Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–15.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.47913

Infinitely Many Solutions for a Class of Fractional Boundary Value Problem with p-Laplacian with Impulsive Effects

M. Abolghasemi and S. Moradi

ABSTRACT: The existence of infinitely many solutions for a class of impulsive fractional boundary value problems with p-Laplacian with Neumann conditions is established. Our approach is based on recent variational methods for smooth functionals defined on reflexive Banach spaces. One example is presented to demonstrate the application of our main results.

Key Words: Fractional $p(x)$-Laplacian, impulsive effects, infinitely many solutions, variational methods.

Contents

1. Introduction

The aim of this paper is to investigate the existence of infinitely many classical solutions for the following nonlinear impulsive fractional boundary value problem (BVP, for short):

$$\begin{cases} D^{\alpha}_{-T}\Phi_p({}^cD^{\alpha}_{0+}u(t)) + |u(t)|^{p-2}u(t) = \lambda f(t,u(t)) + \mu g(t,u(t)), \quad t \neq t_j, \quad t \in (0,T), \\ \Delta(D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0+}u))(t_j) = I_j(u(t_j)) \\ u(0) = u(T) = 0 \end{cases} \tag{1.1}$$

where $\alpha \in (\frac{1}{p}, 1]$, $p > 1$, $\Phi_p(s) = |s|^{p-2}s$ $(s \neq 0)$, D^{α}_{-T} represents the right Riemann–Liouville fractional derivative of order α and ${}^cD^{\alpha}_{0+}$

represents the left Caputo fractional derivative of order α,

$$\Delta(D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0+}u))(t_j) = D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0+}u)(t_j^+) - D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0+}u)(t_j^-),$$

$$D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0+}u)(t_j^+) = \lim_{t\to t_j^+} D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0+}u)(t),$$

$$D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0+}u)(t_j^-) = \lim_{t\to t_j^-} D^{\alpha-1}_{-T}\Phi_p({}^cD^{\alpha}_{0+}u)(t),$$

$\lambda > 0$, $\mu \geq 0$, $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$ and $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \dots, m$ are Lipschitz continuous functions with the Lipschitz constants $L_j > 0$, i.e

$$|I_j(x_2) - I_j(x_1)| \leq L_j|x_2 - x_1|$$

for every $x_1, x_2 \in \mathbb{R}$ and $I_j(0) = 0$.

In [32], Risken introduced an advection-dispersion equation to describe the Brownian motion of particles

$$\frac{\partial C(x,t)}{\partial t} = \left[-v\frac{\partial}{\partial x} + D\frac{\partial^2}{\partial x^2}\right]C(x,t)$$

2010 *Mathematics Subject Classification:* 26A33, 34B15.
Submitted May 14, 2019. Published June 09, 2021

where $C(x,t)$ is a concentration field of space variable x at time t, $D>0$ is the diffusion coefficient and $v>0$ is the drift coefficient. Many laboratory data [3,4] and numerical experiments [12] indicate that solutes moving through a highly heterogeneous aquifer violate the basic assumptions of the local second order theories because of the large deviations due to the stochastic process of Brownian motion. According to [3], an anomalous dispersion process should be described by the following advection-dispersion equation containing the left and the right fractional differential operators

$$\frac{\partial C(x,t)}{\partial t} = -v\frac{\partial C(x,t)}{\partial x} + Dj\frac{\partial^\gamma C(x,t)}{\partial x^\gamma} + D(1-j)\frac{\partial^\gamma C(x,t)}{\partial(-x)^\gamma} \tag{1.2}$$

where C is the expected concentration field of space variable x at time t, v is a constant mean velocity, x is the distance in the direction of the mean velocity, D is a constant dispersion coefficient, $0\le j\le 1$ describes the skewness of the transport process, and γ is the order of left and right fractional differential operators (see [3, Appendix] for details about left and right fractional differential operators). Especially, if $\gamma=2$, the dispersion operator reduces to the classical advection-dispersion operator and (1.2) becomes the classical advection-dispersion equation. On the other hand, if $j=\dfrac{1}{2}$, (1.2) describes symmetric transitions. Define an equivalent Riesz potential symmetric operator [33]

$$2\nabla^\gamma \equiv D_+^\gamma + D_-^\gamma,$$

which gives the mass balance equation for the symmetric fractional advection dispersion

$$\frac{\partial C(x,t)}{\partial t} = -v\nabla C(x,t) + D\nabla^\gamma C(x,t).$$

Fractional differential equations (FDEs) have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc, for instance see [11,24,27] and the references therein. Recently, the existence of solutions to boundary value problems for FDEs have been studied in many papers and we refer the reader to the papers [1,2,14,16,17,23,26,28,37] and the references therein. For example, [37] Zhang et al. by establishing a variational structure and using the critical point theory, investigated the existence of multiple solutions for a class of fractional advection-dispersion equations arising from a symmetric transition of the mass flux. In [17] based on critical point theory and variational methods, the existence of infinitely many solutions for the following perturbed fractional boundary value problem

$$\begin{cases} -\dfrac{d}{dt}\big({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) +{}_t D_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\big) \\ = \lambda f(t,u(t)) + \mu g(t,u(t)) = 0, \text{ a.e. } t\in[0,T], \\ u(0)=u(T)=0 \end{cases}$$

where $T>0$, $\lambda>0$, $\mu\ge 0$, $\frac{1}{2}<\alpha\le 1$, ${}_0D_t^{\alpha-1}$ and ${}_tD_T^{\alpha-1}$ are the left and right Riemann-Liouville fractional derivatives of order $1-\alpha$, respectively, $f,g:[0,T]\times\mathbb{R}\to\mathbb{R}$ are continuous functions, was discussed. Jiang et al. in [26] by using of Avery-Peterson fixed point theorem, obtained the existence of positive solutions for the two-point boundary value problem of fractional differential equations. Galewski and Molica Bisci in [14] by using variational methods, proved that a suitable class of one-dimensional fractional problems admits at least one non-trivial solution under an asymptotical behaviour of the nonlinear datum at zero, their the problem was as the following

$$\begin{cases} \frac{d}{dt}\big({}_0D_t^{\alpha-1}({}_0^cD_t^\alpha u(t)) - {}_tD_T^{\alpha-1}({}_t^cD_T^\alpha u(t))\big) + f(t,u(t)) = 0, \text{ a.e. } t\in[0,T], \\ u(0)=u(T)=0 \end{cases}$$

where $\alpha\in(\frac{1}{2},1]$, ${}_0D_t^{\alpha-1}$ and ${}_tD_T^{\alpha-1}$ are the left and right Riemann-Liouville fractional derivatives of order $1-\alpha$, respectively, ${}_0^cD_t^\alpha$ and ${}_t^cD_T^\alpha$ are the left and right Caputo fractional derivatives of order α

respectively, and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. In [23] the existence of at least one weak solution for the following fractional differential system

$$\begin{cases} {}_tD_T^{\alpha_i}(a_i(t)_0D_t^{\alpha_i}u_i(t)) = F_{u_i}(t, u_1, \dots, u_n) + h_i(u_i(t)), & t \in (0, T), \\ u_i(0) = u_i(T) = 0 \end{cases}$$

for $1 \le i \le n$, where $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ is measurable with respect to t, for all $u \in \mathbb{R}^n$, continuously differentiable in u, for almost every $t \in [0, T]$ such that $F(t, 0, \dots, 0) = 0$ for every $t \in [0, T]$ and $h_i : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function, was studied.

Nonlinear boundary value problems involving p-Laplacian operator Δ_p occur in a variety of physical phenomena, such as: non-Newtonian fluids, reaction-diffusion problems, petroleum extraction, flow through porous media, etc. Thus, the study of such problems and their far reaching generalizations have attracted several mathematicians in recent years. We refer the reader to [5,6,10,13,30] and the references therein.

Impulsive differential equations have become more important in recent years in some mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology, and economics. We mention, for instance, the books [29,34] dealing with impulsive differential equations.

Recently, many researchers pay their attention to impulsive differential equations by variational method and critical point theory, to the best our knowledge, we refer the reader to [22,23,35] and references cited therein.

The study of impulsive fractional boundary value problem has already been extended to the case involving the p-Laplacian. For details, see [19,20,21,36,38] and the references therein. For example, Wang et al. in [36] based on a variant fountain theorem, the existence of infinitely many nontrivial high or small energy solutions for the problem (1.1). Zhao and Tang in [38] by employing critical point theory and variational methods to study, the existence and multiplicity of solutions for the problem (1.1). In [19,21] based on variational methods and critical point theory the existence of multiple solutions for the problem (1.1) in the case $\mu = 0$, was studied. Also in [20] by using variational methods, the existence three classical solutions for the problem (1.1), was discussed.

A special case of our main result, is the following theorem in the case $p = 2$.

Theorem 1.1. *Assume that*

($\mathcal{A}_1$) $F(t, x) \ge 0$ *for each* $(t, x) \in [0, T] \times [0, +\infty)$*;*

($\mathcal{A}_2$)

$$\liminf_{\xi \to +\infty} \frac{\displaystyle\int_0^T \sup_{|x| \le \xi} F(t, x) dt}{\xi^2} < \frac{1 - LT\bar{k}^2}{\bar{k}^2(1 + LT\bar{k}^2)} \limsup_{\xi \to +\infty} \frac{\int_0^T F(t, \xi) dt}{\xi^2}$$

where $\bar{k} = \dfrac{T^{\alpha - \frac{1}{2}}}{\Gamma(\alpha)(2\alpha - 1)^{\frac{1}{2}}}$. *Then, for each*

$$\lambda \in \left(\frac{1}{\frac{2}{1 + LT\bar{k}^2} \limsup_{\xi \to +\infty} \frac{\int_0^T F(t, \xi) dt}{\xi^2}}, \frac{1 - LT\bar{k}^2}{2\bar{k}^2 \liminf_{\xi \to +\infty} \frac{\int_0^T \sup_{|x| \le \xi} F(t, x) dt}{\xi^2}} \right)$$

for every non-negative arbitrary function $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ *which is measurable in* $[0, T]$ *and of class* $C^1(\mathbb{R})$ *satisfying the condition*

$$g_\infty := \limsup_{\xi \to +\infty} \frac{\int_0^T \sup_{|x| \le \xi} G(t, x) dt}{\xi^2} < +\infty$$

and for every $\mu \in [0, \mu_g[$ *where*

$$\mu_g := \frac{1 - LT\bar{k}^2}{2\bar{k}^2 g_\infty}\left(1 - \lambda \frac{2\bar{k}^2}{1 - LT\bar{k}^2} \liminf_{\xi \to +\infty} \frac{\int_0^T \sup_{|x| \le \xi} F(t, x)dt}{\xi^2}\right),$$

the problem

$$\begin{cases} D^\alpha_{-T}({}^c D^\alpha_{0+} u(t)) + u(t) = \lambda f(t, u(t)) + \mu g(t, u(t)), \quad t \neq t_j, \quad t \in (0, T), \\ \Delta(D^{\alpha-1}_{-T}({}^c D^\alpha_{0+} u))(t_j) = I_j(u(t_j)) \\ u(0) = u(T) = 0 \end{cases}$$

has an unbounded sequence of weak solutions in $E_0^{\alpha,2}$.

Motivated by the above works, in the present paper, employing a smooth version of [8, Theorem 2.1], under an appropriate oscillating behaviour of the nonlinear term f, we determine the exact collections of the parameter λ in which the problem (1.1) for every non-negative arbitrary function $g : [0, T] \times \mathbb{R} \to \mathbb{R}$ which is measurable in $[0, T]$ and of class $C^1(\mathbb{R})$ satisfying a certain growth at infinity, choosing μ sufficiently small, admits infinitely many weak solutions (Theorem 3.1). Replacing the oscillating behaviour condition at infinity, by a similar one at zero, we achieve a sequence of pairwise distinct weak solutions which converges to zero (Theorem 3.6). We also list some consequences the main results. The applicability of our results is illustrated by an example.

The present paper is arranged as follows. In Section 2 we recall some basic definitions and preliminary results, while Section 3 is devoted to the existence of multiple weak solutions for the double eigenvalue problem (1.1).

2. Preliminaries

In this section, we formulate our main results on the existence infinitely many weak solutions for the problem (1.1). Our main tool to ensure the results is a smooth version of Theorem 2.1 of [8] which is a more precise version of Ricceri's Variational Principle [31, Theorem 2.5] that we now recall here.

Theorem 2.1. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u \in \Phi^{-1}(-\infty, r)} \frac{\sup_{u \in \Phi^{-1}(-\infty, r)} \Psi(u) - \Psi(u)}{r - \Phi(u)}$$

and

$$\theta := \liminf_{r \to +\infty} \varphi(r), \qquad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then, one has

(a) *for every $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(] - \infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.*

(b) *If $\theta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\theta}[$, the following alternative holds: either*

(b_1) *I_λ possesses a global minimum,*

or

(b_2) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that*

$$\lim_{n \to +\infty} \Phi(u_n) = +\infty.$$

(c) *If $\delta < +\infty$ then, for each $\lambda \in]0, \frac{1}{\delta}[$, the following alternative holds:*

(c_1) *there is a global minimum of Φ which is a local minimum of I_λ,*

(c_2) *there is a sequence of pairwise distinct critical points (local minima) of I_λ which weakly converges to a global minimum of Φ.*

We refer the interested reader to the paper [7,9,15,18] in which Theorem 2.1 has been successfully employed to the existence of infinitely many solutions for boundary value problems.

This section is devoted to introduce some basic notations and results which will be used in the proofs of our main results.

Let $AC[a, b]$ be the space of absolutely continuous functions on $[a, b]$.

Definition 2.2. *[25] Let f be a function defined on $[a, b]$ and $0 < \alpha \leq 1$. The left and right Riemann-Liouville fractional integrals of order α for the function f are defined by*

$$D_{a^+}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds, \quad t \in [a, b],$$
$$D_{b^-}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) ds, \quad t \in [a, b]$$

provided the right-hand sides are pointwise defined on $[a, b]$ where $\Gamma(\alpha)$ is the standard gamma function given by

$$\Gamma(\alpha) = \int_0^{+\infty} z^{\alpha-1} e^{-z} dz.$$

Definition 2.3. *[25] Let f be a function defined on $[a, b]$ and $0 < \alpha \leq 1$. The left and right Riemann-Liouville fractional integrals of order α for the function f are defined by*

$$D_{a^+}^{\alpha} f(t) = \frac{d}{dt} D_{a^+}^{\alpha-1} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t (t-s)^{-\alpha} f(s) ds, \quad t \in [a, b],$$
$$D_{b^-}^{\alpha} f(t) = -\frac{d}{dt} D_{b^-}^{\alpha-1} f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b (s-t)^{-\alpha} f(s) ds, \quad t \in [a, b].$$

Definition 2.4. *[25] Let f be a function defined on $[a, b]$ and $0 < \alpha \leq 1$. The left and right Riemann-Liouville fractional integrals of order α for the function f are defined by*

$${}^cD_{a^+}^{\alpha} f(t) = D_{a^+}^{\alpha-1} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} f'(s) ds, \quad t \in [a, b],$$
$${}^cD_{b^-}^{\alpha} f(t) = -D_{b^-}^{\alpha-1} f(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (s-t)^{-\alpha} f'(s) ds, \quad t \in [a, b].$$

In particular, when $\alpha = 1$, we have ${}^cD_{a^+}^1 f(t) = f'(t)$ and ${}^cD_{b^-}^1 f(t) = -f'(t)$.

Proposition 2.5. *[39]*

(1) *If $u \in L^p([0, T], \mathbb{R})$, $v \in L^q([0, T], \mathbb{R})$ and $p \geq 1$, $q \geq 1$, $\frac{1}{p} + \frac{1}{q} \leq 1 + \theta$ or $p \neq 1$, $q \neq 1$, $\frac{1}{p} + \frac{1}{q} = 1 + \theta$, then we have*

$$\int_a^b [D_t^{-\theta} u(t)] v(t) dt = \int_a^b [v(t) D_b^{-\theta}] u(t) dt, \quad \theta > 0.$$

(2) *If $0 < \alpha \leq 1$, $u \in AC[a, b]$, and $v \in L^p[a, b]$ $(1 \leq p < \infty)$, then*

$$\int_a^b u(t)({}^cD_{a^+}^{\alpha} f(t)) dt = D_b^{\alpha-1} u(t) v(t) \mid_{t=a}^{t=b} + \int_a^b D_b^{\alpha} u(t) v(t) dt.$$

Let $C_0^\infty([0,T],\mathbb{R}^N)$ be the set of all functions $u\in C^\infty([0,T],\mathbb{R}^N)$ with $u(a)=u(b)=0$ and the norm

$$\|u\|_\infty=\max_{t\in[a,b]}|u(t)|.$$

Denote the norm of the space $L^p([0,T],\mathbb{R}^N)$ for $1\le p<\infty$ by

$$\|u\|_{L^p}=\Big(\int_a^b|u(s)|^pds\Big)^{\frac{1}{p}}.$$

The following lemma yields the boundedness of the Riemann-Liouville fractional integral operators from the space $L^p([a,b],\mathbb{R}^N)$ to the space $L^p([a,b],\mathbb{R}^N)$ where $1\le p<\infty$.

Definition 2.6. *Let $0<\alpha\le 1$, $1<p<\infty$. The fractional derivative space $E_0^{\alpha,p}$ is defined by the closure $C_0^\infty([0,T],\mathbb{R})$, that is*

$$E_0^{\alpha,p}=\overline{C_0^\infty([0,T],\mathbb{R})}$$

with respect to the weighted norm

$$\|u\|_{E_0^{\alpha,p}}=\Big(\int_0^T|{}^cD_{0^+}^\alpha u(t)|^pdt+\int_0^T|u(t)|^pdt\Big)^{\frac{1}{p}}\tag{2.1}$$

for every $u\in E_0^{\alpha,p}$.

Remark 2.7. *It is obvious that the fractional derivative space $E_0^{\alpha,p}$ is the space of functions $u\in L^2([0,T],\mathbb{R})$ having an α-order Riemann-Loiuville fractional derivative ${}^cD_t^\alpha u\in L^2([0,T],\mathbb{R})$ and $u(0)=u(T)=0$ for $1\le i\le n$. From [25, Propostion 3.1], we know for $0<\alpha\le 1$, the space $E_0^{\alpha,p}$ is a reflexive and separable Banach space.*

Lemma 2.8. *[39] Let $0<\alpha\le1$ and $1<p<\infty$. For any $u\in E_0^{\alpha,p}$, we have*

$$\|u\|_{L^p}\le\frac{T^\alpha}{\Gamma(\alpha+1)}\|{}^cD_{0^+}^\alpha u(t)\|_{L^p}.\tag{2.2}$$

In addition, for $\dfrac{1}{p}<\alpha\le1$ and $\dfrac{1}{p}+\dfrac{1}{q}=1$, we have

$$\|u\|_\infty\le k\|{}^cD_{0^+}^\alpha u(t)\|_{L^p}\tag{2.3}$$

where $k=\dfrac{T^{\alpha-\frac{1}{2}}}{\Gamma(\alpha)(\alpha q-q+1)^{\frac{1}{q}}}$.

Remark 2.9. *According to Lemma 2.8, it is easy to see that the norm of $E_0^{\alpha,p}$ defined in (2.1) is equivalent to the following norm:*

$$\|u\|_{\alpha,p}=\Big(\int_0^T|{}^cD_{0^+}^\alpha u(t)|^pdt\Big)^{\frac{1}{p}}.\tag{2.4}$$

Lemma 2.10. *Let $\frac{1}{p}<\alpha\le1$. If the sequence $\{u_k\}$ converges weakly to u in $E_0^{\alpha,p}$, i.e., $u_k\rightharpoonup u$, then $u_k\longrightarrow u$ in $C[0,T]$, i.e., $\|u-u_k\|_\infty\longrightarrow0$ as $k\longrightarrow\infty$.*

Lemma 2.11. *A function*

$$u\in\left\{u\in AC[0,T]:\left(\int_{t_j}^{t_{j+1}}|({}^cD_{0^+}^\alpha u(t)|^p+|u(t)|^p)dt\right)<\infty,\ j=1,2,\dots,m\right\}$$

is called a classical solution of BVP (1.1) if

(1) *u satisfies (1.1).*

(2) *The limits* $D_{T^-}^{\alpha-1}\Phi_p({}^cD_{0+}^{\alpha}u)(t_j^+)$, $D_{T^-}^{\alpha-1}\Phi_p({}^cD_{0+}^{\alpha}u)(t_j^-)$ *exist.*

Definition 2.12. *We mean by a (weak) solution of the BVP (1.1), any function* $u \in E_0^{\alpha,p}$ *such that*

$$\int_0^T |{}^cD_{0+}^{\alpha}u(t)|^{p-2}({}^cD_{0+}^{\alpha}u(t))({}^cD_{0+}^{\alpha}v(t))dt + \int_0^T |u(t)|^{p-2}u(t)v(t)dt$$
$$+\sum_{j=1}^{m} I_j(u(t_j))v(t_j) - \lambda\int_0^T f(t,u(t))v(t)dt - \mu\int_0^T g(t,u(t))v(t)dt = 0$$

for every $v \in E_0^{\alpha,p}$.

Lemma 2.13. *[38] If* $u \in E_0^{\alpha,p}$ *is a weak solution of BVP (1.1), then u is a classical solution of BVP (1.1)*

Corresponding to the functions f and g, we introduce the functions $F : [0,T]\times\mathbb{R} \to \mathbb{R}$ and $G : [0,T]\times\mathbb{R}\to\mathbb{R}$, respectively, as follow

$$F(t,\xi) = \int_0^{\xi} f(t,x)dx \quad \text{for all } (t,\xi)\in[0,T]\times\mathbb{R}$$

and

$$G(t,\xi) = \int_0^{\xi} g(t,x)dx \quad \text{for all } (t,\xi)\in[0,T]\times\mathbb{R}.$$

We assume throughout and without further mention, that the following condition holds:

(H) $1 > LTk^p$

where $L = \sum_{j=1}^{n} L_j$.

3. Main Results

In this section, we will state and prove our main results.

For convenience, put

$$A = \liminf_{\xi\to+\infty}\frac{\int_0^T \sup_{|x|\le\xi} F(t,x)dt}{\xi^p},$$

$$B = \frac{p}{1+LTk^p}\limsup_{\xi\to+\infty}\frac{\int_0^T F(t,\xi)dt}{\xi^p},$$

$$\lambda_1 = \frac{1}{B}$$

and

$$\lambda_2 = \frac{1-LTk^p}{pk^pA}.$$

Theorem 3.1. *Assume that*

(A_1) $F(t,x)\ge 0$ *for each* $(t,x)\in[0,T]\times[0,+\infty)$;

(A_2)

$$A < \frac{1-LTk^p}{pk^p}B.$$

Then, for each $\lambda \in]\lambda_1, \lambda_2[$ for every non-negative arbitrary function $g : [0,T]\times\mathbb{R} \to \mathbb{R}$ which is measurable in $[0,T]$ and of class $C^1(\mathbb{R})$ satisfying the condition

$$g_\infty := \limsup_{\xi\to+\infty} \frac{\int_0^T \sup_{|x|\le\xi} G(t,x)dt}{\xi^p} < +\infty \tag{3.1}$$

and for every $\mu \in [0, \mu_{g,\lambda}[$ where

$$\mu_{g,\lambda} := \frac{1-LTk^p}{pk^p g_\infty}\left(1-\lambda\frac{pk^p}{1-LTk^p}A\right), \tag{3.2}$$

the problem (1.1) has an unbounded sequence of classical solutions in $E_0^{\alpha,p}$.

Proof. Our aim is to apply Theorem 2.1 to the problem (1.1). Take $X = E_0^{\alpha,p}$. Let the functionals Φ, Ψ for every $u \in X$, defined by

$$\Phi(u) = \frac{1}{p}\|u\|_{\alpha,p}^p - \sum_{j=1}^m \int_0^{u(t_j)} I_j(s)ds \tag{3.3}$$

and

$$\Psi(u) = \int_0^T F(t,u(t))dt + \frac{\mu}{\lambda}\int_0^T G(t,u(t))dt.$$

Let us prove that the functionals Φ and Ψ satisfy the required conditions in Theorem 2.1. It is well known that Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_0^T f(t,u(t))v(t)dt + \frac{\mu}{\lambda}\int_0^T g(t,u(t))v(t)dt$$

for every $v \in X$, as well as is sequentially weakly upper semicontinuous. Now from the facts $-L_j|\xi| \le I_j(\xi) \le L_j|\xi|$ for every $\xi\in\mathbb{R}$, $j = 1,\dots,n$, and taking (2.3) into account, for every $u \in X$, we have

$$\begin{aligned}\frac{1-LTk^p}{p}\|u\|_{\alpha,p}^p &\le \frac{1}{p}\|u\|_{\alpha,p}^p - \frac{LTk^p}{p}\|u\|_{\alpha,p}^p \le \Phi(u)\\ &\le \frac{1}{p}\|u\|_{\alpha,p}^p + \frac{LTk^p}{p}\|u\|_{\alpha,p}^p \le \frac{1+LTk^p}{p}\|u\|_{\alpha,p}^p,\end{aligned} \tag{3.4}$$

by using the condition (H) and the first inequality in (3.4), it follows $\lim_{\|u\|\to+\infty}\Phi(u) = +\infty$, namely Φ is coercive. Moreover, Φ is continuously differentiable whose differential at the point $u \in X$ is

$$\begin{aligned}\Phi'(u)(v) = &\int_0^T |{}^cD_{0^+}^\alpha u(t)|^{p-2}({}^cD_{0^+}^\alpha u(t))({}^cD_{0^+}^\alpha v(t))dt\\ &+\int_0^T |u(t)|^{p-2}u(t)v(t)dt + \sum_{j=1}^m I_j(u(t_j))v(t_j)\end{aligned}$$

for every $v \in X$. Moreover, Φ is sequentially weakly lower semicontinuous. Therefore, we observe that the regularity assumptions on Φ and Ψ, as requested in Theorem 2.1, are verified.

Let $\{\xi_n\}$ be a real sequence of positive numbers such that $\lim_{n\to+\infty}\xi_n = +\infty$, and

$$A = \lim_{n\to+\infty}\frac{\int_0^T \sup_{|x|\le\xi_n} F(t,x)dt}{\xi_n^p}.$$

Put

$$r_n = \frac{1-LTk^p}{pk^p}\xi_n^p.$$

From the definition of Φ and considering equations (2.3), (3.3) and (3.4) for every $r_n > 0$, one has

$$\Phi^{-1}(-\infty, r_n) = \{u \in X; \Phi(u) < r_n\} \subseteq \{u \in X; |u| \le \xi_n\}. \tag{3.5}$$

Therefore, since $0 \in \Phi^{-1}(-\infty, r_n)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r_n)} \Psi(u)) - \Psi(u)}{r_n - \Phi(u)} \le \frac{\sup_{u\in\Phi^{-1}(-\infty,r_n)} \Psi(u)}{r_n} \\
&= \frac{\sup\limits_{u\in\Phi^{-1}(-\infty,r_n)} \displaystyle\int_0^T \left[F(t,u(t)) + \frac{\mu}{\lambda} G(t,u(t))\right] dt}{r_n} \\
&\le \frac{\displaystyle\int_0^T \sup_{|x|\le\xi_n} F(t,x)dt}{r_n} + \frac{\frac{\mu}{\lambda}\displaystyle\int_0^T \sup_{|x|\le\xi_n} G(t,x)dt}{r_n} \\
&= \frac{pk^p}{1-LTk^p} \frac{\displaystyle\int_0^T \sup_{|x|\le\xi_n} F(t,x)dt}{\xi_n^p} + \frac{\mu}{\lambda}\frac{pk^p}{1-LTk^p} \frac{\displaystyle\int_0^T \sup_{|x|\le\xi_n} G(t,x)dt}{\xi_n^p}
\end{aligned}$$

for all $n \in \mathbb{N}$. Therefore, from assumption (A_2) and the condition (3.1) one has

$$\theta \le \liminf_{n\to+\infty} \varphi(r_n) \le \frac{pk^p}{1-LTk^p}(A + \frac{\mu}{\lambda} g_\infty) < +\infty.$$

Now, let $\{\eta_n\}$ be positive real sequences and for all $n \in \mathbb{N}$, and

$$\lim_{n\to+\infty} \eta_n = +\infty.$$

Define w_n by setting

$$w_n(t) = \begin{cases} 0, & \text{if } t = 0, \\ \eta_n, & \text{if } t \in (0,T), \\ 0, & \text{if } t = 0. \end{cases}$$

Clearly, $w_n \in X$, from (3.3) and (3.4), we have

$$\frac{1-LTk^p}{p}\eta_n^p \le \Phi(w_n) \le \frac{1+LTk^p}{p}\eta_n^p. \tag{3.6}$$

On the other hand, since g is nonnegative and bearing the assumption (A_1) in mind, from (3.3) one has

$$\begin{aligned}
\Psi(w_n) &= \int_0^T F(t,\eta_n)dt + \frac{\mu}{\lambda}\int_0^T G(t,\eta_n)dt \\
&\ge \int_0^T F(t,\eta_n)dt.
\end{aligned}$$

Then,

$$I_\lambda(w_n) = \Phi(w_n) - \lambda\Psi(w_n) \le \frac{1+LTk^p}{p}\eta_n^p - \lambda\int_0^T F(t,\eta_n)dt.$$

Now, consider the following cases.
If $B < +\infty$, let $\epsilon \in]0, B - \frac{1}{\lambda}[$. There exists ν_ϵ such that

$$\int_0^T F(t,\eta_n)dt > (B-\epsilon)\frac{1+LTk^p}{p}\eta_n^p$$

for all $n > \nu_\epsilon$, and so

$$I_\lambda(w_n) < \frac{1+LTk^p}{p}\eta_n^p - \lambda\int_0^T F(t,w_n(t))dt$$
$$= \frac{1+LTk^p}{p}\eta_n^p(1-\lambda(B-\epsilon)).$$

Since $1-\lambda(B-\epsilon) < 0$, and taking into account (3.4) and (3.6) one has

$$\lim_{n\to+\infty} I_\lambda(w_n) = -\infty.$$

If $B = +\infty$, fix $N > \frac{1}{\lambda}$. There exists ν_N such that

$$\int_0^T F(t,\eta_n)dt > N\frac{1+LTk^p}{p}\eta_n^p$$

for all $n > \nu_N$, and moreover,

$$I_\lambda(w_n) < \frac{1+LTk^p}{p}\eta_n^p(1-\lambda N).$$

Since $1-\lambda N < 0$, and arguing as before, we have

$$\lim_{n\to+\infty} I_\lambda(w_n) = -\infty.$$

Taking into account that

$$\left]\frac{1}{B}, \frac{\sigma}{2MA}\right[\subset \left]0, \frac{1}{\theta}\right[,$$

and that I_λ does not possess a global minimum, from part (b) of Theorem 2.1, there exists an unbounded sequence $\{u_n\}$ of critical points which are the classical solutions of (1.1). So, our conclusion is achieved. $\square$

We present an example to illustrate Theorem 3.1.

Example 3.2. *Consider the following problem*

$$\begin{cases} D_{-1}^{\alpha}\Phi_4({}^cD_{0+}^{\alpha}u(t)) + |u(t)|^2u(t) = \lambda f(u) + \mu g(u), \quad t \neq \frac{1}{2}, \quad t \in (0,1), \\ \Delta(D_{-1}^{\alpha-1}\Phi_4({}^cD_{0+}^{\alpha}u))(\frac{1}{2}) = I_1(u(\frac{1}{2})) \\ u(0) = u(1) = 0 \end{cases} \tag{3.7}$$

where $\alpha = \frac{5}{6}$, $I(\zeta) = \dfrac{7\Gamma^4(\frac{5}{6})}{6}\sin(\zeta)$ *for every* $\zeta \in \mathbb{R}$, $f(\xi) = 4\xi^3 + 80\xi^3\sin^2(\xi) + 40\xi^4\sin(\xi)\cos(\xi)$ *and* $g(\xi) = \frac{5}{2}\sqrt{\xi^3}$ *for every* $\xi \in \mathbb{R}$. *By the expressions of* f *and* g, *we have* $F(\xi) = \xi^4(1+20\sin^2(\xi))$ *and* $G(\xi) = \sqrt{\xi^5}$ *for every* $\xi \in \mathbb{R}$. *By simple calculations, we obtain* $k = \dfrac{\sqrt[4]{3}}{\Gamma(\frac{5}{6})\sqrt[4]{7}}$. *By simple calculations, we see that*

$$\liminf_{\xi\to+\infty}\frac{\sup_{|x|\le\xi}F(x)}{\xi^4} = 1,$$
$$\limsup_{\xi\to+\infty}\frac{F(\xi)}{\xi^4} = 21$$

and

$$g_\infty := \limsup_{\xi\to+\infty}\frac{\sup_{|x|\le\xi}G(x)}{\xi^4} = 0 < +\infty.$$

We clearly see that all assumptions of Theorem 3.1 are satisfied. Then, for every $\lambda \in (\dfrac{1}{56}, \dfrac{7\Gamma^4(\frac{5}{6})}{24})$ *and for each* $\mu \in [0,+\infty)$ *the problem* (3.7) *admits a sequence of classical solutions which is unbounded in* $E_0^{\frac{5}{6},4}$.

Remark 3.3. *Under the conditions $A = 0$ and $B = +\infty$, Theorem 3.1 concludes that for every $\lambda > 0$ and for each*

$$\mu \in \Big[0, \frac{\sigma}{2Mg_\infty}\Big[$$

the problem (1.1) admits infinitely many classical solutions in X. Moreover, if $g_\infty = 0$, the result holds for every $\lambda > 0$ and $\mu \geq 0$.

Remark 3.4. *Put*

$$\hat{\lambda}_1 = \lambda_1$$

and

$$\hat{\lambda}_2 = \frac{1}{\lim_{n\to+\infty} \frac{\int_0^T \sup_{|x|\leq c_n} F(t,x)dt - \int_0^T F(t,b_n)dt}{\frac{1-LTk^p}{pk^p}c_n^p - \frac{1+LTk^p}{p}b_n^p}}.$$

We explicitly observe that the assumption (A_2) in Theorem 3.1 could be replaced by the following more general condition
(A_3) there exist two sequence $\{c_n\}$ with $\{b_n\}$ for all $n \in \mathbb{N}$ and

$$b_n^p < \frac{1 - LTk^p}{k^p(1 + LTk^p)} c_n^p$$

for every $n \in \mathbb{N}$ and $\lim_{n\to+\infty} c_n = +\infty$ such that

$$\lim_{n\to+\infty} \frac{\int_0^T \sup_{|x|\leq c_n} F(t,x)dt - \int_0^T F(t,b_n)dt}{\frac{1-LTk^p}{pk^p}c_n^p - \frac{1+LTk^p}{p}b_n^p}$$

$$< \frac{p}{1 + LTk^p} \limsup_{n\to+\infty} \frac{\int_0^T F(t,\eta_n)dt}{\eta_n^p}.$$

Obviously, from (A_3) we obtain (A_2), by choosing $b_n = 0$ for all $n \in \mathbb{N}$. Moreover, if we assume (A_3) instead of (A_2) and set

$$r_n = \frac{1 - LTk^p}{pk^p} c_n^p.$$

for all $n \in \mathbb{N}$, by the same arguing as inside in Theorem 3.1, we obtain

$$\begin{aligned}
\varphi(r_n) &= \inf_{u\in\Phi^{-1}(-\infty,r_n)} \frac{(\sup_{u\in\Phi^{-1}(-\infty,r_n)} \Psi(u)) - \Psi(u)}{r_n - \Phi(u)} \\
&\leq \frac{\sup_{u\in\Phi^{-1}(-\infty,r_n)} \Psi(u) - \left[\int_0^T F(t,u(t))dt + \frac{\mu}{\lambda}\int_0^T G(t,u(t))dt\right]}{r_n - \Phi(u)} \\
&\leq \frac{\int_0^T \sup_{|x|\leq c_n} F(t,x)dt - \int_0^T F(t,b_n)dt}{\frac{1-LTk^p}{pk^p}c_n^p - \frac{1+LTk^p}{p}b_n^p}.
\end{aligned}$$

We have the same conclusion as in Theorem 3.1 with Λ replaced by $\Lambda' :=]\hat{\lambda}_2, \hat{\lambda}_2[$.

Here we point out the following consequence of Theorem 3.1.

Corollary 3.5. *Assume that (A_1) holds and*

(A_4) $\liminf_{\xi\to+\infty} \frac{\int_0^T \sup_{|x|\leq\xi} F(t,x)dt}{\xi^p} < \frac{1-LTk^p}{pk^p}$;

(A_5) $\limsup_{\xi\to+\infty} \frac{\int_0^T F(t,\xi)dt}{\xi^p} > \frac{1+LTk^p}{p}$.

Then, for every non-negative arbitrary function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ which is measurable in $[0,T]$ and of class $C^1(\mathbb{R})$ satisfying the condition (3.1) and for every $\mu \in [0, \mu_{g,1}[$ where

$$\mu_{g,1} := \frac{1 - LTk^p}{pk^p g_\infty}\left(1 - \frac{pk^p}{1 - LTk^p}A\right),$$

the problem

$$\begin{cases} {}_tD_T^{\alpha_i}(a_i(t){}_0D_t^{\alpha_i}u_i(t)) = F_{u_i}(t,u) + \mu G_{u_i}(t,u) + h_i(u_i(t)), \quad t \in (0,T), \\ u_i(0) = u_i(T) = 0 \end{cases}$$

for $1 \le i \le n$, has an unbounded sequence of classical solutions in X.

In the same way as in the proof of Theorem 3.1 but using conclusion (c) of Theorem 2.1 instead of (b), we will obtain the following result.

Theorem 3.6. *Assume that all the hypotheses of Theorem 3.1 hold except for Assumption (A_2). Suppose that*

(B_1)

$$\bar{A} < \frac{1 - LTk^p}{k^p(1 + LTk^p)}\bar{B}$$

where

$$\bar{A} = \liminf_{\xi \to 0^+} \frac{\displaystyle\int_0^T \sup_{|x| \le \xi} F(t,x)dt}{\xi^p}$$

and

$$\bar{B} = \frac{p}{1 + LTk^p} \limsup_{\xi \to 0^+} \frac{\int_0^T F(t,\xi)dt}{\xi^p}.$$

Then, for each $\lambda \in]\lambda_3, \lambda_4[$ where

$$\lambda_3 := \frac{1}{\bar{B}}$$

and

$$\lambda_4 := \frac{1 - LTk^p}{pk^p\bar{A}}$$

for every nonnegative arbitrary function $g : [0,T] \times \mathbb{R} \to \mathbb{R}$ which is measurable in $[0,T]$ and of class $C^1(\mathbb{R})$ satisfying the condition

$$g_0 := \limsup_{\xi \to 0^+} \frac{\int_0^T \sup_{|x| \le \xi} G(t,x)dt}{\xi^p} < +\infty \tag{3.8}$$

and for every $\mu \in [0, \mu_{g_0,\lambda}[$ where

$$\mu_{g_0,\lambda} := \frac{1 - LTk^p}{pk^p g_0}\left(1 - \lambda\frac{pk^p}{1 - LTk^p} \liminf_{\xi \to 0^+} \frac{\int_0^T \sup_{|x| \le \xi} F(t,x)dt}{\xi^p}\right), \tag{3.9}$$

the problem (1.1) has a sequence of pairwise distinct classical solutions which strongly converges to 0 in X.

Proof. We take Φ and Ψ as in the proof of Theorem 3.1 and put $I_{\bar{\lambda}}(u) = \Phi(u) - \bar{\lambda}\Psi(u)$ for $u \in X$. Since

$$\frac{\int_0^T \sup_{|x|\le\xi}[F(t,x) + \frac{\bar{\mu}}{\bar{\lambda}}G(t,x)]dt}{\xi^p} \le \frac{\int_0^T \sup_{|x|\le\xi} F(t,x)dt}{\xi^p} + \frac{\bar{\mu}}{\bar{\lambda}}\frac{\int_0^T \sup_{|x|\le\xi} G(t,x)dt}{\xi^p},$$

taking into account (3.8) one has

$$\liminf_{\xi\to0^+}\frac{\int_0^T \sup_{|x|\le\xi}[F(t,u(t)) + \frac{\bar{\mu}}{\bar{\lambda}}G(t,u(t))]dt}{\xi^p} \le \liminf_{\xi\to0^+}\frac{\int_0^T \sup_{|x|\le\xi} F(t,x)dt}{\xi^p} + \frac{\bar{\mu}}{\bar{\lambda}}G_0.$$

We verify that $\delta < +\infty$. For this, let $\{\xi_n\}$ be a sequence of positive numbers such that $\xi_n \to 0^+$ as $n \to +\infty$ and

$$\lim_{n\to+\infty}\frac{\int_0^T \sup_{|x|\le\xi_n}\left[F(t,x) + \frac{\bar{\mu}}{\bar{\lambda}}G(t,x)\right]dt}{\xi_n^p} < +\infty.$$

Put

$$\bar{A} = \lim_{n\to+\infty}\frac{\int_0^T \sup_{|x|\le\xi_n} F(t,x)dt}{\xi_n^p}$$

and

$$r_n = \frac{1 - LTk^p}{pk^p}\xi_n^p$$

for $n \in \mathbb{N}$. Therefore, from assumption (B_1) and the condition (3.8) one has

$$\delta \le \liminf_{n\to+\infty}\varphi(r_n) \le \frac{pk^p}{1 - LTk^p}(\bar{A} + \frac{\bar{\mu}}{\bar{\lambda}}g_0) < +\infty.$$

Let us show that the functional $I_{\bar{\lambda}}$ does not have a local minimum at zero. For this, let $\{\eta_n\}$ be a sequence of positive such that $\eta_n \to 0^+$ as $n \to +\infty$. Put

$$\bar{B} = \frac{p}{1 + LTk^p}\lim_{n\to0^+}\frac{\int_0^T F(t,\eta_n)dt}{\eta_n^p}. \tag{3.10}$$

Let $\{w_n\}$ be a sequence in X with w_n defined in (3). Moreover, since g is non-negative, from the assumption (A_1) we obtain

$$\begin{aligned}\Psi(w_n) &= \int_0^T F(t,\eta_n)dt + \frac{\bar{\mu}}{\bar{\lambda}}\int_0^T G(t,\eta_n)dt \\ &\ge \int_0^T F(t,\eta_n)dt.\end{aligned}$$

Then,

$$I_{\bar{\lambda}}(w_n) = \Phi(w_n) - \bar{\lambda}\Psi(w_n)$$

$$\leq \frac{1+LTk^p}{p}\eta_n^p - \bar{\lambda}\int_0^T F(t,\eta_n)dt.$$

Consider the following cases.
If $\bar{B} < +\infty$, let $\varepsilon \in]0, \bar{B} - \frac{1}{\bar{\lambda}}[$. By (3.10), there exists ν_ε such that

$$\int_0^T F(t,\eta_n)dt > (\bar{B}-\varepsilon)\frac{1+LTk^p}{p}\eta_n^p$$

for all $n > \nu_\varepsilon$, hence

$$\begin{aligned} I_\lambda(w_n) &< \frac{1+LTk^p}{p}\eta_n^p - \bar{\lambda}(\bar{B}-\varepsilon)\int_0^T F(t,w_n(t))dt \\ &= \frac{1+LTk^p}{p}\eta_n^p(1-\bar{\lambda}(\bar{B}-\varepsilon)). \end{aligned}$$

Since $1-\bar{\lambda}(\bar{B}-\varepsilon) < 0$, and by considering (3.4), one has

$$\lim_{n\to+\infty} I_{\bar{\lambda}}(w_n) = 0.$$

If $\bar{B} = +\infty$, fix $N_0 > \frac{1}{\bar{\lambda}}$. There exists ν_{N_0} such that

$$\int_0^T F(t,\eta_n)dt > N_0\frac{1+LTk^p}{p}\eta_n^p$$

for all $n > \nu_{N_0}$, and moreover,

$$I_{\bar{\lambda}}(w_n) < \frac{1+LTk^p}{p}\eta_n^p(1-\bar{\lambda}N_0).$$

Since $1-\bar{\lambda}N_0 < 0$, and as above, we can say

$$\lim_{n\to+\infty} I_{\bar{\lambda}}(w_n) = 0.$$

Since $I_{\bar{\lambda}} = 0$, this implies that the functional $I_{\bar{\lambda}}$ does not have a local minimum at zero. Hence, part (c) of Theorem 2.1 ensures that there exists a sequence $\{u_n\}$ in X of critical points of $I_{\bar{\lambda}}$ such that $\|u_n\| \to 0$ as $n \to \infty$, and the proof is complete. □

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Mohammad Abolghasemi,
Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran.
E-mail address: m_abolghasemi@razi.ac.ir

and

Shahin moradi,
Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran.
E-mail address: shahin.moradi86@yahoo.com