Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–10.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.63024

Existence of Solutions for an Approximation of the Paneitz Problem on Spheres

K. Ould Bouh*

ABSTRACT: This paper is devoted to studying the nonlinear problem with subcritical exponent (S_ε) : $\Delta^2 u - c_n\Delta u + d_n u = K u^{\frac{n+4}{n-4}-\varepsilon}$, $u > 0$ on S^n, where $n \geq 5$, ε is a small positive parameter and K is a a smooth positive function on S^n. We construct some solutions which blow up at q different critical points of K.

Key Words: Critical points, critical exponent, variational problem, Paneitz curvature.

Contents

1. Introduction

Given (M^4, g), a smooth 4-dimensional Riemannian manifold, let S_g be the scalar curvature of g and Ric_g be the Ricci curvature of g. In [13], Paneitz discovered the following fourth order operator

$$P_g^4\varphi = \Delta_g^2\varphi - div_g(\frac{2}{3}S_g - 2Ric_g)d\varphi$$

This operator is conformally invariant in the sense that if $\widetilde{g} = e^{2u}g$ is a metric conformally equivalent to g, then

$$P_{\widetilde{g}}^4\varphi = e^{-4u}P_g^4\varphi \quad \text{ for all } \varphi \in C^\infty(M),$$

and it can be seen as a natural extension of the conformal Laplacian on 2-manifolds. A generalization of P_g^4 to higher dimensions has been discovered by Branson [7]. Let (M, g) be a smooth compact Riemannian n-manifol, $n \geq 5$. The Paneitz operator P_g^n of [7] is defined by

$$P_g^n u = \Delta_g^2 u - div_g(a_n S_g g + b_n Ric_g)du + \frac{n-4}{2}Q_g^n u,$$

where

$$a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}, \quad b_n = \frac{-4}{n-2}$$

$$Q_g^n = -\frac{1}{2(n-1)}\Delta_g S_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}S_g^2 - \frac{2}{(n-2)^2}|Ric_g|^2.$$

If $\widetilde{g} = u^{4/(n-4)}g$ is a metric conformal to g, then for all $\varphi \in C^\infty(M)$ one has

$$P_g^n(u\varphi) = u^{(n+4)/(n-4)}P_{\widetilde{g}}^n(\varphi)$$

* This work was carried out with the aid of a grant from UNESCO-TWAS and the Swedish International Development Cooperation Agency (Sida). The views expressed herein do not necessarily represent those of UNESCO-TWAS, Sida or its Board of Governors.
2010 *Mathematics Subject Classification:* Primary: 35J20, Secondary: 35J60.
Submitted March 24, 2022. Published June 23, 2022

and

$$P_g^n(u) = \frac{n-4}{2} Q_{\widetilde{g}}^n u^{(n+4)/(n-4)}. \tag{1.1}$$

For more details about the properties of the Paneitz operator, see for example [7], [8], [9], [10], [11]. In view of relation (1.1), it is natural to consider the problem of prescribing the conformal invariant Q^n called the Paneitz curvature, that is : given a function $f : M \to \mathbb{R}$ does there exist a metric $\widetilde{g}$ conformally equivalent to g such that $Q_{\widetilde{g}}^n = f$? By equation (1.1), the problem can be formulated as follows. We look for solutions of

$$P_g^n(u) = \frac{n-4}{2} f u^{(n+4)/(n-4)}, \qquad u > 0 \quad \text{on} \quad M. \tag{1.2}$$

In the case of the standard sphere (S^n, g), $n \geq 5$. Thus, we are reduced to finding a positive solution u of the problem

$$\mathcal{P}u = \Delta^2 u - c_n \Delta u + d_n u = K u^{\frac{n+4}{n-4}}, \qquad u > 0 \quad \text{on} \quad S^n, \tag{1.3}$$

where $c_n = \frac{1}{2}(n^2 - 2n - 4)$, $d_n = \frac{n-4}{16} n(n^2-4)$ and K is a given function defined on S^n.
It is well known that, there are topological obstructions to solve (1.3), based on a Kazdan-Warner type condition, see [9]. Thus a natural question arises : under which conditions on K, does (1.3) admit a solution? In this paper, we give sufficient conditions on K such that (1.3) possesses a solution.
Notice that, problem (1.3) has been widely studied in the last decades. In [5], [9], [10], the authors treated the lower dimensional case (n = 5, 6). In [11], Felli proved a perturbative theorem and some existence results under assumptions of symmetry.

Note that the embedding of $H_2^2(S^n)$ into $L^{2n/(n-4)}(S^n)$ is noncompact. Hence, for the study of problem (1.3), it is interesting to approach it by a family of subcritical problems (S_ε)

$$(S_\varepsilon) \quad \mathcal{P}u = \Delta^2 u - c_n \Delta u + d_n u = K u^{\frac{n+4}{n-4} - \varepsilon}, \qquad u > 0 \quad \text{on} \quad S^n,$$

and we need to study the asymptotic behavior of the solutions (u_ε) as $\varepsilon \to 0$. Observe that, since $\varepsilon > 0$, problem (S_ε) has always a positive solution (u_ε).

In this paper, we aim to construct some sign-changing solutions (u_ε) of (S_ε) which blow up at one or two different points in the interior.

Before stating the result, we need to introduce some notations. For $K \equiv 1$, the solutions of (1.3) form a family $\widetilde{\delta}_{(a,\lambda)}$ defined by

$$\widetilde{\delta}_{(a,\lambda)}(x) = \frac{\beta_n}{2^{\frac{n-4}{2}}} \frac{\lambda^{\frac{n-4}{2}}}{\left(1 + \frac{\lambda^2 - 1}{2}(1 - \cos d(a,x))\right)^{\frac{n-4}{2}}} \tag{1.4}$$

where $a \in S^n$, $\lambda > 0$ and $\beta_n = \Big((n-4)(n-2)n(n+2)\Big)^{(n-4)/8}$.
After performing a stereographic projection π with the point $-a$ as pole, the function $\widetilde{\delta}_{(a,\lambda)}$ is transformed into

$$\delta_{(0,\lambda)} = \beta_n \Big(\frac{\lambda}{1 + \lambda^2 |y|^2}\Big)^{\frac{n-4}{2}}$$

which is a solution of the problem (see [12])

$$\Delta^2 u = u^{\frac{n+4}{n-4}}, \quad u > 0, \quad \text{on } \mathbb{R}^n.$$

The space $H_2^2(S^n)$ is equipped with the norm :

$$\|u\|^2 = \langle u, u \rangle_{\mathcal{P}} = \int_{S^n} \mathcal{P}u.u = \int_{S^n} |\Delta u|^2 + c_n \int_{S^n} |\nabla u|^2 + d_n \int_{S^n} u^2.$$

Our result deals with the construction of some solutions (u_ε) of (S_ε) which blow up at q different points of S^n.

Theorem 1.1. *Let $n = 6$ and y_1, y_2, ..., y_q be nondegenerate critical points of K with $-\Delta K(y_i) > 0$ for $i = 1, 2, ..., q$. Then, there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$, the problem (S_ε) has a solution (u_ε) of the form*

$$u_\varepsilon = \sum_{i=1}^{q} \alpha_i \widetilde{\delta}_{(x_i,\lambda_i)} + v, \tag{1.5}$$

where $\alpha_i \to K(y_i)^{-1/4}$; $\|v\| \to 0$; $x_i \to y_i$, $\lambda_i \to +\infty$; $\lambda_i = \gamma\lambda_j(1 + o(1))$ as $\varepsilon \to 0$.
Here, γ is a positive fixed constant.

Remark 1.2. *Note that, Theorem 1.1 is proved only in dimension $n = 6$. We think that it is also true for $n > 6$ but it is too technical to discuss this problem in this paper.*

The remaind of this paper is organized as follows. In Section 2, we recall some preliminaries. In Section 3, we give some careful expansions of gradient of the associated variational functional I_ε for $(\varepsilon > 0)$. While Section 4 is devoted to the proof of Theorem 1.1.

2. Preliminary results

First, let us introduce the general setting. For $\varepsilon > 0$, we define the functional

$$I_\varepsilon(u) = \int_{S^n} |\Delta u|^2 + c_n \int_{S^n} |\nabla u|^2 + d_n \int_{S^n} u^2 - \frac{1}{\frac{2n}{n-4} - \varepsilon} \int_{S^n} K u^{\frac{2n}{n-4}-\varepsilon},\ u \in H_2^2(S^n). \tag{2.1}$$

Note that if u is a positive critical point of I_ε, then u is a solution of (S_ε), and inversely.
Let

$$E_{(x,\lambda)} = \{w \in H_2^2(S^n)/\langle w, \varphi\rangle = 0 \quad \forall\varphi \in \mathrm{Span}\{\widetilde{\delta}_i, \frac{\partial\widetilde{\delta}_i}{\partial\lambda_i}, \frac{\partial\widetilde{\delta}_i}{\partial x_i^j},\ i \le q; j \le n\}\}.$$

Here, x_i^j denotes the j-th component of x_i. For sake of simplicity, we will write $\widetilde{\delta}_i$ instead of $\widetilde{\delta}_{(x_i,\lambda_i)}$ and therefore, for $u = \sum_i \alpha_i \widetilde{\delta}_{(x_i,\lambda_i)} + v$ we can write $u = \sum_i \alpha_i \widetilde{\delta}_i + v$. In the sequel, we mention that it will be convenient to perform some stereographic projection in order to reduce our problem to $\mathbb{R}^n$.

Lemma 2.1. *[2] For $a \in S^n$ and $\lambda > 0$ large enough. Using the stereographic projection π_{-a}, the function $\widetilde{\delta}_{(a,\lambda)}$ will be transformed to $\delta_{(0,\lambda)}$. Furthermore, we have*

$$\int_{S^n} \mathcal{P}\widetilde{\delta}_i.\widetilde{\delta}_i = \int_{S^n} \widetilde{\delta}_i^{\frac{2n}{n-4}} = \int_{\mathbb{R}^n} \delta_i^{\frac{2n}{n-4}} = S_n, \tag{2.2}$$

$$\langle \widetilde{\delta}_i, \lambda_i \frac{\partial\widetilde{\delta}_i}{\partial\lambda_i}\rangle = 0 \tag{2.3}$$

$$\langle \widetilde{\delta}_i, \frac{1}{\lambda_i}\frac{\partial\widetilde{\delta}_i}{\partial x_i}\rangle = 0. \tag{2.4}$$

Lemma 2.2. *[2] For $a_1, a_2 \in S^n$ and $\lambda_1, \lambda_2 > 0$ large enough, let $b \in S^n$ such that $d(a_1, b) = d(a_2, b)$. Using the stereographic projection π_{-b}, the function $\widetilde{\delta}_{(a_i,\lambda_i)}$ will be transformed to $\delta_{(\widetilde{a_i},\widetilde{\lambda_i})}$ with*

$$\widetilde{a_i} = \frac{(\lambda_i^2 - 1)Proj_{\mathbb{R}^n} a_1}{2 + (\lambda_i^2 - 1)(1 - \cos\theta_0)}, \quad \widetilde{\lambda_i} = \frac{2 + (\lambda_i^2 - 1)(1 - \cos\theta_0)}{\lambda_i}, \quad \theta_0 = \pi - d(a_i, b)$$

Furthermore, we have $i \ne j$,

$$\int_{S^n} \mathcal{P}\widetilde{\delta}_i.\widetilde{\delta}_j = \int_{S^n} \widetilde{\delta}_i^{\frac{n+4}{n-4}} \widetilde{\delta}_j = \int_{\mathbb{R}^n} \delta_i^{\frac{n+4}{n-4}} \widetilde{\delta}_j = c_1\widetilde{\varepsilon}_{ij} + o(\widetilde{\varepsilon}_{ij}) = c_1\varepsilon_{ij} + o(\varepsilon_{ij}), \tag{2.5}$$

where

$$\varepsilon_{ij} = \Big(\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \frac{\lambda_i\lambda_j}{2}(1 - \cos d(a_i, a_j))\Big)^{-\frac{n-4}{2}} \tag{2.6}$$

$$\widetilde{\varepsilon}_{ij} = \Big(\frac{\widetilde{\lambda}_i}{\widetilde{\lambda}_j} + \frac{\widetilde{\lambda}_j}{\widetilde{\lambda}_i} + \widetilde{\lambda}_i\widetilde{\lambda}_j|\widetilde{a}_i - \widetilde{a}_j|^2\Big)^{-\frac{n-4}{2}}. \tag{2.7}$$

3. Expansions of the gradient of the functional I_ε

In this section, we collect some expansions of the gradient of the functional I_ε associated to the problem (S_ε) for $\varepsilon > 0$ which are needed in Section 4. We start by giving the following remark which is proved in [14] when S^n is replaced by a bounded domain of $\mathbb{R}^3$.

Remark 3.1. *Let $\delta_{(a,\lambda)}$ be the function defined in (1.4). Assume that $\varepsilon \log \lambda$ is small enough. For $\varepsilon > 0$, we have*

$$\delta_{(a,\lambda)}^{-\varepsilon}(x) = 1 - \varepsilon \log \delta_{(a,\lambda)} + O(\varepsilon^2 \log^2 \lambda) \quad \text{in } S^n.$$

Now, explicit computations, by Remark 3.1, yield the following propositions

Proposition 3.2. *For $u = \sum_i \alpha_i \widetilde{\delta}_{(x_i,\lambda_i)} + v$ with $v \in E_{x,\lambda}$, we have*

$$\langle \nabla I_\varepsilon, \widetilde{\delta}_i \rangle = \alpha_i S_n (1 - \alpha_i^{\frac{8}{n-4}-\varepsilon} K(x_i)) + O\Big(\varepsilon \log \lambda_i + \frac{1}{\lambda_i^2} + \sum_{j\neq i} \varepsilon_{ij} + \|v\|^2\Big),$$

where $S_n = \int_{\mathbb{R}^n} \delta^{\frac{2n}{n-4}}$.

Proof. We have

$$\langle \nabla I_\varepsilon, h \rangle = \int_{S^n} \mathcal{P}u.h - \int_{S^n} K u^{p-\varepsilon} h. \tag{3.1}$$

A computation similar to the one performed in [1] shows that

$$\int_{S^n} \mathcal{P}\widetilde{\delta}_i.\widetilde{\delta}_i = \int_{S^n} \widetilde{\delta}_i^{\frac{2n}{n-4}} = \int_{\mathbb{R}^n} \delta_i^{\frac{2n}{n-4}} = S_n, \tag{3.2}$$

$$\int_{S^n} \mathcal{P}\widetilde{\delta}_j.\widetilde{\delta}_i = \int_{S^n} \widetilde{\delta}_j^{\frac{n+4}{n-4}} \widetilde{\delta}_i = \int_{\mathbb{R}^n} \delta_j^{\frac{n+4}{n-4}} \delta_i = O(\varepsilon_{ij}). \tag{3.3}$$

For the integral, we write

$$\int_{S^n} K u^{\frac{n+4}{n-4}-\varepsilon} \widetilde{\delta}_i = \int_{S^n} K(\sum_j \alpha_j \widetilde{\delta}_j)^{\frac{n+4}{n-4}-\varepsilon} \widetilde{\delta}_i + O\Big(\sum_{j\neq i} \varepsilon_{ij}^2 \log \varepsilon_{ij}^{-1} + |v|^2\Big). \tag{3.4}$$

We also write

$$\int_{S^n} K(\sum_{j=1} \alpha_j \widetilde{\delta}_j)^{\frac{n+4}{n-4}-\varepsilon} \widetilde{\delta}_i = \sum_{j=1} \int_{S^n} K(\alpha_j \widetilde{\delta}_j)^{\frac{n+4}{n-4}-\varepsilon} \delta_i + (\frac{n+4}{n-4} - \varepsilon) \sum_{j\neq i} \alpha_i^{\frac{8}{n-4}-\varepsilon} \alpha_j \int_{S^n} K \widetilde{\delta}_i^{\frac{n+4}{n-4}-\varepsilon} \widetilde{\delta}_j$$
$$+ O\Big(\sum_{i\neq j} \varepsilon_{ij}^2 \log \varepsilon_{ij}^{-1}\Big). \tag{3.5}$$

Expanding of K around x_i and x_j, $j \neq i$, we get

$$\int_{S^n} K \widetilde{\delta}_i^{\frac{2n}{n-4}-\varepsilon} = \int_{\mathbb{R}^n} \widetilde{K} \delta_i^{\frac{2n}{n-4}-\varepsilon} = K(x_i) S_n + O\Big(\varepsilon \log \lambda_i + \frac{1}{\lambda_i^2}\Big), \tag{3.6}$$

$$\int_{S^n} K \widetilde{\delta}_j^{\frac{n+4}{n-4}-\varepsilon} \widetilde{\delta}_i = \int_{\mathbb{R}^n} \widetilde{K} \delta_j^{\frac{n+4}{n-4}-\varepsilon} \delta_i = O\left(\varepsilon \log \lambda_j + \varepsilon_{ij}\right). \tag{3.7}$$

Combining (3.1)-(3.7), we easily derive our proposition.

Proposition 3.3. *For $u = \sum_i \alpha_i \widetilde{\delta}_{(x_i,\lambda_i)} + v$ with $v \in E_{x,\lambda}$, we have the following expansion:*

$$\langle \nabla I_\varepsilon(u), \lambda_i \frac{\partial \widetilde{\delta}_i}{\partial \lambda_i} \rangle = \Bigg[\alpha_j c_1 \Big(1 - \sum_j \alpha_j^{\frac{8}{n-4}-\varepsilon} K(x_j)\Big) \lambda_i \frac{\partial \varepsilon_{12}}{\partial \lambda_i} + \alpha_i^{\frac{n+4}{n-4}-\varepsilon} \frac{\varepsilon S_n K(x_i)}{n}$$
$$+ \alpha_i^{\frac{n+4}{n-4}-\varepsilon} \frac{4(n-4)c_2}{n} \frac{\Delta K(x_i)}{\lambda_i^2} \Bigg] + O\Big(\varepsilon^2 \log \lambda_i + \frac{\varepsilon \log \lambda_i}{\lambda_i^2} + \frac{1}{\lambda_i^3} + \|v\|^2\Big)$$
$$+ O\Big(\sum_{j\neq i} \Big[\varepsilon \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{\frac{n-4}{n}} + \varepsilon_{ij}^{\frac{n}{n-4}} \log \varepsilon_{ij}^{-1} + \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{\frac{n-4}{n}} (\frac{1}{\lambda_i} + \frac{1}{\lambda_j})\Big]\Big),$$

where

$$c_1 = \beta_n^{\frac{2n}{n-4}} \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{\frac{n+4}{2}}} dx\,, \quad c_2 = \frac{1}{2n}\int_{\mathbb{R}^n} |x|^2 \delta_{(0,1)}^{\frac{2n}{n-4}} dx.$$

Proof. Observe that (see [1])

$$\langle \widetilde{\delta}_i, \lambda_i \frac{\partial \widetilde{\delta}_i}{\partial \lambda_i}\rangle = 0 \tag{3.8}$$

$$\langle \widetilde{\delta}_j, \lambda_i \frac{\partial \widetilde{\delta}_i}{\partial \lambda_i}\rangle = c_1 \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + o(\varepsilon_{ij}). \tag{3.9}$$

For the other part

$$\int_{\mathbb{S}^n} K \widetilde{\delta}_i^{\frac{n+4}{n-4}-\varepsilon} \lambda_i \frac{\partial \widetilde{\delta}_i}{\partial \lambda_i} = \int_{\mathbb{R}^n} \widetilde{K} \delta_i^{\frac{n+4}{n-4}-\varepsilon} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = -\frac{(n-4)c_2}{n}\frac{4\Delta K(x_i)}{\lambda_i^2} - \frac{S_n \varepsilon}{n} K(x_i) + O\Big(\varepsilon^2 \log \lambda_i + \frac{1}{\lambda_i^3} + \frac{\varepsilon \log \lambda_i}{\lambda_i^2}\Big), \tag{3.10}$$

$$\int_{\mathbb{S}^n} K \widetilde{\delta}_j^{\frac{n+4}{n-4}-\varepsilon} \lambda_i \frac{\partial \widetilde{\delta}_i}{\partial \lambda_i} = \int_{\mathbb{R}^n} \widetilde{K} \delta_j^{\frac{n+4}{n-4}-\varepsilon} \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = c_1 K(x_j) \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\Big(\varepsilon \varepsilon_{ij} (\log(\varepsilon_{ij}^{-1}))^{\frac{n-4}{n}} + \frac{1}{\lambda_i^3}\Big) + O\Big(\varepsilon_{ij}^{\frac{n}{n-4}} \log(\varepsilon_{ij}^{-1})\Big), \tag{3.11}$$

$$(\frac{n+4}{n-4} - \varepsilon) \int_{\mathbb{S}^n} K \widetilde{\delta}_i^{\frac{8}{n-4}-\varepsilon} \widetilde{\delta}_j \lambda_i \frac{\partial \widetilde{\delta}_i}{\partial \lambda_i} = (\frac{n+4}{n-4} - \varepsilon) \int_{\mathbb{R}^n} \widetilde{K} \delta_i^{\frac{8}{n-4}-\varepsilon} \delta_j \lambda_i \frac{\partial \delta_i}{\partial \lambda_i} = c_1 K(x_i) \lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} + O\Big(\varepsilon \varepsilon_{ij} (\log(\varepsilon_{ij}^{-1}))^{\frac{n-4}{n}}\Big) + O\Big(\varepsilon_{ij}^{\frac{n}{n-4}} \log(\varepsilon_{ij}^{-1}) + \frac{\varepsilon_{ij}}{\lambda_i} (\log(\varepsilon_{ij}^{-1}))^{\frac{n-4}{n}}\Big). \tag{3.12}$$

Combining (3.1) and (3.8)-(3.12), we derive our proposition.

Proposition 3.4. *For $u = \sum_i \alpha_i \widetilde{\delta}_{(x_i,\lambda_i)} + v$, with $v \in E_{x,\lambda}$, we have*

$$\langle \nabla I_\varepsilon(u), \frac{1}{\lambda_i}\frac{\partial \widetilde{\delta}_i}{\partial x_i}\rangle = \frac{1}{\lambda_i}\Big[\alpha_i c_2 \Big(1 - \sum \alpha_j^{\frac{8}{n-4}-\varepsilon} K(x_j)\Big)\frac{\partial \varepsilon_{12}}{\partial x_i} - \alpha_i^{\frac{n+4}{n-4}-\varepsilon} c_3 \nabla K(x_i)\Big] + O\Big(\frac{\varepsilon \log \lambda_i}{\lambda_i} |\nabla K(x_i)| + \frac{1}{\lambda_i^2} + \|v\|^2 + \lambda_j |x_1 - x_2| \varepsilon_{12}^{\frac{n-1}{n-4}}\Big) + O\Big(\varepsilon \varepsilon_{12} (\log \varepsilon_{12}^{-1})^{2\frac{n-4}{n}} + \varepsilon_{12}^2 \log \varepsilon_{12}^{-1} + \varepsilon_{12} (\log \varepsilon_{12}^{-1})^{\frac{1}{2}} (\frac{1}{\lambda_1} + \frac{1}{\lambda_2})\Big).$$

Proof. An easy computation shows

$$\langle \widetilde{\delta}_i, \frac{1}{\lambda_i}\frac{\partial \widetilde{\delta}_i}{\partial x_i}\rangle = 0, \tag{3.13}$$

$$\langle \widetilde{\delta}_j, \frac{1}{\lambda_i}\frac{\partial \widetilde{\delta}_i}{\partial x_i}\rangle = \frac{c_2}{\lambda_i}\frac{\partial \varepsilon_{ij}}{\partial x_i} + O\left(\varepsilon_{ij}^2 \log(\varepsilon_{ij}^{-1}) + \varepsilon_{ij}^{\frac{n-1}{n-4}} \lambda_i |x_i - x_j|\right). \tag{3.14}$$

For the other part

$$\int_{S^n} K\widetilde{\delta}_i^{\frac{n+4}{n-4}-\varepsilon}\frac{1}{\lambda_i}\frac{\partial\widetilde{\delta}_i}{\partial x_i}=\int_{\mathbb{R}^n}\widetilde{K}\delta_i^{\frac{n+4}{n-4}-\varepsilon}\frac{1}{\lambda_i}\frac{\partial\delta_i}{\partial x_i}=c_3\frac{\nabla K(x_i)}{\lambda_i}+O\Big(\frac{1}{\lambda_i^2}+\varepsilon^2\log\lambda_i\Big),\tag{3.15}$$

$$\int_{S^n} K\widetilde{\delta}_j^{\frac{n+4}{n-4}-\varepsilon}\frac{1}{\lambda_i}\frac{\partial\widetilde{\delta}_i}{\partial x_i}=K(x_j)\frac{c_2}{\lambda_i}\frac{\partial\varepsilon_{12}}{\partial x_i}+O\Big(\varepsilon_{12}^{\frac{n-1}{n-4}}\lambda_i|x_1-x_2|+\varepsilon_{12}^2\log(\varepsilon_{12}^{-1})+\frac{1}{\lambda_i}\varepsilon_{12}\Big(\log(\varepsilon_{12}^{-1})\Big)^{\frac{1}{2}}\Big),\tag{3.16}$$

$$(\frac{n+4}{n-4}-\varepsilon)\int_{S^n} K\widetilde{\delta}_i^{\frac{8}{n-4}-\varepsilon}\widetilde{\delta}_j\frac{1}{\lambda_i}\frac{\partial\widetilde{\delta}_i}{\partial x_i}=K(x_i)\frac{c_2}{\lambda_i}\frac{\partial\varepsilon_{12}}{\partial x_i}+O\Big(\varepsilon_{12}^{\frac{n-1}{n-4}}\lambda_j|x_1-x_2|\Big)$$
$$+O\left(\varepsilon_{12}^2\log(\varepsilon_{12}^{-1})+\frac{1}{\lambda_i}\varepsilon_{12}\left(\log(\varepsilon_{12}^{-1})\right)^{\frac{1}{2}}\right).\tag{3.17}$$

Using (3.1), (3.13)-(3.17), we have our proposition.

4. Proof of Theorem 1.1

The proof uses the same argument than the previous proof. We will focus only on the main points. The first one is concerning the concentration points and speeds. As in the proof of Theorem 1.1, we introduce the set

$$M_{\varepsilon,2}=\Bigg\{m=(\alpha,\lambda,x,v)\in\mathbb{R}^q\times(\mathbb{R}_+^*)^q\times(S^n)^q\times H_2^2(S^n):v\in E_{(x,\lambda)},\|v\|<\nu_0,\lambda_i>\frac{1}{\nu_0},$$
$$\varepsilon\log\lambda_i<\nu_0,\ \left|\frac{\alpha_i^{\frac{8}{n-4}}K(x_i)}{\alpha_j^{\frac{8}{n-4}}K(x_j)}-1\right|<\nu_0,\ c_0<\frac{\lambda_i}{\lambda_j}<c_0^{-1},\ |x_i-x_j|>d_0,\ \forall\, i,j\Bigg\},$$

where σ, c_0, d_0 are some suitable positive constants, ν_0 is a small positive constant. Let us define the function by

$$\Psi_{\varepsilon,2}:M_{\varepsilon,2}\to\mathbb{R};\quad m=(\alpha,\lambda,x,v)\mapsto I_\varepsilon\Big(\sum_{i=1}^q\alpha_i\widetilde{\delta}_{(x_i,\lambda_i)}+v\Big).\tag{4.1}$$

As in [3], using the Euler-Lagrange's coefficients, we easily get the following proposition.

Proposition 4.1. *Let $m=(\alpha,\lambda,x,v)\in M_{\varepsilon,2}$. m is a critical point of $\Psi_{\varepsilon,2}$ if and only if $u=\sum\alpha_i\widetilde{\delta}_i+v$ is a critical point of I_ε, i.e. if and only if there exists $\big(A,B,C\big)\in\mathbb{R}^q\times\mathbb{R}^q\times\big(\mathbb{R}^n\big)^q$ such that the following holds :*

$$(E_{\alpha_i})\quad\frac{\partial\Psi_{\varepsilon,2}}{\partial\alpha_i}=0,\ \ \forall\, i=1,...,q\tag{4.2}$$

$$(E_{\lambda_i})\quad\frac{\partial\Psi_{\varepsilon,2}}{\partial\lambda_i}=B_i\langle\frac{\partial^2\widetilde{\delta}_i}{\partial\lambda_i^2},v\rangle+\sum_{j=1}^nC_{ij}\langle\frac{\partial^2\widetilde{\delta}_i}{\partial x_i^j\partial\lambda_i},v\rangle,\ \ \forall\, i=1,...,q\tag{4.3}$$

$$(E_{x_i})\quad\frac{\partial\Psi_{\varepsilon,2}}{\partial x_i}=B_i\langle\frac{\partial^2\widetilde{\delta}_i}{\partial\lambda_i\partial x_i},v\rangle+\sum_{j=1}^nC_{ij}\langle\frac{\partial^2\widetilde{\delta}_i}{\partial x_i^j\partial x_i},v\rangle,\ \ \forall\, i=1,...,q\tag{4.4}$$

$$(E_v)\quad\frac{\partial\Psi_{\varepsilon,2}}{\partial v}=\sum_{i=1}^q\Big(A_i\widetilde{\delta}_i+B_i\frac{\partial\widetilde{\delta}_i}{\partial\lambda_i}+\sum_{j=1}^nC_{ij}\frac{\partial\widetilde{\delta}_i}{\partial x_i^j}\Big).\tag{4.5}$$

The results of Theorem 1.1 will be obtained through a careful analysis of (4.2)-(4.5) on $M_{\varepsilon,2}$. As usual in this type of problems, we first deal with the v-part of u, in order to show that it is negligible with respect to the concentration phenomenon. The study of (E_v) yields :

Proposition 4.2. *There exists a smooth map which to any* $(\varepsilon,\alpha,\lambda,x)$ *such that* $(\alpha,\lambda,x,0)$ *in* $M_{\varepsilon,1}$ *associates* $\overline{v}\in E_{(x,\lambda)}$ *such that* $\|v\|<\nu_0$ *and* (E_v) *is satisfied for some* $\big(A,B,C\big)\in\mathbb{R}^q\times\mathbb{R}^q\times\big(\mathbb{R}^n\big)^q$. *Such a* $\overline{v}$ *is unique, minimizes* $\Psi_{\varepsilon,2}(\alpha,\lambda,x,v)$ *with respect to* v *in* $\{v\in E_{(x,\lambda)}/\|v\|<\nu_0\}$, *and we have the following estimate*

$$\|\overline{v}\|=O\Big(\varepsilon+\sum_i\Big(\frac{|\nabla K(x_i)|}{\lambda_i}+\frac{1}{\lambda_i^2}\Big)+\sum_{j\neq i}\varepsilon_{ij}^{\min(1,\frac{n+4}{2(n-4)})}(\log\varepsilon_{ij}^{-1})^{\min(\frac{n-4}{n},\frac{n+4}{2n})}\Big). \tag{4.6}$$

Proof. Expanding I_ε with respect to $v\in E_{(x,\lambda)}$, we obtain

$$I_\varepsilon\big(\sum_i\alpha_i\widetilde{\delta}_i+v\big)=c(\alpha,x,\lambda)+\frac{1}{2}Q(v,v)-f(v)+R(v), \tag{4.7}$$

where $Q(.,.)$ is a quadratic form positive definite, $f(.)$ is a linear form and $R(v)$ satisfies $R(v)=o(\|v\|^2)$, $R'(v)=o(\|v\|)$ and $R''(v)=o(1)$.
Since $Q(v,v)$ is positive definite, we derive that the following problem

$$\min\{I_\varepsilon\big(\sum_i\alpha_i\widetilde{\delta}_i+v\big),\ v\in E_{(x,\lambda)}\text{ and }\|v\|<\nu_0\} \tag{4.8}$$

is achieved by a unique function $\overline{v}$ which satisfies $\|\overline{v}\|\leq c\|f\|$. Now, following [5] we get the estimate (4.6). Since $\overline{v}$ is orthogonal to the functions $\{\widetilde{\delta}_i,\partial\widetilde{\delta}_i/\partial\lambda_i,\partial\widetilde{\delta}_i/\partial x_i^j, i\leq q,\ j\leq n\}$, there exist A, B and C such that

$$\frac{\partial\Psi_{\varepsilon,2}}{\partial v}(\alpha,\lambda,x,\overline{v})=\nabla I_\varepsilon(\sum_i\alpha_i\widetilde{\delta}_i+\overline{v})=\sum_{i=1}^q\Big(A_i\widetilde{\delta}_i+B_i\frac{\partial\widetilde{\delta}_i}{\partial\lambda_i}+\sum_{j=1}^nC_{ij}\frac{\partial\widetilde{\delta}_i}{\partial x_i^j}\Big). \tag{4.9}$$

The proposition follows.
Proof of Theorem 1.1. Once $\overline{v}$ is defined by Proposition 4.2, we estimate the corresponding numbers A,B,C by taking the scalar product in $H_2^2(S^n)$ of (E_v) with $\widetilde{\delta}_i$, $\partial\widetilde{\delta}_i/\partial\lambda_i$, $\partial\widetilde{\delta}_i/\partial x_i$ for $1\leq i\leq q$ respectively. Thus we get a quasi-diagonal system whose coefficients are given by

$$\int_{\mathbb{R}^4_+}|\nabla P\delta_i|^2=S_4+O(\frac{1}{\lambda_i^2});\quad\int_{\mathbb{R}^4_+}\nabla P\delta_i\nabla P\delta_j=O(\frac{1}{\lambda_i\lambda_j});\quad\int_{\mathbb{R}^4_+}\nabla P\delta_i\nabla\frac{\partial P\delta_i}{\partial\lambda_i}=O(\frac{1}{\lambda_i^3}),$$
$$\int_{\mathbb{R}^4_+}\nabla P\delta_i\nabla\frac{\partial P\delta_j}{\partial\lambda_j}=O(\frac{1}{\lambda_i\lambda_j^2});\ \int_{\mathbb{R}^4_+}\Big|\nabla\frac{\partial P\delta_i}{\partial\lambda_i}\Big|^2=\frac{\Gamma_1}{\lambda_i^2}+O(\frac{1}{\lambda_i^3});\ \int_{\mathbb{R}^4_+}\nabla\frac{\partial P\delta_i}{\partial\lambda_i}\nabla\frac{\partial P\delta_j}{\partial\lambda_j}=O(\frac{1}{\lambda_i^2\lambda_j^2}),$$
$$\int_{\mathbb{R}^4_+}\nabla\frac{\partial P\delta_i}{\partial\lambda_i}\nabla\frac{\partial P\delta_i}{\partial x_i}=O(\frac{1}{\lambda_i^3});\ \int_{\mathbb{R}^4_+}\Big|\nabla\frac{\partial P\delta_i}{\partial x_i}\Big|^2=\Gamma_2\lambda_i^2+O(\frac{1}{\lambda_i});\ \int_{\mathbb{R}^4_+}\nabla P\delta_i\nabla\frac{\partial P\delta_i}{\partial x_i}=O(\frac{1}{\lambda_i^2}),$$
$$\int_{\mathbb{R}^4_+}\nabla P\delta_i\nabla\frac{\partial P\delta_j}{\partial x_j}=O(\frac{1}{\lambda_i});\quad\int_{\mathbb{R}^4_+}\nabla\frac{\partial P\delta_i}{\partial x_i}\nabla\frac{\partial P\delta_j}{\partial x_j}=O(\frac{1}{\lambda_i}),$$

with Γ_1, Γ_2 are positive constants and where we have used the fact that $|x_1-x_2|\geq c>0$.
The other hand side is given by

$$\frac{\partial\Psi_{\varepsilon,2}}{\partial\alpha_i}=\langle\frac{\partial\Psi_{\varepsilon,2}}{\partial v},\widetilde{\delta}_i\rangle;\ \frac{1}{\alpha_i}\frac{\partial\Psi_{\varepsilon,2}}{\partial\lambda_i}=\langle\frac{\partial\Psi_{\varepsilon,2}}{\partial v},\frac{\partial\widetilde{\delta}_i}{\partial\lambda_i}\rangle;\ \frac{1}{\alpha_i}\frac{\partial\Psi_{\varepsilon,2}}{\partial x_i}=\langle\frac{\partial\Psi_{\varepsilon,2}}{\partial v},\frac{\partial\widetilde{\delta}_i}{\partial x_i}\rangle. \tag{4.10}$$

Using Proposition 3.2, some computations yield

$$\frac{\partial\Psi_{\varepsilon,2}}{\partial\alpha_i}=-\frac{8}{n-4}S_n\beta_i+V_{\alpha_i}(\varepsilon,\alpha,\lambda,x), \tag{4.11}$$

with $\beta=(\beta_1,...,\beta_q)$ where $\beta_i=\alpha_i-1/K(y_i)^{\frac{n-4}{8}}$ and V_{α_i} is a smooth function which satisfies

$$V_{\alpha_i}=O\Big(\beta_i^2+\varepsilon\log\lambda_i+\frac{1}{\lambda_i^2}+|x_i-y_i|^2\Big). \tag{4.12}$$

Observe that, $|x_i - x_j| > c$ for $i \neq j$ implies

$$\varepsilon_{ij} = \frac{2}{\lambda_1\lambda_2(1-\cos d(x_i,x_j))}(1+o(1)) = \frac{2G(x_i,x_j)}{\lambda_i\lambda_j}(1+o(1)),$$

where $G(x_i,x_j) = (1-\cos d(x_i,x_j))^{-1}$, it is the Green's function of $\mathcal{P}$. Thus,

$$\lambda_i \frac{\varepsilon_{ij}}{\partial\lambda_i} = -\varepsilon_{ij}(1+o(1)) = \frac{-2G(x_i,x_j)}{\lambda_i\lambda_j}(1+o(1)),$$

Now, using Proposition 3.3, we get

$$\begin{aligned}\frac{\partial\Psi_{\varepsilon,2}}{\partial\lambda_i} =& \frac{1}{K(y_i)^{n-4/4}}\left(\frac{\varepsilon S_n}{n\lambda_i} + \frac{4(n-4)c_2}{n}\frac{\Delta K(x_i)}{K(x_i)}\frac{1}{\lambda_i^3}\right)\\ &+\sum_{j\neq i}\frac{2c_1}{(K(y_i)K(y_j))^{n-4/8}}\frac{1}{\lambda_i}\frac{G(x_i,x_j)}{\lambda_i\lambda_j} + V_{\lambda_i}(\varepsilon,\alpha,\lambda,x),\end{aligned} \tag{4.13}$$

where c_2 and c_3 are defined in Proposition 3.3 and V_{λ_i} is a smooth function satisfying

$$V_{\lambda_i} = O\left[\frac{1}{\lambda_i}\left(\frac{1}{\lambda_i^3} + \frac{|x_i-y_i|^2}{\lambda_i^2} + \varepsilon^2\log\lambda_i + \frac{\varepsilon\log\lambda_i}{\lambda_i^2}\right) + \Big(|\beta| + |x_i-y_i|^2\Big)\big(\frac{\varepsilon}{\lambda_i} + \frac{1}{\lambda_i^3}\big)\right]. \tag{4.14}$$

Lastly, using Proposition 3.4, we have

$$\frac{\partial\Psi_{\varepsilon,2}}{\partial x_i} = \frac{-c_3}{K(y_i)^{(n-4)/8}}\nabla K(x_i) + V_{x_i}(\varepsilon,\alpha,\lambda,x), \tag{4.15}$$

where V_{x_i} is a smooth function such that

$$V_{x_i} = O\Big(\frac{1}{\lambda_i} + (|\beta| + \varepsilon\log\lambda_i + |x_i-y_i|^2)|x_i-y_i|\Big). \tag{4.16}$$

Notice that these estimates imply

$$\frac{\partial\Psi_{\varepsilon,2}}{\partial\alpha_i} = O\Big(|\beta| + \varepsilon\log\lambda_i + \frac{1}{\lambda_i^2} + |x_i-y_i|^2\Big),\ \frac{\partial\Psi_{\varepsilon,2}}{\partial\lambda_i} = O\Big(\frac{1}{\lambda_i^3} + \frac{\varepsilon}{\lambda_i}\Big),\ \frac{\partial\Psi_{\varepsilon,2}}{\partial x_i} = O\Big(|x_i-y_i| + \frac{1}{\lambda_i}\Big).$$

The solution of the system in A, B and C shows that

$$A_i = O\big(|\beta| + \varepsilon\log\lambda_i + \frac{1}{\lambda_i^2} + |x_i-y_i|^2\big),\quad B_i = O\big(\frac{1}{\lambda_i} + \varepsilon\lambda_i\big),\quad C_i = O\big(\frac{|x_i-y_i|}{\lambda_i^2} + \frac{1}{\lambda_i^3}\big).$$

This allows us to evaluate the right hand side in the equations (E_{λ_i}) and (E_{x_i}), namely

$$B_i\langle\frac{\partial^2\widetilde{\delta}_i}{\partial\lambda_i^2},\overline{v}\rangle + \sum_{j=1}^n C_{ij}\langle\frac{\partial^2\widetilde{\delta}_i}{\partial x_i^j\partial\lambda_i},\overline{v}\rangle = O\Big(\Big(\frac{1}{\lambda_i^3} + \frac{\varepsilon}{\lambda_i} + \frac{|y_i-x_i|}{\lambda_i^2}\Big)\|\overline{v}\|\Big), \tag{4.17}$$

$$B_i\langle\frac{\partial^2\widetilde{\delta}_i}{\partial\lambda_i\partial x_i},\overline{v}\rangle + \sum_{j=1}^n C_{ij}\langle\frac{\partial^2\widetilde{\delta}_i}{\partial x_i^j\partial x_i},\overline{v}\rangle = O\Big(\Big(\frac{1}{\lambda_i} + \varepsilon\lambda_i + |x_i-y_i|\Big)\|\overline{v}\|\Big), \tag{4.18}$$

where we have used the following estimates

$$\Big\|\frac{\partial^2\widetilde{\delta}_i}{\partial\lambda_i^2}\Big\| = O(\frac{1}{\lambda_i^2});\quad \Big\|\frac{\partial^2\widetilde{\delta}_i}{\partial x_i\partial\lambda_i}\Big\| = O(1);\quad \Big\|\frac{\partial^2\widetilde{\delta}_i}{\partial x_i^2}\Big\| = O(\lambda_i^2).$$

Now,we set

$$\frac{1}{\lambda_i} = \varepsilon^{\frac{1}{2}}\Lambda_i(1+\zeta_i);\quad x_i = y_i + \xi_i,$$

where $\zeta_i \in \mathbb{R}$, $\xi_i \in \mathbb{R}^n$ are assumed to be small and for $i, j \in 1, ..., q$, $\Lambda_i = \Lambda_i(y_i)$ verifies

$$\frac{S_n}{n} + \frac{4(n-4)c_2}{n}\Lambda_i^2\frac{\Delta K(y_i)}{K(y_i)} + 2c_1\sum_{j\neq i}\Lambda_i\Lambda_j(\frac{K(y_i)}{K(y_j)})^{(n-4)/8}G(y_i, y_j) = 0.$$

With these changes of variables and using (4.11), (E_{α_i}) is equivalent to

$$\beta_i = V_{\alpha_i}(\varepsilon, \beta, \zeta, \xi) = O(|\beta|^2 + \varepsilon|\log\varepsilon| + |\xi|^2). \tag{4.19}$$

Now, using (4.13), we show by an easy computation

$$\begin{aligned}
&\frac{\varepsilon S_n}{n\lambda_i} + \frac{4(n-4)c_2}{n}\frac{\Delta K(y_i+\xi_i)}{K(y_i+\xi_i)}\frac{1}{\lambda_i^3} + \sum_{j\neq i}\frac{2}{K(y_j)^{\frac{n-4}{8}}}\frac{c_1}{\lambda_i}\frac{G(y_i+\xi_i, y_j+\xi_j)}{\lambda_i\lambda_j}\\
&= \frac{\varepsilon^{3/2}S_n}{n}\Lambda_i(1+\zeta_i) + \frac{4(n-4)c_2}{n}\varepsilon^{3/2}\Lambda_i^3(1+3\zeta_i)\Big(\frac{\Delta K(y_i)}{K(y_i)} + \frac{\nabla\Delta K(y_i)}{K(y_i)}\xi_i\Big)\\
&\sum_{j\neq i}2c_1\varepsilon^{3/2}\Lambda_i^2\Lambda_j\frac{(1+2\zeta_i)(1+\zeta_j)}{K(y_j)^{(n-4)/8}}G(y_i, y_j) + \sum_{j\neq i}\frac{2\varepsilon^{3/2}c_1\Lambda_i^2\Lambda_j}{K(y_j)^{(n-4)/8}}\frac{\partial G(y_i, y_j)}{\partial x_i}\xi_i\\
&\sum_{j\neq i}\frac{2c_1\varepsilon^{3/2}\Lambda_i^2\Lambda_j}{K(y_j)^{(n-4)/8}}\frac{\partial G(y_i, y_j)}{\partial x_j}\xi_j + O\Big(\varepsilon^{3/2}(\zeta_i^2 + |\xi_i|^2)\Big)\\
&= \varepsilon^{3/2}\Big[\frac{8(n-4)c_2\Lambda_i^3}{n}\frac{\Delta K(y_i)}{K(y_i)} + \sum_{j\neq i}\frac{2c_1\Lambda_i^2\Lambda_j}{K(y_j)^{(n-4)/8}}G(y_i, y_j)\Big]\zeta_i + \sum_{j\neq i}\frac{2c_1\varepsilon^{3/2}\Lambda_i^2\Lambda_j}{K(y_j)^{(n-4)/8}}G(y_i, y_j)\zeta_j\\
&+ \varepsilon^{3/2}\Big[\frac{4(n-4)c_2\Lambda_i^3}{n}\frac{\nabla(\Delta K)(y_i)}{K(y_i)} + \sum_{j\neq i}\frac{2c_1\Lambda_i^2\Lambda_j}{K(y_j)^{(n-4)/8}}\frac{\partial G(y_i, y_j)}{\partial x_i}\Big]\xi_i\\
&+ \sum_{j\neq i}\frac{2c_1\varepsilon^{3/2}\Lambda_i^2\Lambda_j}{K(y_j)^{(n-4)/8}}\frac{\partial G(y_i, y_j)}{\partial x_j}\xi_j + O\Big(\varepsilon^{3/2}(|\zeta|^2 + |\xi|^2)\Big).
\end{aligned}$$

This implies that (E_{λ_i}) is equivalent, on account of (4.14) and (4.17), to

$$\begin{aligned}
&\Big[\frac{8(n-4)c_2\Lambda_i^3}{n}\frac{\Delta K(y_i)}{K(y_i)} + \sum_{j\neq i}\frac{2c_1\Lambda_i^2\Lambda_j}{K(y_j)^{\frac{n-4}{8}}}G(y_i, y_j)\Big]\zeta_i + \sum_{j\neq i}\frac{2c_1\Lambda_i^2\Lambda_j}{K(y_j)^{\frac{n-4}{8}}}G(y_i, y_j)\zeta_j\\
&+ \Big[\frac{4(n-4)c_2\Lambda_i^3}{n}\frac{\nabla(\Delta K)(y_i)}{K(y_i)} + \sum_{j\neq i}\frac{2c_1\Lambda_i^2\Lambda_j}{K(y_j)^{\frac{n-4}{8}}}\frac{\partial G(y_i, y_j)}{\partial x_i}\Big]\xi_i\\
&+ \sum_{j\neq i}\frac{2c_1\Lambda_i^2\Lambda_j}{K(y_j)^{\frac{n-4}{8}}}\frac{\partial G(y_i, y_j)}{\partial x_j}\xi_j = V_{\lambda_i}(\varepsilon, \beta, \zeta, \xi) = O(|\beta|^2 + |\zeta|^2 + |\xi|^2 + \varepsilon^{1/2}).
\end{aligned} \tag{4.20}$$

Lastly, using (4.15), (4.16) and (4.18), we see that (E_{x_i}) is equivalent to

$$D^2K(y_i)\xi_i = V_{x_i}(\varepsilon, \beta, \zeta, \xi) = O(\varepsilon^{1/2} + |\beta|^2 + |\zeta|^2 + |\xi|^2). \tag{4.21}$$

We remark that V_{α_i}, V_{λ_i} and V_{x_i} are smooth functions. This system may be written as

$$\begin{cases}\beta = V(\varepsilon, \beta, \zeta, \xi),\\ L(\zeta, \xi) = W(\varepsilon, \beta, \zeta, \xi),\end{cases} \tag{4.22}$$

where L is a fixed linear operator on $\mathbb{R}^{q(n+1)}$ defined by (4.20) and (4.21) and V, W are smooth functions satisfying

$$\begin{cases}V(\varepsilon, \beta, \zeta, \xi) = O(\varepsilon^{1/2} + |\beta|^2 + |\xi|^2),\\ W(\varepsilon, \beta, \zeta, \xi) = O\big(\varepsilon^{1/2} + |\beta|^2 + |\zeta|^2 + |\xi|^2\big).\end{cases}$$

Moreover, a simple computation shows that the determinant of L is not equal to zero. Hence L is invertible, and Brouwer's fixed point theorem shows that (4.22) has a solution $(\beta^\varepsilon, \zeta^\varepsilon, \xi^\varepsilon)$ for ε small enough, such that

$$|\beta^\varepsilon| = O(\varepsilon^{1/2}); \quad |\zeta^\varepsilon| = O\big(\varepsilon^{1/2}\big); \quad |\xi^\varepsilon| = O\big(\varepsilon^{1/2}\big).$$

Hence, we have constructed $m^\varepsilon = (\alpha^\varepsilon, \lambda^\varepsilon, x^\varepsilon)$ such that $u_\varepsilon := \sum_i \alpha_i^\varepsilon \widetilde{\delta}_{(x_i^\varepsilon, \lambda_i^\varepsilon)} + \overline{v}_\varepsilon$, satisfies (4.2)-(4.6). Therefore, by Proposition 4.1, u_ε is a critical point of I_ε, i.e., u_ε satisfies (u_ε). Hence, the proof of Theorem 1.1 is thereby completed.

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K. Ould Bouh,
Institut Supérieur de Comptabilité et d'Administration d'Entreprises, ISCAE, Nouakchott, Mauritania.
E-mail address: `kamal.bouh@iscae.mr & kamal_bouh@yahoo.fr`