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Finite Summation Formulas for the Multivariable A-function

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ABSTRACT: The object of this paper is to evaluate some finite double summations relations for the multivariable A-function using the summation of a double hypergeometric series. The formulas derived in this paper are most general in character, we also provide a few particular cases for derived summation formulas.

Key Words: Multivariable A-function, finite double summation, double hypergeometric series.

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1. Introduction and Preliminaries

The multivariable A-function defined by Gautam et al. [7], and it is an extension of the multivariable H-function [11,13]. The multivariable A-function is given by the following manner:

$$A(z_1,\cdots,z_r)=$$
$$A^{m,n:m_1,n_1;\cdots;m_r,n_r}_{p,q:p_1,q_1;\cdots;p_r,q_r}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{l}\left(a_j;A_j^{(1)},\cdots,A_j^{(r)}\right)_{1,p}:\left(c_j^{(1)},C_j^{(1)}\right)_{1,p_1};\cdots;\left(c_j^{(r)},C_j^{(r)}\right)_{1,p_r}\\ \\ \left(b_j;B_j^{(1)},\cdots,B_j^{(r)}\right)_{1,q}:\left(d_j^{(1)},D_j^{(1)}\right)_{1,q_1};\cdots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,q_r}\end{array}\right)$$
$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi'(t_1,\cdots,t_r)\prod_{i=1}^{r}\theta_i'(t_i)\,z_i^{t_i}\,\mathrm{d}t_1\cdots\mathrm{d}t_r, \tag{1.1}$$

where $\phi'(t_1,\cdots,t_r)$ and $\theta_i'(t_i)$ $(i=1,\cdots,r)$ are given by

$$\phi'(t_1,\cdots,t_r)=\frac{\prod_{j=1}^{m}\Gamma\left(b_j-\sum_{i=1}^{r}B_j^{(i)}t_i\right)\prod_{j=1}^{n}\Gamma\left(1-a_j+\sum_{i=1}^{r}A_j^{(i)}t_j\right)}{\prod_{j=n+1}^{p}\Gamma\left(a_j-\sum_{i=1}^{r}A_j^{(i)}t_j\right)\prod_{j=m+1}^{q}\Gamma\left(1-b_j+\sum_{i=1}^{r}B_j^{(i)}t_j\right)}, \tag{1.2}$$

$$\theta_i'(t_i)=\frac{\prod_{j=1}^{n_i}\Gamma\left(1-c_j^{(i)}+C_j^{(i)}t_i\right)\prod_{j=1}^{m_i}\Gamma\left(d_j^{(i)}-D_j^{(i)}t_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma\left(c_j^{(i)}-C_j^{(i)}t_i\right)\prod_{j=m_i+1}^{q_i}\Gamma\left(1-d_j^{(i)}+D_j^{(i)}t_i\right)}, \tag{1.3}$$

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here, $m, n, p, q, m_i, n_i, p_i, c_i \in \mathbb{N}_0 \ (i = 1, \cdots, r)$; $a_j, b_j, c_j^{(i)}, d_j^{(i)}, A_j^{(i)}, B_j^{(i)}, C_j^{(i)}, D_j^{(i)} \in \mathbb{C}$.

The multiple integral defining the A-function of r variables converges absolutely if

$$|\arg(\Omega_i')\, z_i| < \frac{1}{2}\eta_i'\pi, \quad \xi_i'^{*} = 0, \quad \eta_i' > 0\,(i = 1, \ldots, r), \tag{1.4}$$

where

$$\Omega_i' := \prod_{j=1}^{p}\left\{A_j^{(i)}\right\}^{A_j^{(i)}} \prod_{j=1}^{q}\left\{B_j^{(i)}\right\}^{-B_j^{(i)}} \prod_{j=1}^{q_i}\left\{D_j^{(i)}\right\}^{D_j^{(i)}} \prod_{j=1}^{p_i}\left\{C_j^{(i)}\right\}^{-C_j^{(i)}}, \tag{1.5}$$

$$\xi_i'^{*} := \Im\left(\sum_{j=1}^{p} A_j^{(i)} - \sum_{j=1}^{q} B_j^{(i)} + \sum_{j=1}^{q_i} D_j^{(i)} - \sum_{j=1}^{p_i} C_j^{(i)}\right), \tag{1.6}$$

$$\begin{aligned}\eta_i' \;:=\; \Re\Bigg(&\sum_{j=1}^{n} A_j^{(i)} - \sum_{j=n+1}^{p} A_j^{(i)} + \sum_{j=1}^{m} B_j^{(i)} - \sum_{j=m+1}^{q} B_j^{(i)} + \sum_{j=1}^{m_i} D_j^{(i)} \\ &- \sum_{j=m_i+1}^{q_i} D_j^{(i)} + \sum_{j=1}^{n_i} C_j^{(i)} - \sum_{j=n_i+1}^{p_i} C_j^{(i)}\Bigg).\end{aligned} \tag{1.7}$$

In this paper, we use the following notations.

$$X := m_1, n_1; \cdots; m_r, n_r; \quad Y := p_1, q_1; \cdots; p_r, q_r; \tag{1.8}$$

$$\mathbb{A} := \left(a_j; A_j^{(1)}, \cdots, A_j^{(r)}\right)_{1,p}, \ \mathbb{B} = \left(b_j; B_j^{(1)}, \cdots, B_j^{(r)}\right)_{1,q}, \tag{1.9}$$

$$\mathbb{C} = \left(c_j^{(1)}, C_j^{(1)}\right)_{1,p_1}; \cdots; \left(c_j^{(r)}, C_j^{(r)}\right)_{1,p_r}, \ \mathbb{D} = \left(d_j^{(1)}, D_j^{(1)}\right)_{1,q_1}; \cdots; \left(d_j^{(r)}, D_j^{(r)}\right)_{1,q_r}. \tag{1.10}$$

2. Finite double series formulas

In this section three summation formulas for the multivariable A-function have been derived. Throughout this paper, the number $h_i, k_i \ (i = 1, \cdots, r)$ are positive.

Theorem 2.1. *We have the following summation formula:*

$$\begin{aligned}&\sum_{g=0}^{m}\sum_{k=0}^{n} \frac{(-m)_g\,(-n)_k\,(b)_{g+k}}{g!k!\,(b)_g\,(b)_k} \\ &\times A_{p+2,q+1:Y}^{\mathbf{m},\mathbf{n}+2:X}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array}\middle|\begin{array}{c}(1-d_1-g; h_1, \cdots, h_r), (1-d_2-k; k_1, \cdots, k_r), \mathbb{A} : \mathbb{C} \\ \\ \mathbb{B}, (1-d_1-d_2-g-k; h_1+k_1, \cdots, h_r+k_r) : \mathbb{D}\end{array}\right) \\ &= \frac{(b)_{m+n}}{(b)_m\,(b)_n} A_{p+2,q+1:Y}^{\mathbf{m},\mathbf{n}+2:X}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array}\middle|\begin{array}{c}(1-d_1-n; h_1, \cdots, h_r), (1-d_2-m; k_1, \cdots, k_r), \mathbb{A} : \mathbb{C} \\ \\ \mathbb{B}, (1-d_1-d_2-m-n; h_1+k_1, \cdots, h_r+k_r) : \mathbb{D}\end{array}\right).\end{aligned} \tag{2.1}$$

Proof. At first we express the multivariable A-function occurring on the left hand side of (2.1) in terms of Mellin-Barnes type integrals contour as given in (1.1). Next, by interchanging the order of integration and summation which is permissible as the series involved are finite, and using the following result due to Carlitz [3].

$$\sum_{r=0}^{m}\sum_{s=0}^{n} \frac{(-m)_r\,(-n)_s\,(b)_{r+s}\,(c)_r\,(d)_s}{r!s!\,(b)_r\,(b)_s\,(c+d)_{r+s}} = \frac{(b)_{m+n}\,(d)_m\,(c)_n}{(b)_m\,(b)_n\,(c+d)_{m+n}}, \tag{2.2}$$

then we arrive at the desired result (2.1). □

Theorem 2.2. *We have following formula holds true*

$$\sum_{g=0}^{m}\sum_{k=0}^{n}\frac{(-m)_k(-n)_k(b-e-n+1)_g(b)_k}{g!k!(2-e-m-n)_s(e)_k}$$
$$\times\ A^{\mathbf{m},\mathbf{n}+1:X}_{p+1,q+2:Y}\left(\begin{array}{c|c} z_1 & (1-c-g-k;h_1,\cdots,h_r),\mathbb{A}:\mathbb{C}\\ \cdot & \\ \cdot & \\ z_r & \mathbb{B},(1-c-g;h_1+k_1,\cdots,h_r+k_r),(1-c-k;h_1,\cdots,h_r):\mathbb{D}\end{array}\right)$$
$$=\frac{(b)_m(e-b)_n}{(e)_n(e+n-1)_m}$$
$$\times\ A^{\mathbf{m},\mathbf{n}+1:X}_{p+1,q+2:Y}\left(\begin{array}{c|c} z_1 & (1-c-m-n;h_1,\cdots,h_r),\mathbb{A}:\mathbb{C}\\ \cdot & \\ \cdot & \\ z_r & \mathbb{B},(1-c-m;h_1,\cdots,h_r),(1-c-n;h_1,\cdots,h_r):\mathbb{D}\end{array}\right), \tag{2.3}$$

where, $a-b\notin\mathbb{Z}$ *and* $a=b-e-n+1, b=a-d-m+1$.

Proof. The right side of (2.3) is obtained on interpreting the resulting integrals contour with the help of (1.1). Next we use the following result of Carlitz [4]:

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(-n)_s(a)_r(b)_s(c)_{r+s}}{r!s!(d)_r(e)_s(c)_r(c)_s}=\frac{(b)_m(a)_n(c)_{m+n}}{(b-a)_m(a-b)_n(c)_m(c)_n}. \tag{2.4}$$

Then, the desired result (2.3) can be established in a similar manner. □

Theorem 2.3. *We have*

$$\sum_{g=0}^{m}\sum_{k=0}^{n}\frac{(-m)_g(-n)_k}{g!k!(b-d+m+1)_g(b-c-n+1)_k}$$
$$\times\ A^{\mathbf{m},\mathbf{n}+3:X}_{p+3,q+1:Y}\left(\begin{array}{c|c} z_1 & (1-c-g;h_1,\cdots,h_r),(1-d-k;h_1,\cdots,h_r),\\ \cdot & \\ \cdot & \\ z_r & \mathbb{B},(1-c-d-g-k;2h_1,\cdots,2h_r):\end{array}\right.$$
$$\left.\begin{array}{c} (1-b-g-k;h_1,\cdots,h_r),\mathbb{A}:\mathbb{C}\\ \\ \mathbb{D}\end{array}\right)$$
$$=\frac{1}{(d-b)_m(c-b)_n}A^{\mathbf{m},\mathbf{n}+3:X}_{p+3,q+1:Y}\left(\begin{array}{c|c} z_1 & (1-c-n;h_1,\cdots,h_r),(1-d-m;h_1,\cdots,h_r),\\ \cdot & \\ \cdot & \\ z_r & \mathbb{B},(1-c-d-m-n;2h_1,\cdots,2h_r):\end{array}\right.$$
$$\left.\begin{array}{c} (1+b-c-d-m-n;h_1,\cdots,h_r),\mathbb{A}:\mathbb{C}\\ \\ \mathbb{D}\end{array}\right). \tag{2.5}$$

Proof. The result (2.5) can be established in a similar manner if we use the result due to Carlitz [2], as given by

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{(-m)_r(-n)_s(b)_{r+s}(c)_r(d)_s}{r!s!(c+d)_{r+s}(b-c-n+1)_s(b-d-m+1)_r}=\frac{(c+d-b)_{m+n}(d)_m(c)_n}{(c+d)_{m+n}(d-b)_m(c-b)_n}. \tag{2.6}$$

□

3. Particular case

Corollary 3.1. *If we take $n=0$ in (2.1), then the double finite series reduces to the single finite series for the multivariable A-function.*

$$\sum_{g=0}^{m}\frac{(-m)_g}{g!}A^{\mathbf{m},\mathbf{n}+2:X}_{p+2,q+1:Y}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(1-d_1-g;h_1,\cdots,h_r),(1-d_2;k_1,\cdots,k_r),\mathbb{A}:\mathbb{C}\\ \\ \mathbb{B},(1-d_1-d_2-g;h_1+k_1,\cdots,h_r+k_r):\mathbb{D}\end{array}\right)$$
$$=A^{\mathbf{m},\mathbf{n}+2:X}_{p+2,q+1:Y}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(1-d_1-m;h_1,\cdots,h_r),(1-d_1;k_1,\cdots,k_r),\mathbb{A}:\mathbb{C}\\ \\ \mathbb{B},(1-d_1-d_2-m;h_1+k_1,\cdots,h_r+k_r):\mathbb{D}\end{array}\right). \tag{3.1}$$

4. Multivariable H-function

If we take $\mathbf{m}=0$ and $A_j^{(i)},B_j^{(i)},C_j^{(i)},D_j^{(i)}\in\mathbb{R}$ $(i=1,\cdots,r)$ are real number in (2.1), (2.3), (2.5) and (3.1) then these formulas reduce to the corresponding formulas involving multivariable H-function [5,6,11,13,14].
Here, we use another notations because the following numbers $A_j^{(i)},B_j^{(i)},C_j^{(i)}$ and $D_j^{(i)}$ are reals, they are no longer complex numbers unlike section 2.

$$\mathbb{A}_1=\left(a_j;A_j^{(1)},\cdots,A_j^{(r)}\right)_{1,p},\ \mathbb{B}_1=\left(b_j;B_j^{(1)},\cdots,B_j^{(r)}\right)_{1,q}, \tag{4.1}$$

$$\mathbb{C}_1=\left(c_j^{(1)},C_j^{(1)}\right)_{1,p_1};\cdots;\left(c_j^{(r)},C_j^{(r)}\right)_{1,p_r},\ \mathbb{D}_1=\left(d_j^{(1)},D_j^{(1)}\right)_{1,q_1};\cdots;\left(d_j^{(r)},D_j^{(r)}\right)_{1,q_r}. \tag{4.2}$$

We have the following results:

Corollary 4.1.

$$\sum_{g=0}^{m}\sum_{k=0}^{n}\frac{(-m)_g(-n)_k(b)_{g+k}}{g!k!(b)_g(b)_k}$$
$$\times\ H^{0,\mathbf{n}+2:X}_{p+2,q+1:Y}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(1-d_1-g;h_1,\cdots,h_r),(1-d_2-k;k_1,\cdots,k_r),\mathbb{A}_1:\mathbb{C}_1\\ \\ \mathbb{B}_1,(1-d_1-d_2-g-k;h_1+k_1,\cdots,h_r+k_r):\mathbb{D}_1\end{array}\right)$$
$$=\frac{(b)_{m+n}}{(b)_m(b)_n}H^{0,\mathbf{n}+2:X}_{p+2,q+1:Y}\left(\begin{array}{c}z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(1-d_1-n;h_1,\cdots,h_r),(1-d_2-m;k_1,\cdots,k_r),\mathbb{A}_1:\mathbb{C}_1\\ \\ \mathbb{B}_1,(1-d_1-d_2-m-n;h_1+k_1,\cdots,h_r+k_r):\mathbb{D}_1\end{array}\right), \tag{4.3}$$

under the same existence conditions that stated in theorem 2.1.

Corollary 4.2.

$$\sum_{g=0}^{m}\sum_{k=0}^{n}\frac{(-m)_k(-n)_k(b-e-n+1)_g(b)_k}{g!k!(2-e-m-n)_s(e)_k}$$
$$\times\ H^{0,\mathbf{n}+1:X}_{p+1,q+2:Y}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(1-c-g-k;h_1,\cdots,h_r),\mathbb{A}_1:\mathbb{C}_1\\ \mathbb{B}_1,(1-c-g;h_1+k_1,\cdots,h_r+k_r),(1-c-k;h_1,\cdots,h_r):\mathbb{D}_1\end{array}\right)$$
$$=\frac{(b)_m(e-b)_n}{(e)_n(e+n-1)_m}$$
$$\times\ H^{0,\mathbf{n}+1:X}_{p+1,q+2:Y}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(1-c-m-n;h_1,\cdots,h_r),\mathbb{A}_1:\mathbb{C}_1\\ \mathbb{B}_1,(1-c-m;h_1,\cdots,h_r),(1-c-n;h_1,\cdots,h_r):\mathbb{D}_1\end{array}\right),\tag{4.4}$$

$a-b$ is not integer, and $a=b-e-n+1, b=a-d-m+1$. Also, satisfy the same existence conditions as stated in theorem 2.2.

Corollary 4.3.

$$\sum_{g=0}^{m}\sum_{k=0}^{n}\frac{(-m)_g(-n)_k}{g!k!(b-d+m+1)_g(b-c-n+1)_k}H^{0,\mathbf{n}+3:X}_{p+3,q+1:Y}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(1-c-g;h_1,\cdots,h_r),(1-d-k;h_1,\cdots,h_r),(1-b-g-k;h_1,\cdots,h_r),\mathbb{A}_1:\mathbb{C}_1\\ \mathbb{B}_1,(1-c-d-g-k;2h_1,\cdots,2h_r):\mathbb{D}_1\end{array}\right)$$
$$=\frac{1}{(d-b)_m(c-b)_n}H^{0,\mathbf{n}+3:X}_{p+3,q+1:Y}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(1-c-n;h_1,\cdots,h_r),(1-d-m;h_1,\cdots,h_r),(1+b-c-d-m-n;h_1,\cdots,h_r),\mathbb{A}_1:\mathbb{C}_1\\ \mathbb{B}_1,(1-c-d-m-n;2h_1,\cdots,2h_r):\mathbb{D}_1\end{array}\right),\tag{4.5}$$

under the same conditions that theorem 2.3.

In particular, we have

Corollary 4.4. *If we set $n=0$ in (4.3), the double finite series reduces to the single finite series for the multivariable H-function.*

$$\sum_{g=0}^{m}\frac{(-m)_g}{g!}H^{0,\mathbf{n}+2:X}_{p+2,q+1:Y}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(1-d_1-g;h_1,\cdots,h_r),(1-d_2;k_1,\cdots,k_r),\mathbb{A}_1:\mathbb{C}_1\\ \mathbb{B}_1,(1-d_1-d_2-g;h_1+k_1,\cdots,h_r+k_r):\mathbb{D}_1\end{array}\right)$$
$$=H^{0,\mathbf{n}+2:X}_{p+2,q+1:Y}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ z_r\end{array}\middle|\begin{array}{c}(1-d_1-m;h_1,\cdots,h_r),(1-d_1;k_1,\cdots,k_r),\mathbb{A}_1:\mathbb{C}_1\\ \mathbb{B}_1,(1-d_1-d_2-m;h_1+k_1,\cdots,h_r+k_r):\mathbb{D}_1\end{array}\right).\tag{4.6}$$

5. H-function of two variables

If we take $r=2$, then the multivariable H-function reduces to H-function of two variables defined by Gupta and Mittal [8] (see also, [12]). Here, we note

$$\mathbb{A}_2=\left(a_j;A'_j,A''_j\right)_{1,p},\ \mathbb{B}_2=\left(b_j;B'_j,B''_j\right)_{1,q},\tag{5.1}$$

$$\mathbb{C}_2 = \left(c'_j, C'_j\right)_{1,p_1}; \left(c''_j, C''_j\right)_{1,p_2}, \ \mathbb{D}_2 = \left(d'_j, D'_j\right)_{1,q_1}; \left(d''_j, D''_j\right)_{1,q_2}. \tag{5.2}$$

Then, we have following results:

Corollary 5.1.

$$\sum_{g=0}^{m}\sum_{k=0}^{n} \frac{(-m)_g (-n)_k (b)_{g+k}}{g!k! (b)_g (b)_k} \times H^{0,\mathbf{n}+2:m_1,n_1:m_2,n_2}_{p+2,q+1:p_1,q_1;p_2,q_2}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\middle| \begin{array}{c} (1-d_1-g;h_1,h_2),(1-d_2-k;k_1,k_2),\mathbb{A}_2:\mathbb{C}_2 \\ \mathbb{B}_2,(1-d_1-d_2-g-k;h_1+k_1,h_2+k_2):\mathbb{D}_2 \end{array}\right)$$
$$= \frac{(b)_{m+n}}{(b)_m (b)_n} \times H^{0,\mathbf{n}+2:m_1,n_1:m_2,n_2}_{p+2,q+1:p_1,q_1;p_2,q_2}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\middle| \begin{array}{c} (1-d_1-n;h_1,h_2),(1-d_2-m;k_1,k_2),\mathbb{A}_2:\mathbb{C}_2 \\ \mathbb{B}_2,(1-d_1-d_2-m-n;h_1+k_1,\cdots,h_2+k_2):\mathbb{D}_2 \end{array}\right), \tag{5.3}$$

under the same conditions as stated in theorem 2.1.

Corollary 5.2.

$$\sum_{g=0}^{m}\sum_{k=0}^{n} \frac{(-m)_k (-n)_k (b-e-n+1)_g (b)_k}{g!k! (2-e-m-n)_s (e)_k} \times H^{0,\mathbf{n}+1:m_1,n_1:m_2,n_2}_{p+1,q+2:p_1,q_1;p_2,q_2}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\middle| \begin{array}{c} (1-c-g-k;h_1,h_2);\mathbb{A}_2:\mathbb{C}_2 \\ \mathbb{B}_2,(1-c-g;h_1+k_1,h_2+k_2),(1-c-k;h_1,h_2):\mathbb{D}_2 \end{array}\right)$$
$$= \frac{(b)_m (e-b)_n}{(e)_n (e+n-1)_m} \times H^{0,\mathbf{n}+1:m_1,n_1:m_2,n_2}_{p+1,q+2:p_1,q_1;p_2,q_2}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\middle| \begin{array}{c} (1-c-m-n;h_1,h_2),\mathbb{A}_2:\mathbb{C}_2 \\ \mathbb{B}_2,(1-c-m;h_1,h_2),(1-c-n;h_1,h_2):\mathbb{D}_2 \end{array}\right), \tag{5.4}$$

$a-b \notin \mathbb{Z}, a = b-e-n+1, b = a-d-m+1$. *Also, satisfy the same existence conditions as stated in theorem 2.2.*

Corollary 5.3.

$$\sum_{g=0}^{m}\sum_{k=0}^{n}\frac{(-m)_g\,(-n)_k}{g!k!\,(b-d+m+1)_g\,(b-c-n+1)_k}H^{0,\mathbf{n}+3:m_1,n_1:m_2,n_2}_{p+3,q+1:p_1,q_1;p_2,q_2}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ z_2\end{array}\left|\begin{array}{c}(1-c-g;h_1,h_2)\,,\,(1-d-k;h_1,h_2)\,,(1-b-g-k;h_1,h_2)\,,\mathbb{A}_2:\mathbb{C}_2\\ \mathbb{B}_2,\,(1-c-d-g-k;2h_1,2h_2):\mathbb{D}_2\end{array}\right.\right)$$

$$=\frac{1}{(d-b)_m\,(c-b)_n}H^{0,\mathbf{n}+3:m_1,n_1:m_2,n_2}_{p+3,q+1:p_1,q_1;p_2,q_2}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ z_2\end{array}\left|\begin{array}{c}(1-c-n;h_1,h_2)\,,(1-d-m;h_1,h_2)\,,(1+b-c-d-m-n;h_1,h_2)\,,\mathbb{A}_2:\mathbb{C}_2\\ \mathbb{B}_2,\,(1-c-d-m-n;2h_1,2h_2):\mathbb{D}_2\end{array}\right.\right), \tag{5.5}$$

the above corollary satisfies the same conditions as stated in theorem 2.3.

In particular, we have

Corollary 5.4. *If we take $n=0$ in (5.3), then the double finite series reduces to the single finite series for the H-function of two variables.*

$$\sum_{g=0}^{m}\frac{(-m)_g}{g!}\,H^{0,\mathbf{n}+2:m_1,n_1:m_2,n_2}_{p+2,q+1:p_1,q_1;p_2,q_2}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ z_2\end{array}\left|\begin{array}{c}(1-d_1-g;h_1,h_2)\,,(1-d_2;k_1,k_2)\,,\mathbb{A}_2:\mathbb{C}_2\\ \mathbb{B}_2,(1-d_1-d_2-g;h_1+k_1,h_2+k_2):\mathbb{D}_2\end{array}\right.\right)$$

$$=H^{0,\mathbf{n}+2:m_1,n_1:m_2,n_2}_{p+2,q+1:p_1,q_1;p_2,q_2}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ z_2\end{array}\left|\begin{array}{c}(1-d_1-m;h_1,h_2)\,,(1-d_1;k_1,k_2)\,,\mathbb{A}_2:\mathbb{C}_2\\ \mathbb{B}_2,(1-d_1-d_2-m;h_1+k_1,h_2+k_2):\mathbb{D}_2\end{array}\right.\right). \tag{5.6}$$

6. Concluding Remark

By specializing the various parameters as well variables in the multivariable A-function, we can obtain a large number remarkably double and single sums of useful functions or product of such functions which are expressible in terms of E, F, G, H, I-function of one and several variables, also simpler special functions of one and several variables. Hence the formulas derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of pure & applied mathematics, mathematical physics, and other related area of research.

7. Compliance with Ethical Standards

Conflict of Interest: The authors declare that there is no conflict of interests regarding the publication of this paper.

Ethical approval: This article does not contain any studies with human participants performed by any of the authors.

References

1. F.Y. Ayant and D. Kumar, Generating relations and multivariable Aleph-function, *Analysis* **38**(3) (2018), 137–143.
2. L. Carlitz, A Saalschützian theorem for double series, *J. London Math. Soc.* **38**(1) (1963), 415–418.
3. L.Carlitz, A summation for double hypergeometric series, *Rend. Sem. Math. Univ. Padova* **37** (1967), 230–233.
4. L. Carlitz, Summation of a double hypergeometric series, *Matematiche (Catania)* **22** (1967), 138–142.
5. J. Choi, J. Daiya, D. Kumar and R.K. Saxena, Fractional differentiation of the product of Appell function F_3 and multivariable H-function, *Commun. Korean Math. Soc.* **31**(1) (2016), 115–129.
6. J. Daiya, J. Ram and D. Kumar, The Multivariable H-function and the general class of Srivastava polynomials involving the generalized Mellin-Barnes contour integrals, *Filomat* **30**(6) (2016), 1457–1464.
7. B.P. Gautam, A.S. Asgar and A.N. Goyal, On the multivariable A-function, *Vijnana Parishad Anusandhan Patrika* **29**(4) (1986), 67–81.
8. K.C. Gupta and P.K. Mittal, Integrals involving a generalized function of two variables, *Indian J. Pure Appl. Math.* **5** (1974), 430–437.
9. D. Kumar, F.Y. Ayant and J. Choi, Application of product of the multivariable A-function and the multivariable Srivastava polynomials, *East Asian Math. J.* **34**(3) (2018), 295–303.
10. D. Kumar, F.Y. Ayant and D. Kumar, A new class of integrals involving generalized hypergeometric function and multivariable Aleph-function, *Kragujev. J. Math.* **44**(4) (2020), 539–550.
11. D. Kumar, S.D. Purohit and J. Choi, Generalized fractional integrals involving product of multivariable H-function and a general class of polynomials, *J. Nonlinear Sci. Appl.* **9**(1) (2016), 8–21.
12. J. Ram and D. Kumar, Generalized fractional integration involving Appell hypergeometric of the product of two H-functions, *Vijanana Parishad Anusandhan Patrika* **54**(3) (2011), 33–43.
13. H.M. Srivastava and R. Panda, Some expansion theorems and generating relations for the H-function of several complex variables, *Comment. Math. Univ. St. Paul.* **24** (1975), 119–137.
14. H.M. Srivastava and R. Panda, Some bilateral generating function for a class of generalized hypergeometric polynomials, *J. Reine Angew. Math.* **283/284** (1976), 265–274.

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