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Complete Transversal and Formal Normal Forms of Vector Fields *

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ABSTRACT: Inspired by the complete transversal technique we establish a classification of vector fields by normal forms. In the case of vector fields with non zero linear part in $(\mathbb{C}^2, 0)$ and nilpotent vector fields in $(\mathbb{C}^3, 0)$, we recover the classical normal forms for these vector fields, and we provide a different formal normal form from that presented by Takens in dimension 2. We also get the formal normal form for vector fields in $(\mathbb{C}, 0)$ with a fixed multiplicity.

Key Words: Complete transversal, normal form, vector field.

Contents

1. Introduction

The aim of this work is to recover the classical normal forms of the vector fields using a technique based on the *complete transversal*.
Zariski, O. was interested in the analytic classification of plane branches belonging to a given equisingularity class. He exposed his research in a course taught at the École Polytechnique (see [19]). Bruce *et al.* get the classification of singularities of mappings by the complete transversal (see [1]), later on Hefez, A. and Hernandes, M. establish the analytic classification of plane branches (see [3]). Due to the relevance of the complete transversal in the aforementioned classifications, we establish a theorem called Complete Transversal for vector fields, which will be used to determine the prenormal form of each jet of order k of the vector field in $(\mathbb{C}^n, 0)$, under the action of the Lie group, $\widehat{\mathrm{Diff}}^k_1(\mathbb{C}^n, 0)$.

With the technique implemented, in the case of the vector fields in $(\mathbb{C}, 0)$, we obtain the normal forms for vector fields of any multiplicity, and we get formally linearize the vector fields in $(\mathbb{C}, 0)$ with non zero linear part. Paul, E. in [9] and Loray, F. in [6], show the formal linearization of the vector fields in $(\mathbb{C}, 0)$, unlike from Loray, F. (see [6]), in the case of vector fields $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}, 0)$ of multiplicity $\nu \geq 2$, we get a formal equivalence to a polynomial vector fields of multiplicity ν.

We formally get the classic normal forms of Poincaré ([11]) and Dulac ([2]). In this way, we recover the following normal forms for vector fields $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ with non zero linear part, in the cases described below.

In this work we consider, the vector fields $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$, with non zero linear part such that

$$j^1(\hat{X}) = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \lambda_2 x_2 \frac{\partial}{\partial x_2}. \tag{1.1}$$

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If $\lambda_1/\lambda_2 = -q/p \in \mathbb{Q}^-$(the negative rationals), then the vector field $\hat{X}$ is formally orbitally equivalent to the vector field

$$x_1(1 + A(x_1^p x_2^q))\frac{\partial}{\partial x_1} - \frac{p}{q}x_2(1 + B(x_1^p x_2^q))\frac{\partial}{\partial x_2} \tag{1.2}$$

where $A(t), B(t) \in \mathbb{C}[[t]]$ and $A(0) = B(0) = 0$.

If $\lambda_1 = 0$ y $\lambda_2 \neq 0$, then the vector field $\hat{X}$ is formally orbitally equivalent to the vector field

$$x_1 A(x_1)\frac{\partial}{\partial x_1} + x_2(1 + B(x_1))\frac{\partial}{\partial x_2} \tag{1.3}$$

where $A, B \in \mathbb{C}[[x_1]]$ and $A(0) = B(0) = 0$.

In (1.2) and (1.3), we have the normal forms due to Dulac, H. (see [2]), in (1.3) the normalization of vector fields with saddle-node type singularities. Paul, E. shows a prenormalization of vector fields and for foliations with saddle-node singularities (see [10]). The normal forms described in (1.2) and (1.3) are found in [7].

We show that, the vector field $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ with $j^1(\hat{X}) = (x_1 + x_2)\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$ is formally equivalent to $j^1(\hat{X})$ by a change of coordinates, which is a formal diffeomorphism tangent to the identity. In the case of vector fields $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)(n = 2, 3)$ with nilpotent linear part, we recover the Takens normal form in dimension 2 (see [13], [14], [17], [18]), and we get another formal normal form in this dimension.

Stróżyna, E. and Żoła̧dek, H. in [16], show the non-analyticity of the Takens normal form of certain vector fields in $(\mathbb{C}^3, 0)$ with nilpotent Jordan cell as linear part. In addition, they show the generalization of the non-analyticity of the Takens normal form for dimension greater than or equal to 3.

In this work, we get a normal form for vector fields $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ with nilpotent linear part $x_2\frac{\partial}{\partial x_1}$, by a diffeomorphism tangent to the identity, which can be written as

$$x_2\frac{\partial}{\partial x_1} + \Big(a'(x_1) + x_2 b'(x_1)\Big)\frac{\partial}{\partial x_2}$$

where the multiplicities of a' and b' in $0 \in \mathbb{C}$, are greater than or equal to 2 and 1 respectively.

We detail the nilpotent case in dimension 3, that is, given the vector field $\hat{V} \in \hat{\mathfrak{X}}(\mathbb{C}^3, 0)$ such that $j^1(\hat{V}) = 2x_2\frac{\partial}{\partial x_1} + x_3\frac{\partial}{\partial x_2}$, by a diffeomorphism tangent to the identity, the Takens normal form for the vector field $\hat{V}$ is,

$$\Big(2x_2 + x_1\hat{F}_1(x_1, G_2)\Big)\frac{\partial}{\partial x_1} + \Big(x_3 + x_1\hat{F}_2(x_1, G_2)\Big)\frac{\partial}{\partial x_2} + \hat{F}_3(x_1, G_2)\frac{\partial}{\partial x_3}$$

where $G_2 = x_1x_3 - x_2^2$, $\hat{F}_j$ is a formal power series in x_1 and G_2, such that the multiplicity of $\hat{F}_j$ for $j = 1, 2$ is greater than or equal to 1, and the multiplicity of $\hat{F}_3$ is greater than or equal to 2.

The Takens normal form for any dimension n, can be found in [15] and [16].

2. Preliminaries

We will use the following notation.

$\hat{\mathcal{O}}_n$	is the local ring of formal series defined in a neighborhood of $0 \in \mathbb{C}^n$ and $\hat{\mathcal{M}}$ is the maximal ideal of $\hat{\mathcal{O}}_n$.
$\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$	is the $\hat{\mathcal{O}}_n$ module of formal vector fields with singularity in the origin.
$\hat{\mathfrak{X}}^k(\mathbb{C}^n, 0)$	is the space of the $k-jet$ of formal vector fields in $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$.
$\hat{\mathcal{R}}^k$	will denote $\widehat{\text{Diff}}^k(\mathbb{C}^n, 0)$, that is, the $k-jet$ of the group of formal diffeomorphisms in $(\mathbb{C}^n, 0)$.
$\hat{\mathcal{R}}^k_1$	will denote $\widehat{\text{Diff}}^k_1(\mathbb{C}^n, 0)$, that is, the $k-jet$ of the group of formal diffeomorphisms in $(\mathbb{C}^n, 0)$ tangent to the identity.
$\hat{\mathcal{O}}_{n,n}$	is $\hat{\mathcal{O}}_n \oplus \ldots \oplus \hat{\mathcal{O}}_n$ the $\hat{\mathcal{O}}_n$ free module of rank n.
$\hat{\mathcal{M}}^k_n$	is $\hat{\mathcal{M}}^k \oplus \cdots \oplus \hat{\mathcal{M}}^k$, $k \geq 1$. When $k = 1$ we indicate $\hat{\mathcal{M}}^1_n = \hat{\mathcal{M}}_n$.
$\hat{\mathcal{M}}^k_n$	is the direct sum of $\hat{\mathcal{M}}^k$ n times. When $k = 1$ we indicate $\hat{\mathcal{M}}^1_n = \hat{\mathcal{M}}_n$.
$J^k(n, n)$	is the space of the $k-jet$ of $\hat{\mathcal{M}}_n$.

Considering the coordinates $x = (x_1, \ldots, x_n)$ in a neighborhood of $0 \in \mathbb{C}^n$, we will write a formal vector field in $(\mathbb{C}^n, 0)$ as follows,

$$\hat{X} = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i}$$

where a_i belongs to the ring $\hat{\mathcal{O}}_n$.

We will say that the vector field X has multiplicity ν in $0 \in \mathbb{C}^n$, that is, $m_0(\hat{X}) = \nu$, if $\nu = \min\{m_{{}_0}(a_i)/i = 1, 2, \ldots, n\}$, where $m_{{}_0}(a_i)$ denotes the multiplicity of a_i in $0 \in \mathbb{C}^n$.

We will say that $\hat{X}$ and $\hat{Y}$ in $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ are *equivalent* if there is $h \in \widehat{\text{Diff}}\,(\mathbb{C}^n, 0)$ such that $\hat{Y} = h_*\hat{X}$,

$$h_*\hat{X}(y) = Dh(h^{-1}(y)).\hat{X}(h^{-1}(y)).$$

We say that two vector fields $\hat{X}$ and $\hat{Y}$ in $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ are *orbitally equivalent* if there is a unit $u \in \hat{\mathcal{O}}_n$ such that $\hat{Y} = uh_*\hat{X}$, in this case, the conjugation h maps the orbits of the first field vector on the orbits of the second one, without requiring a conjugation of its flows.

A first result that we obtain is the description of the tangent space to the orbits defined by the action of the Lie groups $\hat{\mathcal{R}}^k$ and $\hat{\mathcal{R}}^k_1$ on $\hat{\mathfrak{X}}^k(\mathbb{C}^n, 0)$, we show that the elements of tangent space $T_{\mathcal{X}}\hat{\mathcal{R}}^k.\mathcal{X}$ and $T_{\mathcal{X}}\hat{\mathcal{R}}^k_1.\mathcal{X}$, are given depending on the Lie bracket.

We will use the action φ of the Lie groups $G = \hat{\mathcal{R}}^k$ and $G = \hat{\mathcal{R}}^k_1$ on $\hat{\mathfrak{X}}^k(\mathbb{C}^n, 0)$, given by,

$$\begin{array}{rcl} \varphi : G \times \hat{\mathfrak{X}}^k(\mathbb{C}^n, 0) & \longrightarrow & \hat{\mathfrak{X}}^k(\mathbb{C}^n, 0) \\ (j^k(h), \mathcal{X}) & \longmapsto & \varphi(j^k(h), \mathcal{X}) = j^k(h_*\hat{X}) \end{array} \tag{2.1}$$

where $h \in \widehat{\text{Diff}}(\mathbb{C}^n, 0)$ (or $h \in \widehat{\text{Diff}}_1(\mathbb{C}^n, 0)$) and $\mathcal{X} = j^k(\hat{X}) \in \hat{\mathfrak{X}}^k(\mathbb{C}^n, 0)$.

2.1. Orbits and Tangent Spaces

To present the Complete Transversal Theorem for vector fields, it is necessary to describe the tangent space to the orbits $\hat{\mathcal{R}}^k.\mathcal{X}$ and $\hat{\mathcal{R}}^k_1.\mathcal{X}$ where $\mathcal{X} \in \hat{\mathfrak{X}}^k(\mathbb{C}^n, 0)$. The following theorem allows us to describe the elements of such tangent spaces.

Theorem 2.1. *If $\mathcal{X} = j^k(\hat{X}) \in \hat{\mathfrak{X}}^k(\mathbb{C}^n, 0)$ then*

1. $T_{\mathcal{X}}\hat{\mathcal{R}}^k\mathcal{X} = \left\{ j^k[\hat{X}, F] \,/ F = \sum_{j=1}^{n} f_j \frac{\partial}{\partial x_j},\ f = (f_1, \ldots, f_n) \in J^k(n, n) \right\}.$
2. $T_{\mathcal{X}}\hat{\mathcal{R}}^k_1\mathcal{X} = \left\{ j^k[\hat{X}, F] \,/ F = \sum_{j=1}^{n} f_j \frac{\partial}{\partial x_j},\ f_j \in \hat{\mathcal{M}}^2 \right\}.$

Proof. 1. We consider the action φ of the group $\hat{\mathcal{R}}^k$ on $\hat{\mathfrak{X}}^k(\mathbb{C}^n, 0)$ defined in (2.1).
Let $\mathcal{X} = j^k(\hat{X})$ and $\hat{X} = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$, we define

$$\begin{array}{cccc} \varphi_{\mathcal{X}} : & \hat{\mathcal{R}}^k & \longrightarrow & \hat{\mathfrak{X}}^k. \\ & j^k(h) & \longmapsto & \varphi_{\mathcal{X}}(j^k(h)) = j^k(h_*\hat{X}) \end{array}$$

Since $\hat{\mathcal{R}}^k$ can be identify with a open set in $\mathbb{C}$-vector space $J^k(n, n)$, we have that $T_e\hat{\mathcal{R}}^k = J^k(n, n)$, where e is the identity.
Let $j^k(f) \in T_e\hat{\mathcal{R}}^k$ and let $\alpha_k(t) \in \hat{\mathcal{R}}^k$ for all $t \in (-\epsilon, \epsilon)$ such that $\alpha_k(0) = e$, $\alpha_k'(0) = j^k(f)$, where

$$\alpha(t)(x) := x + t\, f(x), \text{ for all } t \in (-\epsilon, \epsilon),$$

$f(0) = 0$ and $\alpha_k(t) = j^k(\alpha(t))$.

$$\frac{d}{dt}\Big(\varphi_{\mathcal{X}} \circ \alpha_k(t)\Big)\Big|_{t=0} = \frac{d}{dt} j^k\Big(\alpha(t)_*\hat{X}\Big)\Big|_{t=0} = j^k\left(\frac{d}{dt}\Big(\alpha(t)_*\hat{X}\Big)\Big|_{t=0}\right). \tag{2.2}$$

Note that $\frac{d}{dt}(\varphi_{\mathcal{X}} \circ \alpha_k(t))|_{t=0} \in T_{\mathcal{X}}\hat{\mathcal{R}}^k\mathcal{X}$, and

$$\big(\alpha(t)_*\hat{X}\big)(p) \;=\; \sum_{i=1}^{n} a_i(\beta(t)(p)) \frac{\partial}{\partial p_i} + t \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\beta(t)(p)).a_j(\beta(t)(p)) \frac{\partial}{\partial p_i}$$

where $\beta(t) = (\alpha(t))^{-1}$, $f = (f_1, \ldots, f_n)$. So,

$$\frac{d}{dt}\Big(\alpha(t)_*\hat{X}\Big)(p)\Big|_{t=0} = \sum_{i=1}^{n} Da_i(\beta(0)(p)).\beta'(0)(p) \frac{\partial}{\partial p_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}(\beta(0)(p)).a_j(\beta(0)(p)) \frac{\partial}{\partial p_i}. \tag{2.3}$$

Since $\alpha(t).\beta(t) = e$ and $x = \beta(t)(x) + t\, f(\beta(t)(x))$ for all $t \in (-\epsilon, \epsilon)$, we have that $\beta(0) = e$ and $\beta'(0) = -f$.

From (2.3), we have

$$\frac{d}{dt}\Big(\alpha(t)_*\hat{X}\Big)\Big|_{t=0} = \sum_{i=1}^{n}\sum_{j=1}^{n} \left(a_j \frac{\partial f_i}{\partial x_j} - \frac{\partial a_i}{\partial x_j} f_j \right) \frac{\partial}{\partial x_i}.$$

In (2.2), we get

$$\frac{d}{dt}\Big(\varphi_{\mathcal{X}} \circ \alpha_k(t)\Big)\Big|_{t=0} = j^k \sum_{i=1}^{2}\sum_{j=1}^{2} \left(a_j \frac{\partial f_i}{\partial x_j} - \frac{\partial a_i}{\partial x_j} f_j \right) \frac{\partial}{\partial x_i}.$$

Hence,

$$T_{\mathcal{X}}\hat{\mathcal{R}}^k\mathcal{X} = \left\{ j^k \sum_{i=1}^{n}\sum_{j=1}^{n} \left(a_j \frac{\partial f_i}{\partial x_j} - f_j \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_i} \,/\, f = (f_1, \ldots, f_n) \in J^k(n, n) \right\}.$$

Finally, we define the vector field $F := \sum_{j=1}^{n} f_j \frac{\partial}{\partial x_j}$, and we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \left(a_j \frac{\partial f_i}{\partial x_j} - f_j \frac{\partial a_i}{\partial x_j} \right) \frac{\partial}{\partial x_i} = [\hat{X}, F].$$

2. For $G = \hat{\mathcal{R}}_1^k$ we proceed in a similar way, recalling that $D\alpha_k(t)(0) = e$ for all $t \in (-\epsilon, \epsilon)$. We have that $Df(0) = 0$, then $f \in \hat{\mathcal{M}}_n^2$. □

We have obtained $T_{\mathcal{X}}\hat{\mathcal{R}}^k.\mathcal{X}$ and $T_{\mathcal{X}}\hat{\mathcal{R}}_1^k.\mathcal{X}$, in the Theorem that we describe below, the Lie group that we consider is $\hat{\mathcal{R}}_1^k$, so we are able to apply the Complete Transversal Theorem of [1] for our case. This theorem will be the main tool to obtain the normal forms of the vector fields.
You can supplement details of this theorem in [12].

Theorem 2.2. (Complete Transversal for vector fields). *Let $\hat{\mathcal{R}}_1^k$ be the Lie group acting smoothly on the vector space $\hat{\mathfrak{X}}^k(\mathbb{C}^n,0)$ and W a vector subspace of $\hat{\mathfrak{X}}^k(\mathbb{C}^n,0)$ such that*

$$T_{\mathcal{X}+w}\hat{\mathcal{R}}_1^k.(\mathcal{X}+w) = T_{\mathcal{X}}\hat{\mathcal{R}}_1^k.\mathcal{X}$$

for all $\mathcal{X} \in \hat{\mathfrak{X}}^k(\mathbb{C}^n,0)$ and all $w \in W$. Then

1. *For any $\mathcal{X} \in \hat{\mathfrak{X}}^k(\mathbb{C}^n,0)$,*
$$\mathcal{X} + (T_{\mathcal{X}}\hat{\mathcal{R}}_1^k.\mathcal{X} \cap W) \subset \hat{\mathcal{R}}_1^k.\mathcal{X} \cap (\mathcal{X}+W).$$
2. *If $\mathcal{X} = j^k(\hat{X}) \in \hat{\mathfrak{X}}^k(\mathbb{C}^n,0)$ and T is a vector subspace of W satisfying*
$$W \subset T + T_{\mathcal{X}}\hat{\mathcal{R}}_1^k.\mathcal{X}$$
then for any $w \in W$, there exists $h \in \hat{\mathcal{R}}_1^k$ and $t \in T$ such that
$$h_*(\hat{X}+w) = \hat{X} + t \quad (\textit{module jet of order } k).$$

To apply the previous theorem, we need to determine a vector subspace $W \subset \hat{\mathfrak{X}}^k(\mathbb{C}^n,0)$, satisfying the hypothesis $T_{\mathcal{X}+w}\hat{\mathcal{R}}_1^k.(\mathcal{X}+w) = T_{\mathcal{X}}\hat{\mathcal{R}}_1^k.\mathcal{X}$, for all $w \in W$ and all $\mathcal{X} \in \hat{\mathfrak{X}}^k(\mathbb{C}^n,0)$.

Note that, the hypothesis of Complete Transversal Theorem for vector fields, are satisfied with $\hat{\mathcal{R}}_1^k$ and $W = \mathcal{H}^k(n) \subset \hat{\mathfrak{X}}^k(\mathbb{C}^n,0)$, where
$$\mathcal{H}^k(n) = \left\{ \sum_{i=1}^n P_i^k \frac{\partial}{\partial x_i} / \; P_i^k \text{ is a homogeneous polynomial of degree } k \right\}.$$

By Theorem 2.2, the change of coordinates $h \in \hat{\mathcal{R}}_1^k$ for each $k \geq \nu+1$, allows us to obtain a prenormal form of a vector field $\hat{X}$ with $m_0(\hat{X}) = \nu$, so that depending on the subspace $T \subset W$ all or some terms are eliminated of the homogeneous sum of degree k of the vector field. Following this recursive process, after step $k+1$ we have that the new vector field preserves the $k-$jet of the vector field from the previous step. The composition of all these changes of coordinates will allow us to obtain the formal normal form of the initial vector field $\hat{X}$.

We will say formal normal form of the vector field $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^n,0)$, to a particular representative of the equivalence class of the vector field $\hat{X}$, which can be obtained by Theorem 2.2. The subspace $T \subset W$ such that $W \subset T + T_{\mathcal{X}}\hat{\mathcal{R}}_1^k.\mathcal{X}$ is not unique, therefore the normal form is also not unique. A concrete case is when we study the vector fields $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2,0)$ such that $j^1(\hat{X})$ is nilpotent. This situation will be detailed in the section devoted to the nilpotent vector fields, precisely in the Theorem 4.1 that describes the Takens normal form in dimension 2, and in the Theorem 4.3.

The objective of our technique is to find a subspace T of W low-dimensional. The best case is achieved when $T = \{0\}$, that is, when $W \subset T_{\mathcal{X}}\hat{\mathcal{R}}_1^k.\mathcal{X}$. Such is the case of the vector fields $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2,0)$ whose $j^1(\hat{X})$ is non zero and has its eigenvalues λ_1, λ_2 such that $\lambda_1/\lambda_2 \in \mathbb{C} \setminus (\mathbb{Q}^- \cup \mathbb{N} \cup 1/\mathbb{N}^*)$, or when the $j^1(\hat{X})$ is non diagonalizable, in both cases the normal form of $\hat{X}$ is its $j^1(\hat{X})$. This will be seen in the Theorem 3.5 and the Theorem 3.7.

It is necessary to observe the following:

If $\mathcal{X} = j^k(\hat{X}) \in \hat{\mathfrak{X}}^k(\mathbb{C}^n, 0)$ and $W \subset T + T_{\mathcal{X}}\hat{\mathcal{R}}^k_1.\mathcal{X}$ then there exists $h \in \hat{\mathcal{R}}^k_1$ and $t \in T \subset W$ such that

$$h_*\Big((\hat{X} - X_k) + X_k\Big) = (\hat{X} - X_k) + t \quad \text{(module jet of order k)}$$

where X_k is the homogeneous sum of degree k of the vector field $\hat{X}$.

2.2. Formal prenormalization for vector fields

Prenormalization is the process that consists in obtaining a prenormal form for each jet of order $k \geq \nu + 1$, of the vector field $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ with $m_0(\hat{X}) = \nu \geq 1$. With the Lie group $\hat{\mathcal{R}}^k_1$ acting on space $\hat{\mathfrak{X}}^k(\mathbb{C}^n, 0)$, the main hypothesis of Complete Transversal Theorem for vector fields, is

$$T_{\mathcal{X}+w}\hat{\mathcal{R}}^k_1.(\mathcal{X} + w) = T_{\mathcal{X}}\hat{\mathcal{R}}^k_1.\mathcal{X}$$

for all $w \in W = \mathcal{H}^k(n)$ and all $\mathcal{X} \in \hat{\mathfrak{X}}^k(\mathbb{C}^n, 0)$.

It is important to determine a subspace $T \subset W$ such that

$$W \subset T + T_{\mathcal{X}}\hat{\mathcal{R}}^k_1.\mathcal{X}. \tag{2.4}$$

Therefore, we will analyze the elements of the tangent space $T_{\mathcal{X}}\hat{\mathcal{R}}^k_1.\mathcal{X}$, which verify the condition (2.4), this will allow us to define the subspace T, which will determine the prenormal form for each jet of order k of $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$.

We will describe subspaces T such that

$$W = T \oplus S$$

where S is a subspace such that $S \subset T_{\mathcal{X}}\hat{\mathcal{R}}^k_1.\mathcal{X} \cap \mathcal{H}^k(n)$.

As the elements of the $T_{\mathcal{X}}\hat{\mathcal{R}}^k_1.\mathcal{X}$ are of the form $j^k[\hat{X}, F]$, where $F = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$, $f_i \in \mathcal{M}^2$, in the following Proposition, considering vector fields $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ with multiplicity 1, we characterize the vector fields F such that $j^k[\hat{X}, F] \in \mathcal{H}^k(n) \cap T_{\mathcal{X}}\hat{\mathcal{R}}^k_1.\mathcal{X}$.

Proposition 2.3. *For $k \geq 2$, $j^k(F) \in \mathcal{H}^k(n)$ if and only if $j^k[\hat{X}, F] \in \mathcal{H}^k(n)$.*

Proof. We consider the vector field $\hat{X} = \sum_{j\geq 1} X_j$ and the vector field $F = \sum_{j\geq 2} F_j$, where X_j y F_j are homogeneous vector fields of degree j.

We analyze the jet of order k of Lie bracket of $\hat{X}$ and F,

$$\begin{aligned}
j^k\Big[\hat{X}, F\Big] &= j^k\Big[\textstyle\sum_{j\geq 1} X_j, \sum_{j=2}^k F_j\Big] \\
&= j^k\Big\{\Big[X_1, \textstyle\sum_{j=2}^k F_j\Big] + \Big[X_2, \sum_{j=2}^k F_j\Big] + \cdots + \Big[X_{k-1}, \sum_{j=2}^k F_j\Big]\Big\} \\
&= [X_1, F_2] + \cdots + \Big([X_1, F_{k-1}] + [X_2, F_{k-2}] + \cdots + [X_{k-2}, F_2]\Big) + \\
&\quad + \Big([X_1, F_k] + [X_2, F_{k-1}] + \cdots + [X_{k-1}, F_2]\Big).
\end{aligned}$$

If $j^k(F) = F_k \in \mathcal{H}^k(n)$ then $j^k[\hat{X}, F] = [X_1, F_k] \in \mathcal{H}^k(n)$.

Conversely, if we assume that the vector field $F = \sum_{j\geq 2} F_j$ is such that $F_j \neq 0$, for some $j \in \{2, \ldots, k-1\}$ then $j^k[\hat{X}, F] \notin \mathcal{H}^k(n)$. In fact, note that

$$[X_1, F_2] \in \mathcal{H}^2(n), \ldots, \Big([X_1, F_{k-1}] + [X_2, F_{k-2}] + \cdots + [X_{k-2}, F_2]\Big) \in \mathcal{H}^{k-1}(n),$$
$$\Big([X_1, F_k] + [X_2, F_{k-1}] + \cdots + [X_{k-1}, F_2]\Big) \in \mathcal{H}^k(n).$$

Hence $j^k(F) = F_k \in \mathcal{H}^k(n)$. □

We will write the vector fields $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ with multiplicity 1, as $\hat{X} = X_1 + X_2 + \cdots \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$, where X_j denotes the homogeneous sum of degree j of the vector field $\hat{X}$.

For each $k \geq 2$, we define the map $\mathrm{ad}^k_{X_1}$ of the space $\mathcal{H}^k(n)$ into itself, as follows

$$\mathrm{ad}^k_{X_1}(F) = [X_1, F] \tag{2.5}$$

where $F \in \mathcal{H}^k(n)$.

Note that, if $m_0(\hat{X}) = 1$ we have that $j^k[\hat{X}, F] = [X_1, F]$ for $F \in \mathcal{H}^k(n)$.

Moreover, if $m_0(\hat{X}) = \nu \geq 2$, that is, $\hat{X} = X_\nu + X_{\nu+1} + \cdots \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$, we have that $j^k[\hat{X}, F] = [X_\nu, F]$ where $F \in \mathcal{H}^{(k-\nu)+1}(n)$.

For the vector fields $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ with $m_0(\hat{X}) = 1$, the map defined in (2.5), allows us to obtain the elements of $T_{\mathcal{X}}\hat{\mathcal{R}}^k_1.\mathcal{X} \cap \mathcal{H}^k(n)$ as elements of $\mathrm{Im}\,\mathrm{ad}^k_{X_1}$.

For each $k \geq 2$, we can define the decomposition $\mathcal{H}^k(n) = B^k \oplus C^k$, where $B^k = \mathrm{Im}\,(\mathrm{ad}^k_{X_1})$ and C^k is some complementary space.
In the Complete Transversal Theorem, we will consider $W = B^k \oplus C^k$, and the subspace T of W satisfying (2.4) it will be C^k. This will give us information of the prenormal form for each jet of order k of $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$, that is, we refer to the *formal prenormalization of the vector field* $\hat{X}$, as Paul, E. refers in [10]. In the formal prenormalization that Paul refers, he considers a submodule M of the module of formal vector fields endowed with a graduation by a degree of quasi-homogeneity. For $k \geq 2$, we consider the vectorial space $\mathcal{H}^k(n)$ over $\mathbb{C}$, endowed with the homogeneous graduation of degree k.

3. Some Classic Normal Forms

3.1. Normal forms of vector fields in $(\mathbb{C}, 0)$

In this section, using the Complete Transversal Theorem for formal vector fields in $(\mathbb{C}, 0)$ with multiplicity 1, we get formally linearize, and in the case of the vector fields with multiplicity greater than or equal to 2, we can establish a formal equivalence to a polynomial vector field, in both cases with a formal diffeomorphism tangent to the identity.

Paul, E. shows in [9] that any formal vector field with multiplicity 1 is formally linearizable. If the vector field is analytic, it is analytically linearizable. We stress that our work corresponds to a formal development. In the following theorem, we present the normal forms for the vector fields in $(\mathbb{C}, 0)$ of any multiplicity.

We will write $\hat{\mathfrak{X}}(\mathbb{C}, 0)$ the $\hat{\mathcal{O}}_1$-module of formal vector fields at $0 \in \mathbb{C}$, that is,

$$\hat{\mathfrak{X}}(\mathbb{C}, 0) = \left\{ a(x)x\frac{\partial}{\partial x} / a \in \hat{\mathcal{O}}_1 \right\}.$$

Theorem 3.1. *For $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C},0)$ with $m_0(\hat{X}) = \nu$, we have*

1. *if $\nu = 1$ then the vector field $\hat{X}$ is formally linearizable equivalent to $j^1(\hat{X})$.*
2. *If $\nu \geq 2$ then the vector field $\hat{X}$ is formally equivalent to the polynomial vector field*

$$(a_\nu x^\nu + a' x^{2\nu-1})\frac{\partial}{\partial x}.$$

Proof. Given the vector field $\hat{X} = b(x)\dfrac{\partial}{\partial x} \in \hat{\mathfrak{X}}(\mathbb{C},0)$ and $\mathcal{X} = j^k(\hat{X}) \in \hat{\mathfrak{X}}^k(\mathbb{C},0)$, we consider the following two cases:

First case.- $b(x) = x(a_1 + a_2 x + a_3 x^2 + \cdots)$, $a_1 \neq 0$.

Considering $f(x) = \alpha_k x^k$ for $k \geq 2$, the elements of $T_{\mathcal{X}}\hat{\mathcal{R}}^k_1.\mathcal{X} \cap \mathcal{H}^k(1)$ are given by,

$$(k-1)a_1\alpha_k x^k\frac{\partial}{\partial x}.$$

So, for each $k \geq 2$, we have that $\mathcal{H}^k(1) = W \subset T_{\mathcal{X}}\hat{\mathcal{R}}^k_1.\mathcal{X} + T$ where $T = \{0\}$.

By Theorem 2.2, there exists a change of coordinates tangent to the identity for each $k \geq 2$, thereby the formal normal form for the vector field $\hat{X}$ is $j^1(\hat{X}) = a_1 x\frac{\partial}{\partial x}$. That is, the vector field $\hat{X}$ is formally linearizable.

Second case.- $b(x) = x(a_\nu x^{\nu-1} + a_{\nu+1}x^\nu + a_{\nu+2}x^{\nu+1} + \cdots)$, $a_\nu \neq 0$, $\nu \geq 2$.

Considering $f(x) = \alpha_{k-\nu+1}x^{k-\nu+1}$ for $k \geq \nu+1$, the elements of $T_{\mathcal{X}}\hat{\mathcal{R}}^k_1.\mathcal{X} \cap \mathcal{H}^k(1)$ are given by,

$$(k+1-2\nu)a_\nu\alpha_{k-\nu+1}x^k\frac{\partial}{\partial x}.$$

So, for each $k \geq \nu+1$, we have that $\mathcal{H}^k(1) = W \subset T_{\mathcal{X}}\hat{\mathcal{R}}^k_1.\mathcal{X} + T$.

i) When $k \geq \nu+1$ but $k \neq 2\nu-1$:
$\mathcal{H}^k(1) = W \subset T_{\mathcal{X}}\hat{\mathcal{R}}^k_1.\mathcal{X}$ and $T = \{0\}$.

ii) When $k = 2\nu-1$:
$\mathcal{H}^{2\nu-1}(1) = W \subset T_{\mathcal{X}}\hat{\mathcal{R}}^{2\nu-1}_1.\mathcal{X} + T$, we have $T = \{a'x^{2\nu-1}\frac{\partial}{\partial x};\ a' \in \mathbb{C}\}$.

By Theorem 2.2, there exists a change of coordinates tangent to the identity successively for each $k \geq \nu+1$, so, the formal normal form for the vector field $\hat{X}$ is

$$(a_\nu x^\nu + a' x^{2\nu-1})\frac{\partial}{\partial x}.$$

□

3.2. Normal forms of vector fields in $(\mathbb{C}^2, 0)$

In this section, we show the formal normal forms for some vector fields $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2,0)$, those that will be written in the following way:

$$\hat{X} = X_1 + \sum_{i \geq 2} X_i$$

where $X_1 = j^1(\hat{X})$ is non zero. By a linear change of coordinates, without loss of generality we can assume that X_1 is given by its Jordan canonical form.

Let λ_1, λ_2 be the eigenvalues of X_1. We consider the following cases, analyzing the rank of X_1:

(a) $\text{rank}(X_1) = 2$,

(a.1) λ_1/λ_2 and λ_2/λ_1 does not belong to $\mathbb{Q}^-$, or to $\mathbb{N}^*$(the naturals greater than or equal to 1) (see Theorem 3.5-(i)).

(a.2) λ_1/λ_2 or $\lambda_2/\lambda_1 \in \mathbb{N}$ (see Theorem 3.5-(ii)).

(a.3) $\lambda_1/\lambda_2 \in \mathbb{Q}^-$ (see Theorem 3.6-(i)).

(a.4) $X_1 = (\lambda_1 x_1 + x_2)\frac{\partial}{\partial x_1} + \lambda_1 x_2 \frac{\partial}{\partial x_2}$. Without loss of generality we can assume $\lambda_1 = 1$ (see Theorem 3.7).

(b) $\text{rank}(X_1) = 1$,

(b.1) $\lambda_1 = 0$ ($\lambda_2 \neq 0$) (see Theorem 3.6-(ii)).

The case rank $(X_1) = 1$, when $X_1 = x_2\frac{\partial}{\partial x_1}$ will be considered when we treat the nilpotent vector fields in dimension 2.

To obtain the normal form of the vector field $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$, considering the previously mentioned cases, we stress the importance of the subspace T of W such that $W = B \oplus T$, where $B = \text{Im}\,(\text{ad}^k_{X_1})$ for $k \geq 2$.

We will use the following notation:

$$\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2 \backslash \{0\},\ \ I = (i_1, i_2) \in \mathbb{N}^2,\ \ x^I = x_1^{i_1} x_2^{i_2}.$$
$$<\lambda, I>= \lambda_1 i_1 + \lambda_2 i_2\,,\ \ |I| = i_1 + i_2.$$

Note that $\left\{x^I\frac{\partial}{\partial x_1}, x^I\frac{\partial}{\partial x_2}/|I| = k\right\}$ is a basis for the space of homogeneous vector fields of degree k in dimension 2 denoted by $\mathcal{H}^k(2)$. In the following statements, we relate the elements of this basis with the complex numbers $(<\lambda, I> -\lambda_i),\ i = 1, 2$.

Proposition 3.2. *For $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ with $j^1(\hat{X}) = X_1$, we have*

$$j^k\left[\hat{X}, x^I\frac{\partial}{\partial x_i}\right] = \left[X_1, x^I\frac{\partial}{\partial x_i}\right]$$

for $i = 1, 2$ and $|I| = k \geq 2$.

Proposition 3.3. *Let $\hat{X} = X_1 + \cdots$, where $X_1 = \lambda_1 x_1\frac{\partial}{\partial x_1} + \lambda_2 x_2\frac{\partial}{\partial x_2}$ be a vector field in $(\mathbb{C}^2, 0)$. Then,*

i) $\left[X_1, x^I\frac{\partial}{\partial x_i}\right] = (<\lambda, I> -\lambda_i)x^I\frac{\partial}{\partial x_i}$, *for $i = 1, 2$.*

ii) $j^k\left[\hat{X},\, x^I\frac{\partial}{\partial x_i}\right] = (<\lambda, I> -\lambda_i)x^I\frac{\partial}{\partial x_i}$, *for $|I| = k \geq 2$.*

The proof of the previous propositions is obtained using properties of the Lie bracket, it can also be revised in [12].

Considering $\lambda_1, \lambda_2 \in \mathbb{C}$, we define the complex number

$$\delta_{j,I} = \lambda_j - i_1\lambda_1 - i_2\lambda_2,\ j = 1, 2,\ \text{and}\ |I| \geq 2.$$

With $\delta_{j,I}$ we can establish the following definition,

Definition 3.4. *The pair* $(\lambda_1,\lambda_2)\in\mathbb{C}^2$ *is* resonant *if there exist* $I\in\mathbb{N}^2$ *with* $|I|\geq 2$ *and there exists* $j\in\{1,2\}$ *such that* $\delta_{j,I}=0$. *Otherwise, we will say that* (λ_1,λ_2) *is a* non resonant pair.

Ichikawa, F. in [5], studies the normal forms of the vector fields $\hat{X}\in\hat{\mathfrak{X}}(\mathbb{C}^2,0)$ with $j^1(\hat{X})$ of the form (1.1), he considers the concept of resonance, even when he does not mention it.

Also, Ichikawa, F. in [4], with respect to the eigenvalues of $j^1(\hat{X})$ of the vector fields $\hat{X}\in\hat{\mathfrak{X}}(\mathbb{C}^n,0)$, he establishes definitions such as, the condition of *strong eigenvalue*, the condition of *weak eigenvalue* and a condition of *good eigenvalue* and characterizes finitely determined vector fields $\hat{X}\in\hat{\mathfrak{X}}(\mathbb{C}^n,0)$, if $j^1(\hat{X})$ satisfies the condition of good eigenvalue and does not satisfy the condition of strong eigenvalue.

In the next Theorem, we provide the classical normal forms of vector fields $\hat{X}\in\hat{\mathfrak{X}}(\mathbb{C}^2,0)$ such that $j^1(\hat{X})$ is of the form (1.1), with non resonant and resonant eigenvalues, studied by Poincaré and Dulac, respectively.

Theorem 3.5. (Poincaré-Dulac). *For* $\hat{X}\in\hat{\mathfrak{X}}(\mathbb{C}^2,0)$ *with* $j^1(\hat{X})=\lambda_1x_1\frac{\partial}{\partial x_1}+\lambda_2x_2\frac{\partial}{\partial x_2}$, *we have*

1. *if* $(\lambda_1,\lambda_2)\in\mathbb{C}^2\backslash\{0\}$ *is a non resonant pair, then the vector field* $\hat{X}$ *is formally equivalent to* $j^1(\hat{X})$.
2. *If* $(\lambda_1,\lambda_2)\in\mathbb{C}^2\backslash\{0\}$ *is a resonant pair such that* $\delta_{1,(0,\mathsf{m})}=0$, *then the vector field* $\hat{X}$ *is formally equivalent to the vector field*
$$(\lambda_1x_1+ax_2^{\mathsf{m}})\frac{\partial}{\partial x_1}+\lambda_2x_2\frac{\partial}{\partial x_2}.$$

Proof. According to the prenormalization process, we will establish a change of coordinates for each jet of order k of $\hat{X}\in\hat{\mathfrak{X}}(\mathbb{C}^2,0)$. In this process, for each $k\geq 2$, the subspace $W=\mathcal{H}^k(2)$ will be considered as $W=T\oplus\operatorname{Im}(\operatorname{ad}^k_{X_1})$, where $\operatorname{Im}(\operatorname{ad}^k_{X_1})\subset T_{\mathfrak{X}}\hat{\mathcal{R}}^k_1\mathfrak{X}\cap\mathcal{H}^k(2)$.

Given $F=\sum_{|I|=k}\alpha_Ix^I\frac{\partial}{\partial x_1}+\sum_{|I|=k}\beta_Ix^I\frac{\partial}{\partial x_2}\in\mathcal{H}^k(2)$ for $k\geq 2$, using properties of the Lie bracket, and the Proposition 3.3, we have that the elements of $\operatorname{Im}(\operatorname{ad}^k_{X_1})$ are given by

$$\begin{aligned} j^k[\hat{X},F] &= \sum_{|I|=k}\alpha_I(<\lambda,I>-\lambda_1)x^I\frac{\partial}{\partial x_1}+\sum_{|I|=k}\beta_I(<\lambda,I>-\lambda_2)x^I\frac{\partial}{\partial x_2}\\ &= \sum_{|I|=k}\alpha_I(-\delta_{1,I})x^I\frac{\partial}{\partial x_1}+\sum_{|I|=k}\beta_I(-\delta_{2,I})x^I\frac{\partial}{\partial x_2}.\end{aligned}$$

1. If (λ_1,λ_2) is a non resonant pair then $\delta_{j,I}\neq 0$ for $j=1,2$ and $|I|\geq 2$. So, $\mathcal{H}^k(2)=W\subset\operatorname{Im}(\operatorname{ad}^k_{X_1})$ and $\operatorname{Im}(\operatorname{ad}^k_{X_1})\subset W$, for all $k\geq 2$. Therefore, $T=\{0\}$ for all $k\geq 2$.

2. Since (λ_1,λ_2) is a resonant pair with $\delta_{1,(0,\mathsf{m})}=0$ and $\delta_{j,I}\neq 0$ otherwise for $j=1,2$. When $|I|\geq 2$ and $|I|\neq\mathsf{m}$ we can consider the subspace $T=\{0\}$ as in the non resonant case.

   Considering $|I|=\mathsf{m}$, let us write
$$w=\sum_{|I|=\mathsf{m}}c_{1,I}x^I\frac{\partial}{\partial x_1}+\sum_{|I|=\mathsf{m}}c_{2,I}x^I\frac{\partial}{\partial x_2}$$
   so, it is possible to write the elements $w\in W=\mathcal{H}^{\mathsf{m}}(2)$ as follows,

$$w = (c_{1,(0,\mathsf{m})}x_2^{\mathsf{m}})\frac{\partial}{\partial x_1} + \sum_{\substack{|I| = \mathsf{m} \\ I \neq (0,\mathsf{m})}} \alpha_I(-\delta_{1,I})x^I\frac{\partial}{\partial x_1} + \sum_{|I|=\mathsf{m}} \beta_I(-\delta_{2,I})x^I\frac{\partial}{\partial x_2}$$

where

$(c_{1,(0,\mathsf{m})}x_2^{\mathsf{m}})\frac{\partial}{\partial x_1} \in T$ and $\sum_{\substack{|I| = \mathsf{m} \\ I \neq (0,\mathsf{m})}} \alpha_I(-\delta_{1,I})x^I\frac{\partial}{\partial x_1} + \sum_{|I|=\mathsf{m}} \beta_I(-\delta_{2,I})x^I\frac{\partial}{\partial x_2} \in \mathrm{Im}(\mathrm{ad}^{\mathsf{m}}_{X_1})$.

□

Poincaré and Dulac, establish analytical equivalences of vector fields. By Poincaré, the analytical equivalence is achieved when $\lambda_1/\lambda_2 \notin \mathbb{R}^-$ and (λ_1, λ_2) is a non resonant pair. It is necessary to mention again that our work consists of formal equivalence.

Now, we consider the equivalence relation of foliations, that is, the classification of vector fields up to a unity.

In the following theorem, we show the formal orbital equivalence considering the following cases: $\lambda_1/\lambda_2 = -q/p \in \mathbb{Q}^-$ with $gcd(q,p) = 1$ and the case of singularities saddle-node type, that is, $\lambda_1 = 0$ and $\lambda_2 \neq 0$.

The normal forms described in the following Theorem are due to Dulac (see [2]), such normal forms are also found in [7].

Theorem 3.6. (Dulac). *For $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ such that $j^1(\hat{X}) = \lambda_1 x_1 \frac{\partial}{\partial x_1} + \lambda_2 x_2 \frac{\partial}{\partial x_2}$, we have*

(i) *if $\lambda_1/\lambda_2 = -q/p \in \mathbb{Q}^-$, then the vector field $\hat{X}$ is formally orbitally equivalent to the vector field*

$$x_1(1 + A(x_1^p x_2^q))\frac{\partial}{\partial x_1} - \frac{p}{q}x_2(1 + B(x_1^p x_2^q))\frac{\partial}{\partial x_2}$$

where $A(t), B(t) \in \mathbb{C}[[t]]$ and $A(0) = B(0) = 0$.

(ii) *If $\lambda_1 = 0$ and $\lambda_2 \neq 0$, then the vector field $\hat{X}$ is formally orbitally equivalent to the vector field*

$$x_1 A(x_1)\frac{\partial}{\partial x_1} + x_2(1 + B(x_1))\frac{\partial}{\partial x_2}$$

where $A(x_1), B(x_1) \in \mathbb{C}[[x_1]]$ and $A(0) = B(0) = 0$.

Proof. We will obtain the normal form of vector field $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ with the prenormalization process considered above.

Given $F = \sum_{|I|=k} \alpha_I x^I \frac{\partial}{\partial x_1} + \sum_{|I|=k} \beta_I x^I \frac{\partial}{\partial x_2} \in \mathcal{H}^k(2)$, we have that

$$j^k[\hat{X}, F] = \sum_{|I|=k} \alpha_I(-\delta_{1,I})x^I\frac{\partial}{\partial x_1} + \sum_{|I|=k} \beta_I(-\delta_{2,I})x^I\frac{\partial}{\partial x_2} \in \mathrm{Im}\,(\mathrm{ad}^k_{X_1}) \cap \mathcal{H}^k(2).$$

(i) If $\lambda_1/\lambda_2 = -q/p \in \mathbb{Q}^-$, then $pr\lambda_1 + qr\lambda_2 = 0$, for all $r \in \mathbb{N}^*$.
So, $\delta_{1,(pr+1,qr)} = 0$ and $\delta_{2,(pr,qr+1)} = 0$, for all $r \in \mathbb{N}^*$.
When $k \geq 2$ but $k \neq r(p+q) + 1$ for all $r \in \mathbb{N}^*$, we have $\delta_{j,I} \neq 0$ for $j = 1, 2$ and $|I| = k$.
So, for such values of k, we have that $W \subset \mathrm{Im}\,(\mathrm{ad}^k_{X_1})$ and $\mathrm{Im}\,(\mathrm{ad}^k_{X_1}) \subset W$.
Therefore, $T = \{0\}$.

Now, suppose $k = r'(p+q) + 1$, for some $r' \in \mathbb{N}^*$.

Let $w = \sum_{|I|=k} c_{1,I} x^I \frac{\partial}{\partial x_1} + \sum_{|I|=k} c_{2,I} x^I \frac{\partial}{\partial x_2}$ be an element of $W = \mathcal{H}^k(2)$.

So, we can write any element $w \in W$ as follows:

$$w = c_{1,(pr'+1,qr')} x_1^{pr'+1} x_2^{qr'} \frac{\partial}{\partial x_1} + c_{2,(pr',qr'+1)} x_1^{pr'} x_2^{qr'+1} \frac{\partial}{\partial x_2} + \sum_{I\in\mathfrak{I}_{r'}} \alpha_I(-\delta_{1,I}) x^I \frac{\partial}{\partial x_1} + $$
$$+ \sum_{I\in\mathfrak{J}_{r'}} \beta_I(-\delta_{1,I}) x^I \frac{\partial}{\partial x_2}$$

where

$$\mathfrak{I}_{r'} = \Big\{|I| = k / I \neq (pr'+1, qr')\Big\} \text{ and } \mathfrak{J}_{r'} = \Big\{|I| = k / I \neq (pr', qr'+1)\Big\}.$$

Therefore, $\sum_{I\in\mathfrak{I}_{r'}} \alpha_I(-\delta_{1,I}) x^I \frac{\partial}{\partial x_1} + \sum_{I\in\mathfrak{J}_{r'}} \beta_I(-\delta_{1,I}) x^I \frac{\partial}{\partial x_2} \in \mathrm{Im}\,(\mathrm{ad}^k_{X_1})$ and

$$c_{1,(pr'+1,qr')} x_1^{pr'+1} x_2^{qr'} \frac{\partial}{\partial x_1} + c_{2,(pr',qr'+1)} x_1^{pr'} x_2^{qr'+1} \frac{\partial}{\partial x_2} \in T.$$

Hence, from Complete Transversal Theorem, there exists $\Phi \in \widehat{\mathrm{Diff}}_1(\mathbb{C}^2, 0)$ such that the vector field $\hat{X}$ is formally equivalent to

$$\Phi_* \hat{X} = \left(\lambda_1 x_1 + x_1 \sum_{r'\geq 1} a_{r'} (x_1^p x_2^q)^{r'}\right) \frac{\partial}{\partial x_1} + \left(\lambda_2 x_2 + x_2 \sum_{r'\geq 1} b_{r'} (x_1^p x_2^q)^{r'}\right) \frac{\partial}{\partial x_2}.$$

Finally, the vector field $\hat{X}$ is formally orbitally equivalent to the vector field

$$(1/\lambda_1)\Phi_* \hat{X} \;=\; x_1 \left(1 + \sum_{r'\geq 1} \frac{a_{r'}}{\lambda_1} (x_1^p x_2^q)^{r'}\right) \frac{\partial}{\partial x_1} + \frac{\lambda_2}{\lambda_1} x_2 \left(1 + \sum_{r'\geq 1} \frac{b_{r'}}{\lambda_2} (x_1^p x_2^q)^{r'}\right) \frac{\partial}{\partial x_2}.$$

(ii) Now, we have $X_1 = \lambda_2 x_2 \frac{\partial}{\partial x_2}$. Remark that $\lambda_1 = 0$ implies that $\delta_{1,(i_1,0)} = 0$ for $i_1 \geq 2$ and $\delta_{2,(i_1,1)} = 0$ for $i_1 \geq 1$, in another case $(0, \lambda_2)$ is a non resonant pair.

Given $F = \sum_{|I|=k} \alpha_I x^I \frac{\partial}{\partial x_1} + \sum_{|I|=k} \beta_I x^I \frac{\partial}{\partial x_2} \in \mathcal{H}^k(2)$, we have that

$$j^k[\hat{X}, F] \;=\; \sum_{|I|\,=\,k} \alpha_I(-\delta_{1,I}) x^I \frac{\partial}{\partial x_1} + \sum_{|I|\,=\,k} \beta_I(-\delta_{2,I}) x^I \frac{\partial}{\partial x_2} \in \mathrm{Im}\,(\mathrm{ad}^k_{X_1}) \cap \mathcal{H}^k(2).$$

For $k \geq 2$, let $w = \sum_{|I|=k} c_{1,I} x^I \frac{\partial}{\partial x_1} + \sum_{|I|=k} c_{2,I} x^I \frac{\partial}{\partial x_2}$ be an element of $W = \mathcal{H}^k(2)$.

So, we can write

$$w = c_{1,(k,0)} x_1^k \frac{\partial}{\partial x_1} + c_{2,(k-1,1)} x_1^{k-1} x_2 \frac{\partial}{\partial x_2} +$$
$$+ \sum_{|I|\,=\,k} \alpha_I(-\delta_{1,I}) x^I \frac{\partial}{\partial x_1} + \sum_{|I|\,=\,k} \beta_I(-\delta_{2,I}) x^I \frac{\partial}{\partial x_2} \in T \oplus \mathrm{Im}\,(\mathrm{ad}^k_{X_1}).$$

Hence, for each $k \geq 2$ we have

$$c_{1,(k,0)} x_1^k \frac{\partial}{\partial x_1} + c_{2,(k-1,1)} x_1^{k-1} x_2 \frac{\partial}{\partial x_2} \in T \subset W = \mathcal{H}^k(2).$$

Hence, from Complete Transversal Theorem, there exists $\psi \in \widehat{\mathrm{Diff}}_1(\mathbb{C}^2, 0)$ such that the vector field $\hat{X}$ is formally equivalent to the vector field

$$\psi_*\hat{X} = \left(x_1 \sum_{r\geq 1} a_r x_1^r\right)\frac{\partial}{\partial x_1} + \left(\lambda_2 x_2 + x_2 \sum_{r\geq 1} b_r x_1^r\right)\frac{\partial}{\partial x_2}.$$

Finally, the vector field $\hat{X}$ is formally orbitally equivalent to the vector field

$$(1/\lambda_2)\psi_*\hat{X} = \left(x_1 \sum_{r\geq 1} \mathrm{a}_r\, x_1^r\right)\frac{\partial}{\partial x_1} + x_2\left(1 + \sum_{r\geq 1} \mathrm{b}_r\, x_1^r\right)\frac{\partial}{\partial x_2}.$$

□

Now, we consider $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ with non zero and non diagonal linear part, of the form

$$j^1(\hat{X}) = (x_1 + x_2)\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}.$$

Theorem 3.7. *The vector field $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ with $j^1(\hat{X}) = (x_1 + x_2)\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$ is formally equivalent to $j^1(\hat{X})$, by a formal diffeomorphism tangent to the identity.*

Proof. For each $k \geq 2$, the map defined in (2.5) is surjective.

Given $w = \sum_{|I|=k} c_{1,I}x^I \frac{\partial}{\partial x_1} + \sum_{|I|=k} c_{2,I}x^I \frac{\partial}{\partial x_2} \in \mathcal{H}^k(2)$, we will show that there exists $F \in \mathcal{H}^k(2)$ such that $[X_1, F] = w$.

Let $F = \sum_{|I|=k} \alpha_I x^I \frac{\partial}{\partial x_1} + \sum_{|I|=k} \beta_I x^I \frac{\partial}{\partial x_2}$ an element of $\mathcal{H}^k(2)$, then

$$\begin{aligned}[X_1, F] \quad &= \quad \sum_{|I|=k} \alpha_I\Big((i_1 + i_2 - 1)x^I + i_1 x_1^{i_1-1}x_2^{i_2+1}\Big)\frac{\partial}{\partial x_1} + \\ &\qquad + \sum_{|I|=k} \beta_I\Big(-x^I\frac{\partial}{\partial x_1} + \Big((i_1+i_2-1)x^I + i_1 x_1^{i_1-1}x_2^{i_2+1}\Big)\frac{\partial}{\partial x_2}\Big).\end{aligned}$$

In this way, for each $k \geq 2$ we have the following system of equations:

$$\begin{aligned}\sum_{|I|=k} (\alpha_I(i_1+i_2-1) - \beta_I)x^I + \alpha_I.i_1 x_1^{i_1-1}x_2^{i_2+1} \quad &= \quad \sum_{|I|=k} c_{1,I}x^I \\ \sum_{|I|=k} \beta_I(i_1+i_2-1)x^I + \beta_I.i_1 x_1^{i_1-1}x_2^{i_2+1} \quad &= \quad \sum_{|I|=k} c_{2,I}x^I\end{aligned}$$

where the unknowns are α_I and β_I, for $|I| = k$.

When $|I| = k \geq 2$, the system has the following associated matrix:

$$M = \begin{bmatrix} A & -I \\ 0 & A \end{bmatrix} \in \mathbb{R}^{2(k+1)\times 2(k+1)}$$

where

$$A=\begin{bmatrix} (k-1) & 0 & 0 & \dots\dots & 0 & 0\\ k & (k-1) & 0 & \dots\dots & 0 & 0\\ 0 & (k-1) & (k-1) & \dots\dots & 0 & 0\\ 0 & 0 & (k-2) & \ddots & 0 & 0\\ \vdots & \vdots & & \ddots & \ddots & \vdots\\ 0 & 0 & \dots\dots & & 1 & (k-1)\end{bmatrix}.$$

The matrix $M\in\mathbb{R}^{2(k+1)\times 2(k+1)}$ has non zero determinant, which is equal to $(k-1)^{2(k+1)}$. This proves the surjectivity.

Therefore by the Complete Transversal Theorem, there exists a change of coordinates in $\hat{\mathcal{R}}_1^k$, so that, for each $k\geq 2$, we have that $T=\{0\}\subset\mathcal{H}^k(2)$. □

4. Normal Forms of Nilpotent Vector Fields

We will say that a formal vector field $\hat{X}$ with $m_0(\hat{X})=1$ is nilpotent if $j^1(\hat{X})$ is nilpotent. In this section, we will present normal forms of the nilpotent vector fields in dimension 2 and 3. Using the technique provided by the Complete Transversal Theorem, we recover the *Takens Normal Form* (see [18]) and also show a normal form different of the Takens normal form in dimension 2, both normal forms are obtained with a formal diffeomorphism tangent to the identity.

Proposition 4.1. *Let $\hat{X}$ be the formal vector field in $(\mathbb{C}^2,0)$ such that $j^1(\hat{X})=x_2\frac{\partial}{\partial x_1}$. Then, the dimension of $\mathrm{Ker}(\mathrm{ad}^k_{X_1})$ is 2, for $k\geq 2$.*

Proof. For $k\geq 2$, given $F=\displaystyle\sum_{|I|=k}\alpha_I x^I\frac{\partial}{\partial x_1}+\sum_{|I|=k}\beta_I x^I\frac{\partial}{\partial x_2}\in\mathcal{H}^k(2)$ we have

$$[X_1,F] \;=\; \sum_{|I|=k}(\alpha_{_I}(i_1x_1^{i_1-1}x_2^{i_2+1})-\beta_I x^I)\frac{\partial}{\partial x_1}+\sum_{|I|=k}\beta_{_I} i_1 x_1^{i_1-1}x_2^{i_2+1}\frac{\partial}{\partial x_2}.$$

If $[X_1,F]=0$, then we have the following system of equations:

$$\begin{aligned}\alpha_{_I}(i_1x_1^{i_1-1}x_2^{i_2+1})-\beta_I x^I &= 0\\ \beta_{_I} i_1x_1^{i_1-1}x_2^{i_2+1} &= 0\end{aligned}$$

We have that, $\alpha_I=\beta_I=0$ for $I\neq(0,k)$ and $\alpha_{(0,k)},\beta_{(0,k)}\in\mathbb{C}$.

It is also verified that $\alpha_{(1,k-1)}x_2^k-\beta_{(0,k)}x_2^k=0$, that is, $\alpha_{(1,k-1)}=\beta_{(0,k)}$.

So, $F=\alpha_{(0,k)}x_2^k\frac{\partial}{\partial x_1}+\beta_{(0,k)}\left(x_1x_2^{k-1}\frac{\partial}{\partial x_1}+x_2^k\frac{\partial}{\partial x_2}\right)\in\mathcal{H}^k(2)$. Therefore,

$$\mathrm{Ker}(\mathrm{ad}^k_{X_1})=\left\{\alpha_{(0,k)}x_2^k\frac{\partial}{\partial x_1}+\beta_{(0,k)}\left(x_1x_2^{k-1}\frac{\partial}{\partial x_1}+x_2^k\frac{\partial}{\partial x_2}\right)/\alpha_{(0,k)},\beta_{(0,k)}\in\mathbb{C}\right\}.$$

Hence $\dim\mathrm{Ker}(\mathrm{ad}^k_{X_1})=2$, for $k\geq 2$. □

It follows by Proposition 4.1, that $\dim \operatorname{Im}(\operatorname{ad}^k_{X_1}) = 2k$.

The following theorem give us the Takens normal form for nilpotent vector fields $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$.

Theorem 4.2. *For $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ with $j^1(\hat{X}) = x_2 \frac{\partial}{\partial x_1}$, there exists a formal change of coordinates tangent to the identity reducing it to the form*

$$(x_2 + a(x_1))\frac{\partial}{\partial x_1} + b(x_1)\frac{\partial}{\partial x_2} \tag{4.1}$$

where $m_0(a) \geq 2$ and $m_0(b) \geq 2$.

Proof. We consider the prenormalization process considered above.

For each $k \geq 2$, let $\mathcal{H}^k(2) = T \oplus \operatorname{Im}(\operatorname{ad}^k_{X_1})$ where $X_1 = j^1(\hat{X}) = x_2 \frac{\partial}{\partial x_1}$.

We have that $\dim \mathcal{H}^k(2) = \dim T + \dim \operatorname{Im}(\operatorname{ad}^k_{X_1})$ and $\dim \mathcal{H}^k(2) = 2(k+1)$.

Therefore, $\dim T = \dim \operatorname{Ker}(\operatorname{ad}^k_{X_1}) = 2$, for each $k \geq 2$.

Given $F = \sum_{|I|=k} \alpha_I x^I \frac{\partial}{\partial x_1} + \sum_{|I|=k} \beta_I x^I \frac{\partial}{\partial x_2} \in \mathcal{H}^k(2)$, we have that

$$[X_1, F] = \sum_{|I|=k} (\alpha_I (i_1 x_1^{i_1 - 1} x_2^{i_2+1}) - \beta_I x^I)\frac{\partial}{\partial x_1} + \sum_{|I|=k} \beta_I i_1 x_1^{i_1-1} x_2^{i_2+1} \frac{\partial}{\partial x_2} \in \operatorname{Im}(\operatorname{ad}^k_{X_1}).$$

Let $w = \sum_{|I|=k} c_{1,I} x^I \frac{\partial}{\partial x_1} + \sum_{|I|=k} c_{2,I} x^I \frac{\partial}{\partial x_2}$ be an element of $W = \mathcal{H}^k(2)$. We can write

$$\begin{aligned} w = {} & (c_{1,(k,0)} + \beta_{(k,0)}) x_1^k \frac{\partial}{\partial x_1} + c_{2,(k,0)} x_1^k \frac{\partial}{\partial x_2} + \Big(-\beta_{(k,0)} x_1^k + (k\alpha_{(k,0)} - \beta_{(k-1,1)}) x_1^{k-1} x_2 + \cdots + \\ & + (\alpha_{(1,k-1)} - \beta_{(0,k)}) x_2^k \Big) \frac{\partial}{\partial x_1} + \Big(k\beta_{(k,0)} x_1^{k-1} x_2 + (k-1)\beta_{(k-1,1)} x_1^{k-2} x_2^2 + \cdots + \beta_{(1,k-1)} x_2^k \Big) \frac{\partial}{\partial x_2} \end{aligned}$$

where

$$\begin{aligned} & \Big(-\beta_{(k,0)} x_1^k + (k\alpha_{(k,0)} - \beta_{(k-1,1)}) x_1^{k-1} x_2 + \cdots + (\alpha_{(1,k-1)} - \beta_{(0,k)}) x_2^k \Big) \frac{\partial}{\partial x_1} + \Big(k\beta_{(k,0)} x_1^{k-1} x_2 + \\ & + (k-1)\beta_{(k-1,1)} x_1^{k-2} x_2^2 + \cdots + \beta_{(1,k-1)} x_2^k \Big) \frac{\partial}{\partial x_2} \in \operatorname{Im}(\operatorname{ad}^k_{X_1}). \end{aligned}$$

So, $(c_{1,(k,0)} + \beta_{(k,0)}) x_1^k \frac{\partial}{\partial x_1} + c_{2,(k,0)} x_1^k \frac{\partial}{\partial x_2} \in T \subset \mathcal{H}^k(2)$.

Therefore, by Complete Transversal Theorem for each jet of order $k \geq 2$ of $\hat{X}$, there exists $h_{(k)} \in \hat{\mathcal{R}}_1^k$ such that

$$h_{(k)}{*}(\hat{X}) = \left(x_2 + \sum_{i=2}^{k} a_i x_1^i \right) \frac{\partial}{\partial x_1} + \sum_{i=2}^{k} b_i x_1^i \frac{\partial}{\partial x_2}.$$

With this recursive process, we find a formal diffeomorphism tangent to the identity such that $\hat{X}$ is equivalent to (4.1), that corresponds to the Takens normal form. $\square$

Notice that it is possible to write the elements of $\operatorname{Im}(\operatorname{ad}^k_{X_1})$, in a different way of the expression described in the proof of the Theorem 4.1. So, we can get many distinct subspace T complementary in $\mathcal{H}^k(2)$, that is, for nilpotent vector fields $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2,0)$, it is possible to find different normal form. In the next theorem we illustrate this situation presenting another normal form.

Theorem 4.3. *For $\hat{X} \in \hat{\mathfrak{X}}(\mathbb{C}^2,0)$ with $j^1(\hat{X}) = x_2\frac{\partial}{\partial x_1}$, by a formal diffeomorphism tangent to the identity, the vector field $\hat{X}$ is formally equivalent to the vector field*

$$x_2\frac{\partial}{\partial x_1} + \Big(a'(x_1) + x_2 b'(x_1)\Big)\frac{\partial}{\partial x_2},$$

where $m_0(a') \geq 2$ and $m_0(b') \geq 1$.

Proof. Recall that, the elements of $\operatorname{Im}(\operatorname{ad}^k_{X_1})$ are of the form $[X_1, F]$, where $F \in \mathcal{H}^k(2)$.
Given $F = \sum\limits_{|I|=k} \alpha_I x^I \dfrac{\partial}{\partial x_1} + \sum\limits_{|I|=k} \beta_I x^I \dfrac{\partial}{\partial x_2} \in \mathcal{H}^k(2)$, we have

$$\begin{aligned}
[X_1,F] &= \sum_{|I|=k} \alpha_{_I}(i_1 x_1^{i_1-1}x_2^{i_2+1})\frac{\partial}{\partial x_1} + \sum_{|I|=k}\Big(-\beta_I x^I\frac{\partial}{\partial x_1} + \beta_{_I}(i_1 x_1^{i_1-1}x_2^{i_2+1})\frac{\partial}{\partial x_2}\Big)\\
&= \alpha_{(k,0)} k x_1^{k-1} x_2\frac{\partial}{\partial x_1} + \cdots + \alpha_{(2,k-2)} 2x_1 x_2^{k-1}\frac{\partial}{\partial x_1} + \alpha_{(1,k-1)} x_2^k\frac{\partial}{\partial x_1} +\\
&\quad + \Big(-\beta_{(k,0)}x_1^k\frac{\partial}{\partial x_1} + \beta_{(k,0)} k x_1^{k-1}x_2\frac{\partial}{\partial x_2}\Big) + \cdots +\\
&\quad + \Big(-\beta_{(1,k-1)}x_1 x_2^{k-1}\frac{\partial}{\partial x_1} + \beta_{(1,k-1)}x_2^k\frac{\partial}{\partial x_2}\Big) + -\beta_{(0,k)}x_2^k\frac{\partial}{\partial x_1}.
\end{aligned}$$

That is,

$$\begin{aligned}
[X_1,F] &= \alpha_{(k,0)} k x_1^{k-1} x_2\frac{\partial}{\partial x_1} + \cdots + \alpha_{(2,k-2)} 2x_1 x_2^{k-1}\frac{\partial}{\partial x_1} + (\alpha_{(1,k-1)} - \beta_{(0,k)}) x_2^k\frac{\partial}{\partial x_1} +\\
&\quad + \Big(-\beta_{(k,0)}x_1^k\frac{\partial}{\partial x_1} + \beta_{(k,0)} k x_1^{k-1}x_2\frac{\partial}{\partial x_2}\Big) + \cdots\cdots +\\
&\quad + \Big(-\beta_{(1,k-1)}x_1 x_2^{k-1}\frac{\partial}{\partial x_1} + \beta_{(1,k-1)}x_2^k\frac{\partial}{\partial x_2}\Big) \in \operatorname{Im}(\operatorname{ad}^k_{X_1}).
\end{aligned}$$

Let $w = \sum\limits_{|I|=k} c_{1,I} x^I\dfrac{\partial}{\partial x_1} + \sum\limits_{|I|=k} c_{2,I}x^I\dfrac{\partial}{\partial x_2}$ be an element of $W = \mathcal{H}^k(2)$. So, we can write

$$\begin{aligned}
&w = c_{2,(k,0)}x_1^k\frac{\partial}{\partial x_2} + (c_{2,(k-1,1)} - k\beta_{(k,0)})x_1^{k-1}x_2\frac{\partial}{\partial x_2} +\\
&+\Big(-\beta_{(k,0)}x_1^k + (k\alpha_{(k,0)} - \beta_{(k-1,1)})x_1^{k-1}x_2 + \cdots + (\alpha_{(1,k-1)} - \beta_{(0,k)})x_2^k\Big)\frac{\partial}{\partial x_1} +\\
&+\Big(k\beta_{(k,0)}x_1^{k-1}x_2 + (k-1)\beta_{(k-1,1)}x_1^{k-2}x_2^2 + \cdots + \beta_{(1,k-1)}x_2^k\Big)\frac{\partial}{\partial x_2} \in T \oplus \operatorname{Im}(\operatorname{ad}^k_{X_1})
\end{aligned}$$

where $c_{2,(k,0)}x_1^k\dfrac{\partial}{\partial x_2} + (c_{2,(k-1,1)} - k\beta_{(k,0)})x_1^{k-1}x_2\dfrac{\partial}{\partial x_2} \in T.$

Hence, we have another choice for the subspace T and consequently another way of expressing the normal form for the nilpotent vector fields in dimension 2 that presented in the previous theorem.

By Complete Transversal Theorem for each jet of order $k \geq 2$ of vector field $\hat{X}$, there exists a diffeomorphism $g_{(k)} \in \hat{\mathcal{R}}_1^k$ such that

$$g_{(k)} * (\hat{X}) = x_2 \frac{\partial}{\partial x_1} + \left(\sum_{i=2}^{k} a_i' x_1^i + x_2 (b_i' x_1^{i-1}) \right) \frac{\partial}{\partial x_2}.$$

□

Let us consider the vector fields $\hat{V} \in \hat{\mathfrak{X}}(\mathbb{C}^3, 0)$ of the form $\hat{V} = \mathsf{X} + h.o.t.$ where

$$\mathsf{X} = 2x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}$$

is a linear nilpotent vector field.

Our goal is to present the normal form of such vector fields. For this purpose we define the following linear vector fields

$$\mathsf{Y} = x_1 \frac{\partial}{\partial x_2} + 2x_2 \frac{\partial}{\partial x_3}, \ \mathsf{H} = -2x_1 \frac{\partial}{\partial x_1} + 2x_3 \frac{\partial}{\partial x_3}.$$

The vector fields X and Y treated as a differentiation of the ring $\mathbb{C}[x] = \mathbb{C}[x_1, x_2, x_3]$ are called *locally nilpotent derivation*(see [15]).

For the vector field Y we associate its ring of constants,

$$\mathbb{C}[x]^{\mathsf{Y}} = \{ f \in \mathbb{C}[x] / \mathsf{Y} f = 0 \}.$$

We have $\mathbb{C}[x]^{\mathsf{Y}} = \mathbb{C}[x_1, G_2]$, where $G_2 = x_1 x_3 - x_2^2$ (see Remark 2 in [15]).
Note that $G_2 = x_1 x_3 - x_2^2$ is also first integral for the vector field X, and therefore first integral for $\mathsf{H} = -2x_1 \frac{\partial}{\partial x_1} + 2x_3 \frac{\partial}{\partial x_3}$, since those first integrals for the vector field Y which are also first integrals for the vector field X are first integrals for the vector field H.

Let $\mathbb{C}[x]_k$ be the subspace of $\mathbb{C}[x]$ consisting of homogeneous polynomials of degree k. In the space $\mathbb{C}[x]_k$ we have,

$$\operatorname{Ker} \mathsf{Y} \oplus \operatorname{Im} \mathsf{X} = \mathbb{C}[x]_k.$$

See the proof of this result in [15].

Remark that, we have

$$\operatorname{Im} \mathsf{X} = span \Big\{ 2i_1 x_1^{i_1 - 1} x_2^{i_2 + 1} x_3^{i_3} + i_2 x_1^{i_1} x_2^{i_2 - 1} x_3^{i_3 + 1} / i_1 + i_2 + i_3 = k \Big\} \subset \mathbb{C}[x]_k.$$

Theorem 4.4. *For $\hat{V} \in \hat{\mathfrak{X}}(\mathbb{C}^3, 0)$ such that $j^1(\hat{V}) = 2x_2 \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_2}$, there exists a formal change of coordinates tangent to the identity reducing it to the Takens normal form*

$$\Big(2x_2 + x_1 \hat{F}_1(x_1, G_2) \Big) \frac{\partial}{\partial x_1} + \Big(x_3 + x_1 \hat{F}_2(x_1, G_2) \Big) \frac{\partial}{\partial x_2} + \hat{F}_3(x_1, G_2) \frac{\partial}{\partial x_3}$$

where $\hat{F}_j$ is a formal power series in x_1 and G_2 such that $m_0(\hat{F}_j) \geq 1$ for $j = 1, 2$ and $m_0(\hat{F}_3) \geq 2$.

Proof. We consider $W = \mathcal{H}^k(3) = \sum_{j=1}^{3} \operatorname{Ker} \mathsf{Y} \oplus \operatorname{Im} \mathsf{X} \frac{\partial}{\partial x_j}$ in the prenormalization process for each $k \geq 2$. The prenormal form for each jet of order k of $\hat{V} \in \hat{\mathfrak{X}}(\mathbb{C}^3, 0)$, is given by the subspace T such that $W = \operatorname{Im}(\mathrm{ad}_{\mathsf{X}}^k) \oplus T$. We calculate $\operatorname{Im}(\mathrm{ad}_{\mathsf{X}}^k)$, that is, $[\mathsf{X}, F]$ where $F \in \mathcal{H}^k(3)$.

Given $F = \sum_{|I|=k} \alpha_I x^I \frac{\partial}{\partial x_1} + \sum_{|I|=k} \beta_I x^I \frac{\partial}{\partial x_2} + \sum_{|I|=k} \gamma_I x^I \frac{\partial}{\partial x_3} \in \mathcal{H}^k(3)$, we have that

$$\begin{aligned}[\mathsf{X}, F] = & \sum_{|I|=k} \Big(-2\beta_I x^I + \alpha_I \Big(2i_1 x_1^{i_1-1} x_2^{i_2+1} x_3^{i_3} + i_2 x_1^{i_1} x_2^{i_2-1} x_3^{i_3+1} \Big)\Big) \frac{\partial}{\partial x_1} + \\ & + \sum_{|I|=k} \Big(-\gamma_I x^I + \beta_I \Big(2i_1 x_1^{i_1-1} x_2^{i_2+1} x_3^{i_3} + i_2 x_1^{i_1} x_2^{i_2-1} x_3^{i_3+1} \Big)\Big) \frac{\partial}{\partial x_2} + \\ & + \sum_{|I|=k} \gamma_I \Big(2i_1 x_1^{i_1-1} x_2^{i_2+1} x_3^{i_3} + i_2 x_1^{i_1} x_2^{i_2-1} x_3^{i_3+1} \Big) \frac{\partial}{\partial x_3}.\end{aligned}$$

Therefore, we have

$$[\mathsf{X}, F] = (-2F_2 + \mathsf{X}(F_1)) \frac{\partial}{\partial x_1} + (-F_3 + \mathsf{X}(F_2)) \frac{\partial}{\partial x_2} + \mathsf{X}(F_3) \frac{\partial}{\partial x_3}.$$

• For $k = 2r+1$, for $r \in \mathbb{N}$: $\operatorname{Ker}\mathsf{Y} = span\Big\{ x_1^{2r+1}, x_1^{2r-1}G_2, \ldots, x_1 G_2^r \Big\}$.
Remark that the third component of the action of $\mathrm{ad}_{\mathsf{X}}^k$ on F is $\mathsf{X}(F_3)$, and $\operatorname{Ker}\mathsf{Y}$ is complementary to $\mathsf{X}(F_3)$ in the space $\mathbb{C}[x]_k$. In the same way, $\operatorname{Ker}\mathsf{Y}$ is complementary to the first and the second component of the action of $\mathrm{ad}_{\mathsf{X}}^k$ on F. So, $T = \sum_{j=1}^3 P_j(x_1, G_2) \frac{\partial}{\partial x_j}$, where $P_j \in \operatorname{Ker}\mathsf{Y} \subset \mathbb{C}[x]_{2r+1}$.

• For $k = 2r$, for $r \in \mathbb{N}$: $\operatorname{Ker}\mathsf{Y} = span\Big\{ x_1^{2r}, x_1^{2r-2}G_2, \ldots, G_2^r \Big\}$.
We can consider $(F_3 + \widetilde{F}_3)$ a homogeneous polynomial of degree k, that is, we can write $\mathsf{X}(F_3) = \mathsf{X}(F_3 + \widetilde{\gamma}_{(r,0,r)} G_2^r)$. So, we can proceed an additional cancellation in the second component of the action of $\mathrm{ad}_{\mathsf{X}}^k$ on $(F_3 + \widetilde{\gamma}_{(r,0,r)} G_2^r)$, that is, we have $(-F_3 - \widetilde{\gamma}_{(r,0,r)} G_2^r + \mathsf{X}(F_2))$, and its complement in the space $\mathbb{C}[x]_k$ is $span\{x_1^{2r}, x_1^{2r-2}G_2, \ldots, x_1^2 G_2^{r-1}\}$. Similarly we can also consider $\mathsf{X}(F_2) = \mathsf{X}(F_2 + \widetilde{\beta}_{(r,0,r)} G_2^r)$ to achieve an additional cancellation in the first component of the action of $\mathrm{ad}_{\mathsf{X}}^k$ on $(F_2 + \widetilde{\beta}_{(r,0,r)} G_2^r)$.
So, $T = \sum_{j=1}^3 Q_j(x_1, G_2) \frac{\partial}{\partial x_j}$, $Q_j \in span\Big\{ x_1^{2r}, x_1^{2r-2}G_2, \ldots, x_1^2 G_2^{r-1} \Big\}$, for $j \in \{1, 2\}$ and $Q_3 \in \operatorname{Ker}\mathsf{Y} \subset \mathbb{C}[x]_{2r}$.

By Theorem 2.2, there exits a change of coordinates tangent to the identity, successively for each $k \geq 2$ such that we get the Takens normal form in dimension 3. □

Remark 4.5. *Our method is in accordance with the method of Stróżyna, E. and Żołądek, H. developed in [15], because they use the complement of* $\operatorname{Im} \mathrm{ad}_{\mathsf{X}}^k$*, to obtain the Takens normal form. We stress the relevance of the Complete Transversal Theorem for vector fields, because we get a change of polynomial coordinates for each jet of order* k *of the vector field.*

Remark 4.6. *For the normal form of the vector fields* $\hat{V} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ *such that*

$$\hat{V} = \mathtt{X} + h.o.t. \tag{4.2}$$

where

$$\mathtt{X} = (n-1)x_2 \frac{\partial}{\partial x_1} + (n-2)x_3 \frac{\partial}{\partial x_2} + \cdots + x_n \frac{\partial}{\partial x_{n-1}},$$

define the following vector fields

$$\begin{aligned}\mathtt{Y} &= x_1 \tfrac{\partial}{\partial x_2} + 2x_2 \tfrac{\partial}{\partial x_3} + \cdots + (n-1)x_{n-1} \tfrac{\partial}{\partial x_n} \\ \mathtt{H} &= -(n-1)x_1 \tfrac{\partial}{\partial x_1} - (n-3)x_2 \tfrac{\partial}{\partial x_2} + \cdots + (n-1)x_n \tfrac{\partial}{\partial x_n}.\end{aligned}$$

The vector field $\mathtt{H}$ *defining a quasi-homogeneous graduation in* $\mathbb{C}[x_1,\ldots,x_n]$, $deg_{\mathtt{H}}\, x_j = 2j\text{–}n\text{–}1$. *Stróżyna, E. y Żoła̧dek, H. show in [16] that the Takens normal form is unique for the vector fields* (4.2). *For* $n \geq 2$, *we get*

$$\mathtt{X} + F_1\frac{\partial}{\partial x_1} + \cdots + F_n\frac{\partial}{\partial x_n}$$

where the series F_j *satisfy* $\mathtt{Y}F_j \equiv 0$ *and the series* $F_1,\ldots,F_{n-1}$ *contain only terms with* $deg_{\mathtt{H}} < 0$.

However, when $n = 2\,(\mathbb{C}[x_1,x_2]^{\mathtt{Y}} = \mathbb{C}[x_1])$, *we show in Theorem 4.3, that the Takens normal form, is not unique because it depends of the subspace* T *which is complementary to* $\operatorname{Im}(\operatorname{ad}^k_{\mathtt{X}})$ *in* $W = \mathcal{H}^k(2)$.

In the normal form of the vector field $\hat{V} \in \hat{\mathfrak{X}}(\mathbb{C}^3,0)$ *given by Theorem 4.4, we use that* $\mathbb{C}[x_1,x_2,x_3]^{\mathtt{Y}} = \mathbb{C}[x_1,G_2]$, *this result allowed us to obtain the Takens normal form for* $n = 3$. *For* $n = 4$, *the ring of constants of the derivation* $\mathtt{Y}$ *is not equal the polynomial ring of three polynomials.*

Stróżyna, E. and Żoła̧dek, H. get an expression for the ring $\mathbb{C}[x_1,\ldots,x_n]^{\mathtt{Y}}$, *which is used to determine the Takens normal form of the vector field given in (4.2) for* $n \geq 4$ *(see [15]).*

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