 \to L_2$ which is not g-derivation for each $g \geq f$. See the next example.

Example 3.3. *Let $L_1 = L_2 = \mathbb{R}$. Consider the function $f : \mathbb{R} \to \mathbb{R}$ by $f(1) = 2$ and $f(x) = 3$, for all $x \neq 1$. Then we have $f(-1) = f(1 \wedge -1) \neq f(1) \wedge f(-1)$. Thus f is not f-derivation. Now let $g > f$ be a function from $\mathbb{R}$ to $\mathbb{R}$. Then we have $(f(1) \wedge g(2)) \vee (g(1) \wedge f(2)) > 2 = f(1) = f(1 \wedge 2)$.*

Definition 3.4. *Let $f, g : L_1 \to L_2$ be two maps. Define $f \vee g$ and $f \wedge g$ from L_1 to L_2, by $(f \vee g)(x) = f(x) \vee g(x)$ and $(f \wedge g)(x) = f(x) \wedge g(x)$, respectively.*

Proposition 3.5. *Let L_2 be a distributive lattice. The following statements hold.*

1. *If f_1 and f_2 are two g-derivation from L_1 to L_2, then $f_1 \vee f_2$ is a g-derivation.*

2. *If f is g_1 and g_2-derivation from L_1 to L_2, then f is a $g_1 \vee g_2$-derivation.*

3. *If f is g_1 and g_2-derivation from L_1 to L_2, then f is a $g_1 \wedge g_2$-derivation.*

Proof. (1) As f_1 and f_2 are two g-derivations from L_1 to L_2 and L_2 is a distributive lattice, we conclude that;

$$\begin{aligned}(f_1 \vee f_2)(x \wedge y) &= f_1(x \wedge y) \vee f_2(x \wedge y)\\ &= [(f_1(x) \wedge g(y)) \vee (g(x) \wedge f_1(y))] \vee [(f_2(x) \wedge g(y)) \vee (g(x) \wedge f_2(y))]\\ &= ((f_1(x) \vee f_2(x)) \wedge g(y)) \vee (g(x) \wedge (f_1(y) \vee f_2(y))\\ &= ((f_1 \vee f_2)(x) \wedge g(y)) \vee (g(x) \wedge (f_1 \vee f_2)(y)).\end{aligned}$$

So we are done.
(2) By hypothesis, for $x, y \in L_1$, $f(x \wedge y) = (f(x) \wedge g_1(y)) \vee (g_1(x) \wedge f(y))$ and also $f(x \wedge y) = (f(x) \wedge g_2(y)) \vee (g_2(x) \wedge f(y))$. Thus we have;

$$\begin{aligned}f(x \wedge y) &= [(f(x) \wedge g_1(y)) \vee (g_1(x) \wedge f(y))] \vee [(f(x) \wedge g_2(y)) \vee (g_2(x) \wedge f(y))]\\ &= (f(x) \wedge (g_1(y) \vee g_2(y))) \vee ((g_1(x) \vee g_2(x)) \vee f(y))\\ &= (f(x) \wedge (g_1 \vee g_2)(x)) \vee ((g_1 \vee g_2)(y) \wedge f(y)).\end{aligned}$$

(3) By hypothesis, for $x, y \in L_1$, $f(x \wedge y) = (f(x) \wedge g_1(y)) \vee (g_1(x) \wedge f(y))$ and $f(x \wedge y) = (f(x) \wedge g_2(y)) \vee (g_2(x) \wedge f(y))$. This is also important to know that $f(x) \leq g_1(x)$ (resp., $f(y) \leq g_1(y)$) and $f(x) \leq g_2(x)$ (resp., $f(y) \leq g_2(y)$), for all $x \in L_1$ (resp., for all $y \in L_1$). This implies that;

$$\begin{aligned}f(x \wedge y) &= f(x \wedge y) \wedge f(x \wedge y)\\ &= [(f(x) \wedge g_1(y)) \vee (g_1(x) \wedge f(y))] \wedge [(f(x) \wedge g_2(y)) \vee (g_2(x) \wedge f(y))]\\ &= (f(x) \wedge g_1(y) \wedge g_2(y)) \vee (f(x) \wedge g_1(y) \wedge g_2(x) \wedge f(y)) \vee\\ &\quad (g_1(x) \wedge f(y)) \wedge f(x) \wedge g_2(y)) \vee (g_1(x) \wedge f(y) \wedge g_2(x) \wedge f(y)\\ &= (f(x) \wedge (g_1 \wedge g_2)(y)) \vee (f(x) \wedge f(y)) \vee (f(x) \wedge f(y)) \vee\\ &\quad ((g_1 \wedge g_2)(x) \wedge f(y)).\end{aligned}$$

As $f(y) \leq (g_1 \wedge g_2)(y)$, the above equality coincides to the $(f(x) \wedge (g_1 \wedge g_2)(y)) \vee ((g_1 \wedge g_2)(x) \wedge f(y))$. So we are done. □

The above proposition implies the next result. Whenever there exists $g : L_1 \to L_2$ with f is a g-derivation, we say f is a derivation from L_1 to L_2.

Corollary 3.6. *Let $f : L_1 \to L_2$ be a derivation, $(L_2, \leq, \vee, \wedge)$ a distributive lattice and $C = \{g : L_1 \to L_2 |\ f$ is a g-derivation $\}$. Then $(C, \leq, \vee, \wedge)$ is a distributive lattice.*

Proposition 3.7. *The following statements hold.*

1. *Let $f : L_1 \to L_2$. Then f is an f-derivation and a joinitive mapping if f is a homomorphism.*

2. *Let $f, g : L_1 \to L_2$ and f be a g-derivation. Then $f = g$ and is a joinitive mapping if $f(x \vee y) = (f(x) \vee g(y)) \wedge (g(x) \vee f(y))$, for all $x, y \in L_1$.*

Proof. (1) $\Rightarrow$. The f-derivation of f implies $f(x \wedge y) = f(x) \wedge f(y)$. And since f is a joinitive mapping, $f(x \vee y) = f(x) \vee f(y)$. So f is a homomorphism.
$\Leftarrow$. By hypothesis, $f(x \vee y) = f(x) \vee f(y)$ and $f(x \wedge y) = f(x) \wedge f(y)$. Thus f is a joinitive mapping and an f-derivation.
(2) $\Rightarrow$. f is g-derivation and $f = g$ is a joinitive map, hence

$$\begin{aligned}(f(x) \vee g(y)) \wedge (g(x) \vee f(y)) &= f(x) \vee f(y)\\ &= f(x \vee y).\end{aligned}$$

$\Leftarrow$. Put $x = y$. Then we have $f(x) = f(x \vee y) = f(x) \vee g(x)$. This implies $f \geq g$. As f is a g-derivation, $f \leq g$. Thus $f = g$, and by hypothesis, $f(x \vee y) = f(x) \vee f(y)$, i.e., $f = g$ is a joinitive mapping. □

Theorem 3.8. *Let L_2 be a modular lattice and f be a g-derivation from L_1 to L_2. Then the following conditions are equivalent:*

1. *f is isotone.*

2. *$f(x \wedge y) = f(x) \wedge f(y)$.*

3. *f is f-derivation.*

Proof. (1) $\Rightarrow$ (2). Assume f is isotone. Then $f(x \wedge y) \leq f(x)$ and $f(x \wedge y) \leq f(y)$. Then $f(x \wedge y) \leq f(x) \wedge f(y)^{(1)}$. On the other hand, since L is modular and $f(x) \wedge g(y) \leq g(x)$, we have

$$\begin{aligned} f(x \wedge y) &= (f(x) \wedge g(y)) \vee (g(x) \wedge f(y)) \\ &= ((f(x) \wedge g(y)) \vee f(y)) \wedge g(x) \\ &= (f(x) \vee f(y)) \wedge g(y) \wedge g(x) \\ &\geq f(x) \wedge f(y) \wedge g(x) \wedge g(y) \\ &= f(x) \wedge f(y)^{(2)}. \end{aligned}$$

Combining (1) and (2) we get $f(x \wedge y) = f(x) \wedge f(y)$.

(2) $\Rightarrow$ (1). Let $x \leq y$. Then $x = x \wedge y$, and so

$$\begin{aligned} f(x) &= f(x \wedge y) \\ &= f(x) \wedge f(y). \end{aligned}$$

It follows that $f(x) \leq f(y)$. This shows that f is isotone.

(2) $\Leftrightarrow$ (3). Trivially, f is f-derivation if

$$\begin{aligned} f(x \wedge y) &= (f(x) \wedge f(y)) \vee (f(y) \wedge f(x)) \\ &= f(x) \wedge f(y). \end{aligned}$$

□

Theorem 3.9. *Let L_2 be a distributive lattice and f be a g-derivation from L_1 to L_2. Then the following conditions are equivalent:*

1. *f is isotone.*

2. *$f(x \wedge y) = f(x) \wedge f(y)$.*

3. *f is f-derivation.*

4. *$f(x \vee y) \wedge (g(x) \vee g(y)) = f(x) \vee f(y)$.*

Proof. Since a distributive lattice is a modular lattice, the conditions (1), (2) and (3) are equivalent by Theorem 3.8. Now we prove (1) $\Leftrightarrow$ (4).

(1) $\Rightarrow$ (4). Assume f is isotone. Since f is g-derivation, we have

$$\begin{aligned} f(x) &= f((x \vee y) \wedge x) \\ &= (f(x \vee y) \wedge g(x)) \vee (g(x \vee y) \wedge f(x)) \\ &= (f(x \vee y) \wedge g(x)) \vee f(x) \\ &= f(x \vee y) \wedge g(x). \end{aligned}$$

Also

$$\begin{aligned} f(y) &= f((x \vee y) \wedge y) \\ &= (f(x \vee y) \wedge g(y)) \vee (g(x \vee y) \wedge f(y)) \\ &= (f(x \vee y) \wedge g(y)) \vee f(y) \\ &= f(x \vee y) \wedge g(y). \end{aligned}$$

Thus

$$\begin{aligned} f(x) \vee f(y) &= (f(x \vee y) \wedge g(x)) \vee (f(x \vee y) \wedge g(y)) \\ &= f(x \vee y) \wedge (g(x) \vee g(y)). \end{aligned}$$

(4) $\Rightarrow$ (1). Let $x \leq y$. Then $y = x \vee y$, and so $f(y) = f(x \vee y)$ and $f(y) \vee f(x) = f(y) \wedge (g(x) \vee g(y)) = f(y)$. It follows that $f(x) \leq f(y)$. This shows that f is isotone. □

Definition 3.10. *Let $f, g : L^3 \to L$ be two permuting maps. We say f is g-tri-derivation on a lattice L if for each $x, y, z, w \in L$, we have*

$$f(x \wedge w, y, z) = (f(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge f(w, y, z)),$$

$$f(x, y \wedge w, z) = (f(x, y, z) \wedge g(x, w, z)) \vee (g(x, y, z) \wedge f(x, w, z)),$$

$$f(x, y, z \wedge w) = (f(x, y, z) \wedge g(x, y, w)) \vee (g(x, y, z) \wedge f(x, y, w)).$$

Example 3.11. *1. If $g(x) = x \wedge y \wedge z$ and f satisfies in the above definition, then f is a tri-derivation.*

2. Let $a, b \in L$, $b \leq a$, $f, g : L^3 \to L$ be two functions by $f(x, y, z) = x \wedge y \wedge z \wedge b$ and $g(x, y, z) = x \wedge y \wedge z \wedge a$, respectively. Then we can see that f is a g-tri-derivation.

A mapping $d_f : L \to L$ defined by $d_f(x) = f(x, x, x)$ is called the trace of f, where f is a g-derivation for some $g : L^3 \to L$. Let $f, g : L^3 \to L$. Define two maps $f_1, g_1 : L \to L$, by $f_1(x) = f(x, y, z)$ and $g_1(x) = g(x, y, z)$.

Proposition 3.12. *Let $f, g : L^3 \to L$ and f be a g-tri-derivation. The following statements hold.*

1. *For all $x, y, z \in L$, $f(x, y, z) \leq g(x, y, z)$.*

2. *For all $x \in L$, $d_f(x) \leq d_g(x)$.*

3. *f_1 is a g_1-derivation.*

Proof. (1) f is g-derivation, so

$$\begin{aligned} f(x, y, z) &= f(x \wedge x, y, z) \\ &= (f(x, y, z) \wedge g(y, y, z)) \vee (g(x, y, z) \wedge f(y, y, z)) \\ &= f(x, y, z) \wedge g(y, y, z). \end{aligned}$$

This shows $f(x, y, z) \leq g(x, y, z)$
(2) Put $x = y = z$ in Part (1).
(3) Since f is a g-derivation, we have

$$\begin{aligned} f_1(x \wedge w) &= f(x \wedge w, y, z) \\ &= (f(x, y, z) \wedge g(w, y, z)) \vee (g(x, y, z) \wedge f(w, y, z)) \\ &= (f_1(x) \wedge g_1(w)) \vee (g_1(x) \wedge f_1(w)). \end{aligned}$$

So we are done. □

4. Generalized permuting tri-derivations on Lattice

Definition 4.1. *Let L be a lattice. A mapping $D\colon L\times L\times L\to L$ be a permuting tri-derivation and $\Delta\colon L\times L\times L\to L$ be a permuting mapping. Δ is called generalized permuting tri-derivation associated with D, if it satisfies the following condition*

$$\Delta(x\wedge w,y,z)=(\Delta(x,y,z)\wedge w)\vee(x\wedge D(w,y,z))$$

for all $x,y,z,w\in L$.
It is obvious that $\Delta(x,y\wedge w,z)=(\Delta(x,y,z)\wedge w)\vee(y\wedge D(x,w,z))$ and $\Delta(x,y,z\wedge w)=(\Delta(x,y,z)\wedge w)\vee(z\wedge D(x,y,w))$.

Example 4.2. (i) *Let D be a tri-derivation on a lattice L. Then D is a generalized permuting tri-derivation associated with D.*
(ii) *Let L be a lattice $a,b\in L$ and $b>a$. Consider two maps $\Delta(x,y,z)=[(x\wedge y)\wedge z]\wedge b$ and $D(x,y,z)=[(x\wedge y)\wedge z]\wedge a$. Then, D is a permuting tri-derivation on L and we can see that Δ is a generalized permuting tri-derivation associated with D on L.*
(iii) *Let L be a lattice with the least element 0. The mapping defined on L by $D(x,y,z)=0$ is a permuting tri-derivation on L. Define the mapping Δ on L by $\Delta(x,y,z)=[(x\wedge y)\wedge z]\wedge a$ for all $x,y,z,a\in L$. Then, we can see that Δ is a generalized permuting tri-derivation associated with D on L.*

Example 4.3. *Let L be a lattice. The mapping $\Delta(x,y,z)=(x\wedge y)\vee z$ is not a generalized permuting tri-derivation on L associated with $D(x,y,z)=0$.*

Proposition 4.4. *Let Δ be a generalized permuting tri-derivations associated with a permuting tri-derivations D. Then the mappings $f_1\colon L\to L, f_1(x)=\Delta(x,y,z)$, $f_2\colon L\to L, f_2(y)=\Delta(x,y,z)$, and $f_3\colon L\to L, f_3(z)=\Delta(x,y,z)$ are generalized derivations on L.*

Proof. We have

$$\begin{aligned}f_1(x\wedge a)&=\Delta(x\wedge a,y,z)\\&=(\Delta(x,y,z)\wedge a)\vee(x\wedge D(a,y,z))\\&=(f_1(x)\wedge a)\vee(x\wedge g_1(a)).\end{aligned}$$

In this equation, the mapping $g_1\colon L\to L$, $g_1(a)=D(a,y,z)$ is a derivation on L, where D is the permuting tri-derivation. So, the mapping f_1 is a generalized derivation on L.

Theorem 4.5. *Let L be a lattice, Δ be a generalized permuting tri-derivation associated with permuting tri-derivation D, δ be the trace of Δ and d be the trace of D. Then*

(i) *$D(x,y,z)\leq\Delta(x,y,z)$ for all $x,y,z\in L$;*

(ii) *$\Delta(x,y,z)\leq x$ and $\Delta(x,y,z)\leq y$ and $\Delta(x,y,z)\leq z$,*

(iii) *$\Delta(x,y,z)\leq(x\wedge y)\wedge z$,*

(iv) *$d(x)\leq\delta(x)\leq x$,*

(v) *$d(x)=x\Rightarrow\delta(x)=x$,*

for all $x,y,z\in L$.

Proof.
(i) Using the proposition 2.9 (i), we have

$$\begin{aligned}\Delta(x,y,z)&=\Delta(x\wedge x,y,z)\\&=(\Delta(x,y,z)\wedge x)\vee(x\wedge D(x,y,z))\\&=(\Delta(x,y,z)\wedge x)\vee D(x,y,z)\end{aligned}$$

It follows that $D(x,y,z) \leq \Delta(x,y,z)$, for all $x,y,z \in L$
(ii) We have

$$\begin{aligned}\Delta(x,y,z) &= \Delta(x \wedge x, y, z)\\ &= (\Delta(x,y,z) \wedge x) \vee (x \wedge D(x,y,z))\end{aligned}$$

And by (i), $D(x,y,z) \leq \Delta(x,y,z)$ for all $x,y,z \in L$, we obtain
$x \wedge D(x,y,z) \leq x \wedge \Delta(x,y,z)$, it then $\Delta(x,y,z) = \Delta(x,y,z) \wedge x$. Which implies that $\Delta(x,y,z) \leq x$ for all $x,y,z \in L$.
Since Δ is permuting, we can conclude that $\Delta(x,y,z) \leq y$ and $\Delta(x,y,z) \leq z$.
(iii) Using (ii) we can see that $\Delta(x,y,z) \leq (x \wedge y) \wedge z$.
(iv) Using (i) together with (ii), for $x = y = z$ we obtain $d(x) \leq \delta(x) \leq x$ for all $x \in L$
(v) The proof is clear by (iv). □

By Theorem 4.5 (2), we find the following corollary:

Corollary 4.6. *Let L be a lattice with a least element 0, Δ be a generalized permuting tri-derivation associated with a permuting tri-derivation D, then $\Delta(x,y,z) = 0$ if at least one of the component is 0.*

Proof. The proof is clear. □

Theorem 4.7. *Let L be a lattice, Δ be a generalized permuting tri-derivation associated with a permuting tri-derivation D, δ be the trace of Δ and d be the trace of D. Then*

$$\delta(x \wedge y) = (\delta(x) \wedge y) \vee D(x,y,y) \vee D(x,x,y) \vee (x \wedge d(y)).$$

for all $x,y \in L$.

Proof.

$$\begin{aligned}\delta(x \wedge y) &= \Delta(x \wedge y, x \wedge y, x \wedge y)\\ &= (\Delta(x, x \wedge y, x \wedge y) \wedge y) \vee (x \wedge D(y, x \wedge y, x \wedge y))\\ &= \Delta(x, x \wedge y, x \wedge y) \vee D(y, x \wedge y, x \wedge y)\\ &= (\Delta(x, x, x \wedge y) \wedge y) \vee (x \wedge D(x, y, x \wedge y)) \vee (D(y, x, x \wedge y) \wedge y)\\ &\vee (x \wedge D(y, y, x \wedge y))\\ &= \Delta(x, x, x \wedge y) \vee D(x, y, x \wedge y) \vee D(y, x, x \wedge y) \vee D(y, y, x \wedge y)\\ &= (\Delta(x,x,x) \wedge y) \vee (x \wedge D(x,x,y)) \vee (D(x,y,x) \wedge y) \vee (x \wedge D(x,y,y))\\ &\vee (D(y,x,x) \wedge y) \vee (x \wedge D(y,x,y)) \vee (D(y,y,x) \wedge y) \vee (x \wedge D(y,y,y))\\ &= (\delta(x) \wedge y) \vee D(x,x,y) \vee D(x,y,y) \vee (x \wedge d(y)).\end{aligned}$$

And our proof is complete. □

Corollary 4.8. *Let L be a lattice, and Δ be a generalized permuting tri-derivation associated with a permuting tri-derivation D, δ be the trace of Δ and d be the trace of D, then*

(i) $\delta(x) \wedge y \leq \delta(x \wedge y)$;

(ii) $x \wedge d(y) \leq \delta(x \wedge y)$;

(iii) $\delta^2(x) = \delta(x)$;

for all $x,y \in L$.

Proof. (i) and (ii) are clear by Theorem 4.7.
(iii)

$$\begin{aligned}\delta^2(x) &= \delta(\delta(x)) \\ &= \delta(x \wedge \delta(x)) \\ &= (\delta(x) \wedge \delta(x)) \vee D(x, x, \delta(x)) \vee D(x, \delta(x), \delta(x)) \vee (x \wedge d(\delta(x)) \\ &= \delta(x).\end{aligned}$$

For all $x \in L$. □

Theorem 4.9. *Let L be a lattice and Δ be a generalized permuting tri-derivation associated with a permuting tri-derivation D, δ be the trace of Δ and d be the trace of D. Then*

$$d(x) \wedge d(y) \leq \delta(x) \wedge \delta(y) \leq \delta(x \wedge y)$$

for all $x, y \in L$.

Proof.
Since $d(x) \leq \delta(x)$ and $d(y) \leq \delta(y)$, by Theorem 4.5 (iv), we obtain $d(x) \wedge d(y) \leq \delta(x) \wedge \delta(y)$ for all $x, y \in L$. By Corollary 4.8 (i), we get $\delta(x) \wedge y \leq \delta(x \wedge y)$. From Theorem 4.5 (iv), we can deduce $\delta(y) \leq y$ then $\delta(x) \wedge \delta(y) \leq \delta(x) \wedge y$ for all $x, y \in L$, hence we can conclude that $\delta(x) \wedge \delta(y) \leq \delta(x \wedge y)$ for all $x, y \in L$. □

Theorem 4.10. *Let L be a lattice and Δ be a generalized permuting tri-derivation associated with a permuting tri-derivation D, δ be the trace of Δ and d be the trace of D. Let the greatest element of L be* 1*, then*

(i) *If $x \leq \delta(1)$, then $\delta(x) = x$,*

(ii) *If $x \geq \delta(1)$, then $\delta(x) \geq \delta(1)$*

(iii) *If $x \leq y$ and $\delta(y) = y$, then $\delta(x) = x$,*

For all $x, y \in L$.

Proof. (i) By applying Corollary 4.8 (i), we have $x \wedge \delta(1) \leq \delta(x \wedge 1)$. Since 1 be the greatest element of L, we get $x \wedge \delta(1) \leq \delta(x)$ and by our hypothesis, we arrive at $x \leq \delta(x)$. From Theorem 4.5 (iv), we have $\delta(x) \leq x$, which forces that $\delta(x) = x$ for all $x \in L$.

(ii) Suppose that $x \geq \delta(1)$. By corollary 3.2 (i), we have $x \wedge \delta(1) \leq \delta(x \wedge 1)$, which gives $\delta(1) \leq \delta(x)$ for all $x \in L$.

(iii) By Corollary 4.8 (i), we find that $x \wedge \delta(y) \leq \delta(x \wedge y)$. Since $\delta(y) = y$ and $x \wedge y = x$, we get $x \leq \delta(x)$ and by Theorem 4.5 (iv), we conclude that $\delta(x) = x$. □

Corollary 4.11. *Let L be a lattice and Δ be a generalized permuting tri-derivation associated with a permuting tri-derivation D, δ be the trace of δ and the greatest element of L be* 1*, then $\delta(1) = 1$ if and only if δ is an identity mapping on L.*

Proof. By hypothesis we have $x \leq 1$ for all $x \in L$ and $\delta(1) = 1$, using Theorem 4.10 (iii) we get $\delta(x) = x$ for all $x \in L$. Conversely, it is clear that if δ is an identity mapping on L, then $\delta(1) = 1$. □

Definition 4.12. *Let L be a lattice, the map $\Delta\colon L \times L \times L \to L$ is called jointive mapping ($\vee$-homomorphism) if it satisfies*

$$\Delta(x \vee w, y, z) = \Delta(x, y, z) \vee \Delta(w, y, z)$$

$$\Delta(x, y \vee w, z) = \Delta(x, y, z) \vee \Delta(x, w, z)$$

$$\Delta(x, y, z \vee w) = \Delta(x, y, z) \vee \Delta(x, y, w)$$

For all $x, y, z, w \in L$.

Theorem 4.13. *Let L be a lattice and Δ be a jointive and permuting tri-derivation with the trace δ on L, then*

(i) $\delta(x \vee y) = \delta(x) \vee \delta(y) \vee \Delta(x,x,y) \vee \Delta(x,y,y)$;

(ii) $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$,

for all $x, y \in L$.

Proof. (i) We have

$$\begin{aligned}
\delta(x \vee y) &= \Delta(x \vee y, x \vee y, x \vee y) \\
&= \Delta(x, x \vee y, x \vee y) \vee \Delta(y, x \vee y, x \vee y) \\
&= \Delta(x,x,x \vee y) \vee \Delta(x,y,x \vee y) \vee \Delta(y,x,x \vee y) \vee \Delta(y,y,x \vee y) \\
&= \Delta(x,x,x \vee y) \vee \Delta(x,y,x \vee y) \vee \Delta(y,y,x \vee y) \\
&= \Delta(x,x,x) \vee \Delta(x,x,y) \vee \Delta(x,y,x) \vee \Delta(x,y,y) \\
&\vee \Delta(y,y,x) \vee \Delta(y,y,y) \\
&= \Delta(x,x,x) \vee \Delta(x,x,y) \vee \Delta(x,y,y) \vee \Delta(y,y,y)
\end{aligned}$$

Then $\delta(x \vee y) = \delta(x) \vee \Delta(x,x,y) \vee \Delta(x,y,y) \vee \delta(y)$. For all $x, y, z \in L$.
(ii) It is obvious from (i). □

Notation 4.1. *Let L be a lattice, and δ be a mapping of L. We set*

$$Fix_\delta(L) = \{x \in L : \delta(x) = x\} \quad and \quad B_\delta(L) = \{x \in L : x \leq \delta(1)\}.$$

Theorem 4.14. *Let L be a lattice, and Δ be a generalized permuting tri-derivation with the trace δ, and 1 be the greatest element of L. Then the following properties holds:*

(i) $\delta(x) \in B_\delta(L)$ *for all* $x \in B_\delta(L)$;

(ii) $B_\delta(L) \subset Fix_\delta(L)$;

(iii) *If Δ is jointive, then $Fix_\delta(L)$ and $B_\delta(L)$ are ideals of L;*

(iv) $1 \in B_\delta$ *if and only if δ is an identity mapping on L;*

(v) $x \leq \delta(1) \leq y$ *implies* $\delta(x) \leq \delta(y), \delta(x \wedge y) = x \wedge y$ *and* $\delta(x \vee y) = \delta(x) \vee \delta(y)$,

For all $x, y \in L$.

Proof. (i) Let $x \in B_\delta(L)$, then $x \leq \delta(1)$, from Theorem 4.10 (i), we obtain $\delta(x) \leq \delta(1)$, then $\delta(x) \in B_\delta(L)$.
(ii) Let $x \in B_\delta(L)$, then $x \leq \delta(1)$. By Theorem 4.10 (i), we get $\delta(x) = x$. Then $x \in Fix_\delta(L)$ so $B_\delta(L) \subset Fix_\delta(L)$.
(iii) By Ttheorem 4.10 (iii), if $y \leq x$ and $x \in Fix_\delta(L)$, then $y \in Fix_\delta(L)$.
Let $x, y \in Fix_\delta(L)$, from the Theorem 4.13 (ii), we obtain $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$, since $\delta(x) = x$ and $\delta(y) = y$, we get $x \vee y \leq \delta(x \vee y)$, and by Theorem 4.5 (iv), we conclude that $\delta(x \vee y) = x \vee y$, then $x \vee y \in Fix_\delta(L)$, and so $Fix_\delta(L)$ is an ideal of L.
It is easy to proof that $B_\delta(L)$ is an ideal of L.
(iv) The proof is clear by Corollary 4.11.
(v) The condition $x \leq \delta(1) \leq y$, implies $\delta(x) \leq \delta(1) \leq \delta(y)$, by Theorem 4.10 (i) and (ii).
It is obvious to see that $\delta(x \wedge y) = x \wedge y$.
The proof of the result $\delta(x \vee y) = \delta(x) \vee \delta(y)$, it suffice to use the fact that $\delta(x \vee y) = \delta(y)$ and $\delta(x) \vee \delta(y) = \delta(y)$. □

Proposition 4.15. *Let L be a lattice, Δ_1 and Δ_2 are a two generalized permuting tri-derivation associated with a same permuting tri-derivation D, δ_1 be the trace of Δ_1, δ_2 be the trace of Δ_2 and d be the trace of D, then If $\delta_2 \leq \delta_1$, then $\delta_1(\delta_2(x)) = \delta_2(x)$ for all $x \in L$.*

Proof. *From Theorem 4.5 (iv) and Theorem 4.7, we get*

$$\begin{aligned}
\delta_1(\delta_2(x)) &= \delta_1(x \wedge \delta_2(x)) \\
&= (\delta_1(x) \wedge \delta_2(x)) \vee D_1(x, x, \delta_2(x)) \vee D_1(x, \delta_2(x), \delta_2(x)) \vee \\
&\quad (x \wedge d_1(\delta_2(x))) \\
&= \delta_2(x) \text{ for all } x \in L.
\end{aligned}$$

□

Proposition 4.16. *Let L be a distributive lattice. The following statements hold.*

1. *If Δ_1 and Δ_2 are generalized permuting tri-derivations associated with D, then $\Delta_1 \vee \Delta_2$ is too.*

2. *If Δ_1 and Δ_2 are generalized permuting tri-derivations associated with D, then $\Delta_1 \wedge \Delta_2$ is too.*

3. *If Δ is generalized permuting tri-derivation associated with D_1 and D_2, then Δ is generalized permuting tri-derivation associated with $D_1 \vee D_2$.*

Proof. (1) By hypothesis for $x, y, w, z \in L$, we have

$$\Delta_1(x \wedge w, y, z) = (\Delta_1(x, y, z) \wedge w) \vee (x \wedge D(w, y, z)$$

and

$$\Delta_2(x \wedge w, y, z) = (\Delta_2(x, y, z) \wedge w) \vee (x \wedge D(w, y, z)).$$

Thus

$$\begin{aligned}
\Delta_1 \vee \Delta_2(x \wedge w, y, z) &= \Delta_1(x \wedge w, y, z) \vee \Delta_2(x \wedge w, y, z) \\
&= ((\Delta_1(x, y, z) \wedge w) \vee (x \wedge D(w, y, z))) \vee ((\Delta_2(x, y, z) \wedge w) \vee \\
&\quad (x \wedge D(w, y, z))) \\
&= ((\Delta_1(x, y, z) \vee \Delta_2(x, y, z)) \wedge w) \vee (x \wedge D(w, y, z)) \\
&= ((\Delta_1 \vee \Delta_2)(x, y, z) \wedge w) \vee (x \wedge D(w, y, z)).
\end{aligned}$$

(2) Similar to the Part (1), we have

$$\begin{aligned}
\Delta_1 \wedge \Delta_2(x \wedge w, y, z) &= \Delta_1(x \wedge w, y, z) \wedge \Delta_2(x \wedge w, y, z) \\
&= ((\Delta_1(x, y, z) \wedge w) \vee (x \wedge D(w, y, z))) \wedge ((\Delta_2(x, y, z) \wedge w) \vee \\
&\quad (x \wedge D(w, y, z))) \\
&= (x \wedge D(w, y, z)) \vee ((\Delta_1(x, y, z) \wedge \Delta_2(x, y, z)) \wedge w) \\
&= ((\Delta_1 \wedge \Delta_2)(x, y, z) \wedge w) \vee (x \wedge D(w, y, z)).
\end{aligned}$$

(3) By hypothesis for $x, y, w, z \in L$, we have

$$\Delta(x \wedge w, y, z) = (\Delta(x, y, z) \wedge w) \vee (x \wedge D_1(w, y, z))$$

and

$$\Delta(x \wedge w, y, z) = (\Delta(x, y, z) \wedge w) \vee (x \wedge D_2(w, y, z)).$$

Thus

$$\begin{aligned}
\Delta(x \wedge w, y, z) &= ((\Delta(x, y, z) \wedge w) \vee (x \wedge D_1(w, y, z))) \vee ((\Delta(x, y, z) \wedge w) \vee \\
&\quad (x \wedge D_2(w, y, z)) \\
&= ((\Delta(x, y, z) \wedge w) \vee (x \wedge (D_1(w, y, z) \vee D_2(w, y, z))) \\
&= (\Delta(x, y, z) \wedge w) \vee (x \wedge (D_1 \vee D_2(w, y, z))).
\end{aligned}$$

□

Corollary 4.17. *Let L be a distributive lattice, D be a tri-derivation on L and $\Gamma = \{\Delta : L^3 \to L | \ \Delta$ is generalized permuting tri-derivation associated with $D\}$. Then $(\Gamma, \vee, \wedge)$ is a distributive lattice.*

Definition 4.18. *Let L be a lattice, Δ be a generalized permuting tri-derivation associated with a permuting tri-derivation D and δ be the trace of Δ. δ is called an isotone mapping, if $x \leq y$ implies $\delta(x) \leq \delta(y)$.*

Proposition 4.19. *Let L be a distributive lattice and Δ be a generalized permuting tri-derivation related to a permuting tri-derivation D, δ be the trace of Δ and d be the trace of D. If 1 is the greatest element of L, then the following conditions are equivalent:*

(i) *δ is an isotone mapping,*

(ii) $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$,

(iii) $\delta(x \wedge y) = \delta(x) \wedge \delta(y)$,

(iv) $\delta(x) = x \wedge \delta(1)$,

for all $x, y \in L$.

Proof. $(i) \Rightarrow (ii)$. We have $x \leq x \vee y$ and $y \leq x \vee y$. Since δ is an isotone mapping, we obtain $\delta(x) \leq \delta(x \vee y)$ and $\delta(y) \leq \delta(x \vee y)$, so $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$ for all $x, y \in L$.
(ii) $\Rightarrow$ (i). Suppose that $x \leq y$ and $\delta(x) \vee \delta(y) \leq \delta(x \vee y)$. Using the fact that $x \vee y = y$, then $\delta(x) \vee \delta(y) \leq \delta(y)$. Since $\delta(y) \leq \delta(x) \vee \delta(y)$, we find that $\delta(y) = \delta(x) \vee \delta(y)$, which assures that $\delta(x) \leq \delta(y)$ for all $x, y \in L$.
(i) $\Rightarrow$ (iii). From Theorem 4.9, we have $\delta(x \wedge y) \geq \delta(x) \wedge \delta(y)$. Since $x \wedge y \leq x, x \wedge y \leq y$ and δ is an isotone mapping, we get $\delta(x \wedge y) \leq \delta(x)$ and $\delta(x \wedge y) \leq \delta(y)$, so $\delta(x \wedge y) \leq \delta(x) \wedge \delta(y)$, which conclude that $\delta(x \wedge y) = \delta(x) \wedge \delta(y)$ for all $x, y \in L$.
$(iii) \Rightarrow (i)$. Let $\delta(x \wedge y) = \delta(x) \wedge \delta(y)$ and $x \leq y$, then $\delta(x) = \delta(x) \wedge \delta(y)$, so $\delta(x) \leq \delta(y)$ for all $x, y \in L$, which implies that δ is an isotone mapping.
$(i) \Rightarrow (iv)$. Since $x \leq 1$ for all $x \in L$, and δ is an isotone mapping, we have $\delta(x) \leq \delta(1)$. From Theorem 4.5 (iv), it follows that $\delta(x) \leq x$, then $\delta(x) \leq x \wedge \delta(1)$ for all $x \in L$. By Corollary 4.8 (ii), we have $\delta(1) \wedge x \leq \delta(x)$ for all $x \in L$, which forces that $\delta(x) = x \wedge \delta(1)$ for all $x \in L$.
$(iv) \Rightarrow (i)$. Assume that $x \leq y$ and $\delta(x) = x \wedge \delta(1)$, then

$$\begin{aligned}
\delta(x) &= \delta(x \wedge y) \\
&= (x \wedge y) \wedge \delta(1) \\
&= (x \wedge \delta(1)) \wedge (y \wedge \delta(1)) \\
&= \delta(x) \wedge \delta(y),
\end{aligned}$$

so $\delta(x) \leq \delta(y)$ for all $x, y \in L$. □

Remark 4.20. *Let L be a lattice, Δ be a generalized permuting tri-derivation associated with a permuting tri-derivation D and δ the trace of Δ. We can easilly see that $Fix_\delta(L) = B_\delta(L)$ if δ is an isotone mapping.*

Theorem 4.21. *Let L be a lattice, Δ be a generalized permuting tri-derivation associated with a permuting tri-derivation D and δ be the trace of Δ. The following conditions are equivalent:*

(i) *δ is the identity derivation;*

(ii) $\delta(x \vee y) = (\delta(x) \vee y) \wedge (x \vee \delta(y))$ *for all $x, y \in L$;*

(iii) *δ is a monomorphic derivation;*

(iv) *δ is an epic derivation.*

Proof. $(i) \Rightarrow (ii)$. If $\delta(x) = x$ for all $x \in L$, then $\delta(x \vee y) = x \vee y$ and

$$\begin{aligned} (\delta(x) \vee y) \wedge (x \vee \delta(y)) &= (x \vee y) \wedge (x \vee y) \\ &= x \vee y, \end{aligned}$$

thus $\delta(x \vee y) = (\delta(x) \vee y) \wedge (x \vee \delta(y))$ for all $x, y \in L$.
(ii) $\Rightarrow$ (i). Suppose that $\delta(x \vee y) = (\delta(x) \vee y) \wedge (x \vee \delta(y)$ for all $x, y \in L$, then

$$\begin{aligned} \delta(x) &= \delta(x \vee x) \\ &= (\delta(x) \vee x) \wedge (x \vee \delta(x)) \\ &= x \wedge x \\ &= x \text{ for all } x \in L, \end{aligned}$$

which implies that δ is the identity derivation.
$(i) \Rightarrow (iii)$ is obvious.
$(i) \Rightarrow (iv)$ is obvious.
$(iii) \Rightarrow (i)$ Let δ be a monomorphic derivation. Suppose there exists $a \in L$, such that $\delta(a) \neq a$, then $\delta(a) < a$. Denote $a_1 = \delta(a)$, then $a_1 < a$. From Theorem 4.7, Proposition 2.9 (1) and Theorem 4.5 (iv), we get

$$\begin{aligned} \delta(a_1) &= \delta(a_1 \wedge a) \\ &= (\delta(a_1) \wedge a) \vee D(a_1, a_1, a) \vee D(a_1, a, a) \vee (a_1 \wedge d(a)) \\ &= (\delta^2(a) \wedge a) \vee D(a_1, a_1, a) \vee D(a_1, a, a) \vee (\delta(a) \wedge d(a)) \\ &= (\delta(a) \wedge a) \vee D(a_1, a_1, a) \vee D(a_1, a, a) \vee d(a) \\ &= \delta(a) \vee D(a_1, a_1, a) \vee D(a_1, a, a) \vee d(a) \\ &= \delta(a), \end{aligned}$$

which contradicts with the fact that δ is monomorphic and $a_1 \neq a$.
(iv) $\Rightarrow$ (i). Suppose that δ is an epic derivation, then $\delta(L) = L$. Thus for any $x \in L$, there exists $y \in L$, such that $x = \delta(y)$. By Corollary 4.8 (iii), we conclude that $\delta(x) = \delta(\delta(y)) = \delta^2(y) = \delta(y) = x$ for all $x \in L$. Which shows that δ is the identity derivation on L. $\square$

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