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A Generalized Fixed Point Theorem in Fuzzy $b-$Metric Spaces and Applications

Mohamed Dahhouch, Noreddine Makran and Brahim Marzouki

ABSTRACT: In this paper, we are interested to prove a general fixed point theorem for a mapping in fuzzy $b-$metric spaces. The results in this paper generalize the Banach fixed point theorem in fuzzy $b-$metric spaces. To show the significance of our result an application is presented to establish the existence of a solution for an integral equation.

Key Words: s-nondecreasing, fuzzy $b-$metric space, t-norm, fixed point.

Contents

1. Introduction

The concept of fuzzy sets was introduced initially by Zadeh [22] in 1965. A fuzzy set M in X is a function with domain X and values in $[0,1]$. The notion of fuzzy maps was introduced by Heilpern [10] where some fixed point theorems for fuzzy maps are also established.

In 1975, Kramosil and Michalek [11] initiated the idea of a fuzzy distance between two elements of a nonempty set by using the concepts of a fuzzy set and a t-norm.

A binary operation $T : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm, if it satisfies the following conditions :
i) T is continuous, associative and commutative.
ii) $T(a,1) = a$ for all $a \in [0,1]$.
iii) for all $a,b,c,d \in [0,1]$ if $a \leq c$ and $b \leq d$ then $T(a,b) \leq T(c,d)$.

Typical examples of a continuous t-norm are $T_p(a,b) = a.b$, $T_{min}(a,b) = \min\{a,b\}$ and $T_L(a,b) = \max\{a+b–1,0\}$. George and Veeramani [7] generalized the concept of fuzzy metric spaces introduced by Kramosil and Michalek [11]. Given a non empty set X, and T is a continuous t-norm, the 3-tuple (X,M,T) is said to be a fuzzy metric space [7],[8] if M is a fuzzy set on $X \times X \times (0,\infty)$ satisfying the following conditions for all $x,y,z \in X$ $t,u > 0$:

$$\begin{cases} 1) & M(x,y,t) > 0, \\ 2) & M(x,y,t) = M(y,x,t) = 1 \iff x = y, \\ 3) & M(x,z,t+u) \geq T(M(x,y,t), M(y,z,u)), \\ 4) & M(x,y,.) \quad (0,\infty) \to [0,1] \text{is left continuous function.} \end{cases}$$

The study of fixed point theory in metric spaces has several applications in mathematics, especially in solving differential and integral equations. In 1989, Bakhtin [3] introduced a new class of generalized metric space called b-metric space which has been studied by many authors. For example, see ([1]- [2], [4]- [5], [12]). The relation between b-metric and fuzzy metric spaces is considered in [9]. On the other hand, in [20] the notion of a fuzzy b-metric space was introduced, where the triangle inequality is replaced by $M(x,z,s(t+u)) \geq T(M(x,y,t), M(y,z,u))$ with $s \geq 1$.

In this paper, we prove the existence and uniqueness of a fixed point in fuzzy b-metric spaces. To show the significance of our result an application is presented to establish the existence and uniqueness of solution for an integral equation.

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Definition 1.1. *[20] A 3-tuple (X, M, T) is called a fuzzy $b-$metric space if X is an arbitrary nonempty set, T is a continuous t-norm, and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions for all $x, y, z \in X$, $t, u > 0$ and a given real number $s \geq 1$:*

(b_1) $M(x, y, t) > 0$,

(b_2) $M(x, y, t) = 1 \Longleftrightarrow x = y$,

(b_3) $M(x, y, t) = M(y, x, t)$,

(b_4) $M(x, z, s(t+u)) \geq T(M(x, y, t), M(y, z, u))$,

(b_5) $M(x, y, .) : (0, \infty) \to [0, 1]$ *is continuous.*

Remark 1.2. *In this paper we will further use a fuzzy b-metric space in the sense of Definition 1.1 with additional condition* $\lim\limits_{t\to\infty} M(x, y, t) = 1$.

Note that every fuzzy metric space is a fuzzy $b-$metric space with $s = 1$. But the converse need not be true as is shown in the following example.

Example 1.3. *[6] Let $X = \mathbb{R}$, $M(x, y, t) = e^{-\frac{|x-y|^p}{t}}$, where $p > 1$ is a real number, and $T(a, b) = a.b$ for all $a, b \in [0, 1]$. Then (X, M, T) is a fuzzy b-metric space with $s = 2^{p-1}$.*

Definition 1.4. *[6] A function $f : \mathbb{R} \to \mathbb{R}$ is called $s-$nondecreasing, if $x > sy$ implies $fx \geq fy$ for all $x, y \in \mathbb{R}$.*

Lemma 1.5. *[6] Let (X, M, T) be a fuzzy b-metric space with constant s. Then $M(x, y, t)$ is $s-$non-decreasing with respect to t, for all $x, y \in X$. Also,*

$$M(x, y, s^n t) \geq M(x, y, t), \quad \forall n \in \mathbb{N}.$$

We recall the notions of convergence and completeness in a fuzzy b-metric space.

Definition 1.6 ([20], [21])**.** .

(i) *A sequence (x_n) converges to x if $M(x_n, x, t) \to 1$ as $n \to \infty$ for each $t > 0$. In this case, we write* $\lim\limits_{n\to\infty} x_n = x$.

(ii) *A sequence (x_n) is called a Cauchy sequence if for all $\varepsilon \in (0, 1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.*

(iii) *The fuzzy b-metric space (X, M, T) is said to be complete if every Cauchy sequence is convergent.*

Lemma 1.7 ([20], [21])**.** *In a fuzzy $b-$metric space (X, M, T) we have*

(i) *If a sequence (x_n) in X converges to x, then x is unique.*

(ii) *If a sequence (x_n) in X converges to x, then it is a Cauchy sequence.*

Lemma 1.8. *[19] If for some $\lambda \in (0, 1)$ and $x, y \in X$,*

$$M(x, y, t) \geq M\left(x, y, \frac{t}{\lambda}\right), \quad \forall t > 0,$$

then $x = y$.

Lemma 1.9. *[19] Let (x_n) be a sequence in a fuzzy b-metric space (X, M, T) with constant s. Suppose that there exists $\lambda \in (0, \frac{1}{s})$ such that*

$$M(x_n, x_{n+1}, t) \geq M\left(x_{n-1}, x_n, \frac{t}{\lambda}\right), \qquad \forall n \in \mathbb{N}, \ \forall t > 0,$$

and there exist $x_0, x_1 \in X$ and $\upsilon \in (0, 1)$ such that
$\lim\limits_{n\to\infty} T_{i=n}^{\infty} M(x_0, x_1, \frac{t}{\upsilon^i}) = 1, \quad t > 0$. *Then (x_n) is a Cauchy sequence.*

2. Main results

In this section, we demonstrate the Lemma 1.9 [19], with $\lambda \in]0, \frac{1}{2s}[$ and neglect the condition "$\upsilon \in (0,1)$ such that $\lim_{n\to\infty} T_{i=n}^{\infty} M(x_0, x_1, \frac{t}{\upsilon^i}) = 1, \quad t > 0$." As an application we demonstrate a result of existence and uniqueness for a fixed point.

Lemma 2.1. *Let (x_n) be a sequence in a fuzzy b-metric space (X, M, T) with constant s. Suppose that there exists $\lambda \in]0, \frac{1}{2s}[$ such that*

$$M(x_n, x_{n+1}, t) \geq M\left(x_{n-1}, x_n, \frac{t}{\lambda}\right), \qquad \forall n \in \mathbb{N}^*,\ \forall t > 0.$$

Then (x_n) is a Cauchy sequence.

Proof.
We have

$$M(x_n, x_{n+1}, t) \geq M\left(x_{n-1}, x_n, \frac{t}{\lambda}\right), \qquad n \in \mathbb{N}^*,\ t > 0,$$

consequently for every $n \in \mathbb{N}^*$ we get

$$M(x_n, x_{n+1}, t) \geq M\left(x_0, x_1, \frac{t}{\lambda^n}\right), \qquad t > 0. \tag{2.1}$$

Therefore, for any $n, m \in \mathbb{N}^*$, we have

$$\begin{aligned} M(x_n, x_{n+m}, t) &\geq T\left(M\left(x_n, x_{n+1}, \frac{t}{2s}\right), M\left(x_{n+1}, x_{n+m}, \frac{t}{2s}\right)\right) \\ &\geq T\left(M\left(x_n, x_{n+1}, \frac{t}{2s}\right), T\left(M\left(x_{n+1}, x_{n+2}, \frac{t}{(2s)^2}\right), M\left(x_{n+2}, x_{n+m}, \frac{t}{(2s)^2}\right)\right)\right) \\ &. \\ &. \\ &. \\ &\geq T\left(\begin{array}{c} M\left(x_n, x_{n+1}, \frac{t}{2s}\right), T(M(x_{n+1}, x_{n+2}, \frac{t}{(2s)^2}), ..., \\ T(M(x_{n+m-2}, x_{n+m-1}, \frac{t}{(2s)^{m-1}}), M(x_{n+m-1}, x_{n+m}, \frac{t}{(2s)^{m-1}}))...) \end{array}\right). \end{aligned}$$

So, by (2.1) we get

$$M(x_n, x_{n+m}, t) \geq T\left(\begin{array}{c} M\left(x_0, x_1, \frac{t}{2s\lambda^n}\right), T(M(x_0, x_1, \frac{t}{(2s)^2\lambda^{n+1}}), ..., \\ T(M(x_0, x_1, \frac{t}{(2\lambda s)^{m-1}\lambda^{n-1}}), M(x_0, x_1, \frac{t}{(2\lambda s)^{m-1}\lambda^n}))...) \end{array}\right).$$

Letting $n \to \infty$ we obtain

$$\lim_{n\to\infty} M(x_n, x_{n+m}, t) \geq T(1, T(1, ..., T(1,1))...) = 1, \quad m \in \mathbb{N}^*, t > 0.$$

From where $\lim_{n\to\infty} M(x_n, x_{n+m}, t) = 1$, for $m \in \mathbb{N}^*$. Then (x_n) is a Cauchy sequence.

Theorem 2.2. *Let (X, M, T) be a complete fuzzy b-metric space with constant s, and let $f : X \longrightarrow X$. Suppose that there exists $\lambda \in]0, \frac{1}{2s}[$ such that*

$$M(fx, fy, t) \geq M\left(x, y, \frac{t}{\lambda}\right), \tag{2.2}$$

for all $x, y \in X$, $t > 0$. Then f has a unique fixed point in X.

Proof.

Existence.
Let $x_0 \in X$, define the sequence (x_n) of elements from X such that $x_{n+1} = fx_n$ for every $n \in \mathbb{N}$. According to (2.2), with $x = x_{n-1}$ and $y = x_n$ we have

$$M\left(fx_{n-1}, fx_n, t\right) \geq M\left(x_{n-1}, x_n, \frac{t}{\lambda}\right).$$

This implies

$$M\left(x_n, x_{n+1}, t\right) \geq M\left(x_{n-1}, x_n, \frac{t}{\lambda}\right), \qquad n \in \mathbb{N}^*,\ t > 0. \tag{2.3}$$

By Lemma 2.1 we deduce that (x_n) is a Cauchy sequence. Since (X, M, T) is complete, hence there exists $x \in X$ such that

$$\lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} M(x, x_n, t) = 1, \quad t > 0.$$

Next we show that $x = fx$, indeed, by (2.2) we have

$$\begin{aligned} M\left(fx, x, t\right) &\geq T\left(M\left(fx, x_n, \frac{t}{2s}\right), M\left(x_n, x, \frac{t}{2s}\right)\right) \\ &\geq T\left(M\left(fx, fx_{n-1}, \frac{t}{2s}\right), M\left(x_n, x, \frac{t}{2s}\right)\right) \\ &\geq T\left(M\left(x, x_{n-1}, \frac{t}{2\lambda s}\right), M\left(x_n, x, \frac{t}{2s}\right)\right) \end{aligned}$$

letting $n \to \infty$ we obtain

$$M\left(fx, x, t\right) \quad \geq \quad T\left(1, 1\right) = 1$$

thus $fx = x$.
Unicity.
Suppose that there exists $y \in X$ another fixed point of f, then by (2.2) we have

$$M\left(fx, fy, t\right) \geq M\left(x, y, \frac{t}{\lambda}\right).$$

Then

$$M\left(x, y, t\right) \geq M\left(x, y, \frac{t}{\lambda}\right).$$

So, by Lemma 1.8 we have $x = y$.
As a consequence of Theorem 2.2 we obtain Theorem 2.4 [19].

3. Application

Let $X = C([a, b], \mathbb{R})$ be the set of real continuous functions defined on $[a, b]$, and $T(c, d) = c.d$ for all $c, d \in [0, 1]$ and let (X, M, T) a complete fuzzy $b-$metric space with $s = 2$ and fuzzy $b-$metric

$$M(x, y, t) = e^{-\frac{\sup\limits_{u\in[a,b]} |x(u)-y(u)|^2}{t}}, \quad x, y \in X,\ t > 0.$$

Consider the following integral equation

$$x(u) = g(u) + \int_a^b G(u, v) f(v, x(v)) dv, \quad u \in [a, b], \tag{3.1}$$

where $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, $g : \mathbb{R} \longrightarrow \mathbb{R}$ and $G : [a, b] \times [a, b] \longrightarrow \mathbb{R}^+$ are two functions such that $G(u, .) \in L^1([a, b])$ for all $u \in [a, b]$.

Consider the operator F: $X \longrightarrow X$ defined by

$$Fx(u) = g(u) + \int_a^b G(u, v)f(v, x(v))dv, \quad u \in [a, b]. \tag{3.2}$$

Theorem 3.1. *Suppose that the following conditions are satisfied:*
(H_1) *There exists $\theta \in (0, +\infty)$ such that*

$$| \, f(v, x(v)) - f(v, y(v)) \, | \leq \; \theta \; | \, x(v) - y(v) \, | \quad \forall x, y \in X, \quad \forall v \in [a, b].$$

(H_2) *There exists $\lambda \in]0, \frac{1}{4}[$, such that*

$$\sup_{u\in[a,b]} \int_a^b G(u, v)dv \leq \frac{\sqrt{\lambda}}{\theta}.$$

Then the integral equation (3.1) has a unique solution in X.

Proof.

It is clear that any fixed point of (3.2) is a solution of (3.1). By conditions (H_1) and (H_2), we have

$$\begin{aligned}
\sup_{u\in[a,b]} |Fx(u) - Fy(u)|^2 &= \sup_{u\in[a,b]} \left| \int_a^b G(u, v)f(v, x(v))dv - \int_a^b G(u, v)f(v, y(v))dv \right|^2 \\
&= \sup_{u\in[a,b]} \left| \int_a^b G(u, v)[f(v, x(v)) - f(v, y(v))]dv \right|^2 \\
&\leq \sup_{u\in[a,b]} \left(\int_a^b G(u, v)\theta \, |x(v) - y(v)| \, dv \right)^2 \\
&\leq \theta^2 \sup_{u\in[a,b]} |x(u) - y(u)|^2 \times \left(\sup_{u\in[a,b]} \int_a^b G(u, v))dv \right)^2 \\
&\leq \lambda \sup_{u\in[a,b]} |x(u) - y(u)|^2 .
\end{aligned}$$

This implies

$$e^{-\frac{\sup_{u\in[a,b]} |Fx(u)-Fy(u)|^2}{t}} \geq e^{\frac{-\lambda \sup_{u\in[a,b]} |x(u)-y(u)|^2}{t}}, \quad x, y \in X, \ t > 0.$$

Therefore

$$M \left(Fx, Fy, t\right) \quad \geq \quad M \left(x, y, \frac{t}{\lambda}\right) \quad x, y \in X, \ t > 0.$$

Then all conditions of Theorem 2.2 are satisfied, so the operator F has a unique fixed point, that is the integral equation has a unique solution in X.

Example 3.2. *The following integral equation has a solution in $X = (C[1, e], \mathbb{R})$.*

$$x(u) = \frac{1}{1 + u^2} + \sqrt{\alpha} \int_1^e \frac{\ln(u.v)}{e} x(v)dv, \quad u \in [1, e], \quad 0 < \alpha < \frac{1}{4}. \tag{3.3}$$

Proof.

Let $F\colon\ X \longrightarrow X$ defined by

$$Fx(u) = \frac{1}{1+u^2} + \sqrt{\alpha}\int_1^e \frac{\ln(u.v)}{e} x(v)dv, \quad u \in [1,e], \quad 0 < \alpha < \frac{1}{4}.$$

By specifying $G(u,v) = \sqrt{\alpha}\dfrac{\ln(u.v)}{e}$, $\quad f(v,x) = x$ and $\quad g(u) = \frac{1}{1+u^2}$ in Theorem 3.1, we get :
(1) For all $x(.), y(.) \in X$, it is clear that the condition (H_1) in Theorem 3.1 is satisfied with $\theta = 1$.
(2)

$$\begin{aligned}
\sup_{u\in[1,e]} \int_1^e \sqrt{\alpha}\frac{\ln(u.v)}{e}dv &= \frac{1}{e}\sqrt{\alpha}\sup_{u\in[1,e]}\int_1^e (\ln(v)+\ln(u))dv \\
&= \frac{1}{e}\sqrt{\alpha}\sup_{u\in[1,e]}\left[v\ln(v) - v + v\ln(u)\right]_1^e \\
&= \frac{1}{e}\sqrt{\alpha}\sup_{u\in[1,e]}\left(\ln(u)(e-1)+1\right) \\
&= \sqrt{\alpha} \leq \frac{\sqrt{\lambda}}{\theta}, \quad \lambda \in \left[\alpha, \frac{1}{4}\right[, \quad \theta = 1.
\end{aligned}$$

Therefore, all conditions of Theorem 3.1 are satisfied, hence the mapping F has a fixed point in X, which is a solution to the integral equation (3.3).

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Mohamed Dahhouch, Noreddine Makran, Brahim Marzouki,
Department of Mathematical Sciences,
Mohammed Premier University, Oujda,
Morocco.
E-mail address: dahmohammed1996@gmail.com, makranmakran83@gmail.com, b.marzouki@ump.ac.ma