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A Note on Closure Spaces Determined by Intersections *

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abstract: In this work, we study a kind of closure systems (c.s.) that are defined by means of *intersections* of subsets of a support X with a (fixed) closed set T. These systems (which will be indicated by $M(T)$-spaces) can be understood as a generalization of the usual *relative subspaces*. Several results (referred to continuity and to the ordered structure of families of $M(T)$-spaces) are shown here. In addition, we study the *transference of properties* from the "original closure spaces $(X, \overline{K})$" to the spaces $(X, M(T))$. Among them, we are interested mainly in finitariness and in structurality. In this study of transference, we focus our analysis on the c.s. usually known as *abstract logics*, and we show some results for them.

Key Words: Closure Spaces, abstract logics, transference of properties.

Contents

1. Introduction and Preliminaries

The well-known standard notion of *relative subspace* Y of a given closure space (CSP) $(X, \overline{K})$ (being $\overline{K}$ a closure system; c.s.) allows us to obtain another one, of the form $(Y, \overline{K}_Y)$, where $\overline{K}_Y$ is defined as follows: $\overline{K}_Y = \{A \subseteq Y : A = Y \cap Z \text{ for some } Z \in \overline{K}\}$. This notion allows us to characterize many properties of $\overline{K}_Y$ in terms of $\overline{K}$. This definition is motivated by the applications of certain topological spaces and formalizes in an adequate way the definition of $(I\!R, \tau)$ as a subspace of $(I\!R^2, \tau)$, for instance. Note here that, for $Y \subseteq X$, the c.s. $\overline{K}_Y$ has the set Y as its support. So, an interesting question here is to obtain a natural generalization of the already mentioned idea, in such a way that the "new" closure system has the own set X as its support . With this motivation, we define the c.s. $M(T)$ which verifies the previous requirements. Concretely, given a (fixed) closure space $(X, \overline{K})$, and a set $T \in \overline{K}$, the *meet-closure system* $M(T)$ (determined by $(X, \overline{K})$ and T) is defined in this way: $M(T) = \{Y \subseteq X : Y \cap T \in \overline{K}\}$. We will show that $(X, M(T))$ is a CSP that is naturally related with the CSP $(Y, \overline{K}_Y)$, as it was desired. In addition, we will give several properties of $M(T)$-spaces, and we will show several elucidating examples that motivate our study, already initialized in [7].

Besides that, we will investigate if certain properties of $(X, \overline{K})$ (finitariness, structurality) are *transferred* to $(X, M(T))$. By the way, in this study of transference, we will focus our analysis on *abstract logics*. That is, closure spaces of the form $(\mathbf{A}, \overline{K})$, being $\mathbf{A}$ any algebra (and, a particular case, when $\mathbf{A}$ is a *sentential language* of the form $\mathbf{A} = \mathbf{L}(S_{\mathcal{L}})$). With this idea, the results of this note can be understood as a "first step" in the study of transference of properties between closure spaces (abstract logics), that will be developed in future works.

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We start this paper with a brief reminder of the notion of *closure space*, since it is the main subject to be developed here. For that, we are based on [3], [12] and on [14] (classical reference on Abstract Logic), applying several notational changes which are focused on the self-contention of this work.

Definition 1.1. Let X be a non-empty set:
(a) A **closure operator on** X (c.o.) is a map $Cl : \wp(X) \longrightarrow \wp(X)$ satisfying (for every A, $B \subseteq X$):
a.1) $A \subseteq Cl(A)$ (Extensiveness)
a.2) $Cl(Cl(A)) \subseteq Cl(A)$ (Idempotency)
a.3) If $A \subseteq B$, then $Cl(A) \subseteq Cl(B)$ (Monotonicity).
In addition:
(b) A **closure relation on** X (c.r.) is a relation $\vdash \subseteq \wp(X) \times X$ verifying (for every A, $B \subseteq X$, $x \in X$):
b.1) If $x \in A$ then $A \vdash x$
b.2) If $B \vdash x$, and $A \vdash y$ for every $y \in B$, then $A \vdash x$.
b.3) If $A \vdash x$ and $A \subseteq B$, then $B \vdash x$.
Finally:
(c) A **closure system on** X (c.s.) is a family $K \subseteq \wp(X)$ closed by arbitrary intersections (note that, by vacuity, $X \in K$ for every c.s.).

All the previous notions can be interdefined, as it is well-known.

Proposition 1.2. If Cl is a c.o. on X, then $\vdash_{Cl} \subseteq \wp(X) \times X$ defined by: $A \vdash_{Cl} x$ iff $x \in Cl(A)$ is a c.r. on X. Reciprocally, if $\vdash$ is a c.r. on X, the map $Cl_{\vdash} : \wp(X) \to \wp(X)$ defined as: $Cl(A) = \{x \in X : A \vdash x\}$ is a c.o. on X. Moreover, $Cl_{\vdash_{Cl}} = Cl$, and $\vdash_{Cl_{\vdash}} = \vdash$.

Proposition 1.3. If Cl is a c.o. on X, then $K_{Cl}{:=}\{Y \subseteq X : Y = Cl(Y)\}$ is a c.s. on X. Reciprocally, for every closure system $K \subseteq \wp(X)$, the map $Cl_K : \wp(X) \longrightarrow \wp(X)$ defined as (for every $A \subseteq X$):

$$Cl_K(A) = \bigcap_{B \in \mathcal{F}_A^K} B$$

(where $\mathcal{F}_A^K{:=}\{B \in K : A \subseteq B\}$) is a c.o. on X. Moreover, $K_{Cl_K} = K$, and $Cl_{K_{Cl}} = Cl$.

From the previous results, the following definition makes sense:

Definition 1.4. A **closure space** (CSP) is a pair (X, K), being K a c.s. on X. Equivalently, a CSP is a pair (X, Cl), or a pair $(X, \vdash)$, without risk of confusion.

Proposition 1.5. Given a CSP (X, K), the pair $(K, \subseteq)$ is a complete lattice, with $\bigwedge_{i \in I}^{K} A_i = \bigcap_{i \in I} A_i$, and $\bigvee_{i \in I}^{!A_i} = Cl_K(\bigcup_{i \in I} A_i)$. Here, the greatest (lowest) element of the lattice $(K, \subseteq)$ is $A_1^K = X$ ($A_0^K = Cl_K(\emptyset)$).

All the notions above indicated will be used several times along this paper, depending of the results to be discussed (anyway, the more usual formalization of CSP to be considered here is the based on c. s. (X, K)). Only when we need to indicate if every one of such characterizations is related to another one, we will denote the CSP by (X, Cl_K), (X, K_{Cl}), $(X, \vdash_{Cl})$, and so on. Besides that, when we need to relate two different CSP, they will be distinguished by means of subscripts and/or superscripts.

2. Relative Subspaces and the M(T)-Spaces

The definition of the $M(T)$-spaces is motivated by the notion of *relative closure space*, as it was already indicated. Let us recall a mimimum about it, taking as starting point a fixed CSP $(X, \overline{K})$: let $(X, \overline{K})$ be a CSP, and consider $Y \subseteq X$. The **closure space relative to** Y **(induced by** $\overline{K}$**)** is the pair $(Y, \overline{K}_Y)$, being the c.s. $\overline{K}_Y$ defined by: $\overline{K}_Y{:=}\{G \subseteq Y : \ exists\ F \in \overline{K}$ verifying $G = F \cap Y\}$. The c.o. (c.r.) respective will be denoted by $Cl_{\overline{K}_Y}$ ($\vdash_{\overline{K}_Y}$).
It is easy to see that the pair $(Y, \overline{K}_Y)$, as defined above, is a CSP, indeed. In this context, $\bigcap \emptyset = Y$. Besides that, many properties about relative spaces are direct adaptation from General Topology (see [6], for instance):

Proposition 2.1. Let $(X, \overline{K})$ be a c.s., $(Y, \overline{K}_Y)$ a relative subspace and $A \subseteq Y$. Then:
(1) $Cl_{\overline{K}_Y}(A) = Y \cap Cl_{\overline{K}}(A)$.
(2) $A \vdash_{\overline{K}_Y} x$ iff $A \vdash_{Cl_{\overline{K}}} x$ and $x \in Y$.
(3) If $(Y, \overline{K}_Y)$ is a subspace of $(X, \overline{K})$, $H \in \overline{K}_Y$ and $Y \in \overline{K}$ then $H \in \overline{K}$.
(4) For every $A \subseteq Y$, $A \in \overline{K}_Y$ iff $A \in \overline{K}$.

Again, we emphasize that the subset $Y \subseteq X$ defines a CSP taking the own set Y as its support. On the other hand, the last property above indicated will be important to us because it induces the following problem: it is possible to determine a CSP verifying such property *but considering X as its support*? In other words, we wish to solve the following:

Problem 1. Given a CSP $(X, \overline{K})$ and $T \subseteq X$, find another one (X, K') such that:
$(*)$ For every $A \subseteq T$, $A \in \overline{K}$ iff $A \in K'$.

The more natural solution to this problem is to consider $\overline{K}^*_T{:=}\overline{K}_T \cup \{X\}$. It is easy to see that $\overline{K}^*_T$ is a c.s., indeed. Moreover, it is the *smaller* c.s. verifying $(*)$ (i.e., for every CSP (X, K) verifying such property, $\overline{K}^*_T \subseteq K$). So, our original motivation can be refined now: ¿What is the *higher* closure system (if there exists one) verifying $(*)$ in Problem 1? The answer (which will depends of the belonging of T to $\overline{K}$) is given in the sequel.

Definition 2.2. Let $(X, \overline{K})$ be a c.s., and $T \subseteq X$. The family $M(T) \subseteq \wp(X)$ is defined as $M(T) = \{A \subseteq X : A \cap T \in \overline{K}\}$. If $M(T)$ is closed by arbitrary intersections, then the pair $(X, M(T))$ will be called the **meet-closure space determined by** T. By extension, every c.s. (X, K') such that there is $T \subseteq X$ with $K' = M(T)$ will be called informally as an $M(T)$**-space**.

Proposition 2.3. The pair $(X, M(T))$ is a closure space if, and only if, $T \in \overline{K}$.

Proof. Suppose $T \in \overline{K}$. Then $M(T) \neq \emptyset$ (moreover, $X \in M(T)$). Suppose now that $\{A_i\}_{i \in I} \subseteq M(T)$. That is, $A_i \cap T \in \overline{K}$, for every $i \in I$. Then, $(\bigcap_{i \in I} A_i) \cap T = \bigcap_{i \in I}(A_i \cap T) \in \overline{K}$. On the other hand, if $M(T)$ is a c.s., then $X \in M(T)$, which means that $X \cap T = T \in \overline{K}$. □

From the previous result, when we will talk about any CSP $(X, M(T))$, it will be clear that $T \in \overline{K}$. Note that the members of $M(T)$ are included in X, as it was previously recquired. By the way, it should be obvious that the name "meet-closure" comes from the fact that we are using intersections to define $M(T)$. In addition, note that every c.s. $M(T)$ should be denoted in a more accurate way indicating the family $\overline{K}$ (because the set T can be considered as belonging to different c.s. K_i). Anyway, only in the case in which the specification of the family $\overline{K}$ is neccesary, we will denote $M(T)$ by $M^{\overline{K}}(T)$. Finally, the c.o. (resp. c. r.) determined by $M(T)$ will be indicated by $Cl_{M(T)}$ (resp. $\vdash_{M(T)}$), from now on.

Proposition 2.4. Given a closure space $(X, \overline{K})$, for every T, T_1, $T_2 \in \overline{K}$, it holds:
(a) For every $A \supseteq T$, $A \in M(T)$.
(b) $T_1 \subseteq T_2$ implies $M(T_2) \subseteq M(T_1)$ (and therefore $Cl_{M(T_1)}(A) \subseteq Cl_{M(T_2)}(A)$, for every $A \subseteq X$).
(c) $M(X){=}\overline{K}$.
(d) $\overline{K} \subseteq M(T)$, for every $T \in \overline{K}$ (and thus, for every $A \subseteq X$, $Cl_{M(T)}(A) \subseteq Cl_{\overline{K}}(A)$).
(e) For every $A \subseteq X$, $Cl_{M(A_0^{\overline{K}})}(A) = A \cup A_0^{\overline{K}}$. Therefore, $A \cup A_0^{\overline{K}} \supseteq Cl_{M(T)}(A)$.
(f) $Cl_{M(T)}(A) \subseteq A \cup T$, for every $A \subseteq X$.
(g) For every $A \subseteq T$, it holds:
  $(g.i)$ $A \in M(T)$ iff $A \in \overline{K}$.
  $(g.ii)$ $Cl_{M(T)}(A) \subseteq T$.
  $(g.iii)$ $Cl_{M(T)}(A) = Cl_{\overline{K}}(A)$
(h) The first elements of $\overline{K}$ and of $M(T)$ coincide. That is, $A_0^{\overline{K}} = A_0^{M(T)}$.

Proof. Let $(X, \overline{K})$ be a CSP fixed. Then: (a), (b) and (c) are obvious. Property (d) is valid from (b) and (c). To prove (e): since $A_0^{\overline{K}} = (A \cup A_0^{\overline{K}}) \cap A_0^{\overline{K}}$, we have $A_0^{\overline{K}} \cup A \in \mathcal{F}_A^{M(A_0^{\overline{K}})}$. Now, if $P \in \mathcal{F}_A^{M[A_0^{\overline{K}}]}$, then $A \subseteq P$, obviously. In addition $A_0^{\overline{K}} \cap P \in \overline{K}$, which implies $A_0^{\overline{K}} \subseteq P$. Thus, $A \cup A_0^{\overline{K}} \subseteq P$. All this implies $Cl_{M(A_0^{\overline{K}})}(A) = A_0^{\overline{K}} \cup A$. On the other hand, (f) is valid because $A \cup T \in \mathcal{F}_A^{M(T)}$. With respect to (g): since $T \in \mathcal{F}_A^{M(T)}$, $(g.i)$ and $(g.ii)$ are obvious. Let us prove $(g.iii)$ by double inclusion: first of all, since $Cl_{\overline{K}}(A) \in \overline{K} \subseteq M(T)$, we have $Cl_{\overline{K}}(A) \in \mathcal{F}_A^{M(T)}$. Hence, $Cl_{M(T)}(A) \subseteq Cl_{\overline{K}}(A)$. On the other hand, by $(g.ii)$, $Cl_{M(T)}(A) \subseteq T$. So, from $(g.i)$, $Cl_{M(T)}(A) \in \mathcal{F}_A^{\overline{K}}$ and, thus, $Cl_{\overline{K}}(A) \subseteq Cl_{M(T)}(A)$. Finally, to prove (h): Let $A_0^{\overline{K}}$ $(A_0^{M(T)})$ be the first element of $\overline{K}$ $(M(T))$. On one hand $A_0^{M(T)} \subseteq A_0^{\overline{K}}$, since $A_0^{\overline{K}} \in M(T)$, by (d). On the other hand, note that $A_0^{M(T)} \subseteq T$ (since $T \in M(T)$), and $A_0^{M(T)} \in M(T)$, obviously. So, by $(g.i)$, $A_0^{M(T)} \in \overline{K}$. Then, $A_0^{\overline{K}} \subseteq A_0^{M(T)}$. □

From (g) above, $M(T)$ share the same closed sets as $\overline{K}$ "*inside* T". Moreover, $M(T)$ is the "finest" c.s. (with support X) that satisfies this property, as it was recquired in Problem 1.

Proposition 2.5. Let $(X, \overline{K})$ be a c.s., and let $T \in \overline{K}$. For every c.s. (X, K) satisfying $(*)$: For every $A \subseteq T$, $A \in \overline{K}$ iff $A \in K$, it holds that $K \subseteq M(T)$.

Proof. Let $A \in K$. Since $T \in K$ too, $A \in M(T)$. □

Note, in addition, that the $M(T)$-spaces can be related with the "traditional relative subspaces", again applying (g) of Proposition 2.4 (and Proposition 2.1):

Proposition 2.6. Consider $(X, \overline{K})$ and $T \in \overline{K}$ as before. For every $B \subseteq T$, $(B, \overline{K}_B)$=$(B, M(T)_B)$. In particular, $(T, \overline{K}_T)$=$(T, M(T)_T)$.

We conclude this section characterizing directly $Cl_{M(T)}$ by means of $Cl_{\overline{K}}$ (this result will be useful in the next sections).

Lemma 2.7. $Cl_{M(T)}(A) = Cl_{\overline{K}}(A \cap T) \cup A$ for every $A \subseteq X$.

Proof. Since $A \cap T \subseteq T \in \overline{K}$, $Cl_{\overline{K}}(A \cap T) \cap T = Cl_{\overline{K}}(A \cap T)$. From this, $(Cl_{\overline{K}}(A \cap T) \cup A) \cap T = (Cl_{\overline{K}}(A \cap T) \cap T) \cup (A \cap T) = Cl_{\overline{K}}(A \cap T) \cup (A \cap T) = Cl_{\overline{K}}(A \cap T) \in \overline{K}$. Hence, $(Cl_{\overline{K}}(A \cap T) \cup A) \in M(T)$. So, it is obvious that $(Cl_{\overline{K}}(A \cap T) \cup A) \in \mathcal{F}_A^{M(T)}$. Now, for every $B \in \mathcal{F}_A^{M(T)}$, $A \cap T \subseteq B \cap T \in \overline{K}$, which implies $Cl_{\overline{K}}(A \cap T) \subseteq B$ and then $Cl_{\overline{K}}(A \cap T) \cup A \subseteq B$. Apply Proposition 1.3 now. □

3. Ordered Structures referred to $M(T)$-spaces

The definition of $M(T)$-spaces suggests some order-theoretic questions to be answered. For that, let us remember some well-known about the internal lattice-theoretic structure of the closure systems. First of all, it is obvious (from Proposition 1.5) that $(M(T), \subseteq)$ is a complete lattice, for every $T \in \overline{K}$. Moreover, from Lemma 2.7 it follows straightforwardly that (given $\{A_i\}_{i \in I} \subseteq M(T)$):

$$\bigwedge_{i\in I}^{M(T)} A_i = [\bigwedge_{i\in I}^{\overline{K}} (A_i \cap T)] \cup \bigcap_{i\in I} A_i;$$

$$\bigvee_{i\in I}^{M(T)} A_i = Cl_{\overline{K}}(\bigcup_{i\in I} (A_i \cap T)) \cup \bigcup_{i\in I} A_i = [\bigvee_{i\in I}^{\overline{K}} (A_i \cap T)] \cup \bigcup_{i\in I} A_i.$$

That is, $\bigwedge_{i\in I}^{M(T)}$ $(\bigvee_{i\in I}^{M(T)})$ can be naturally expressed in terms of $\bigwedge_{i\in I}^{\overline{K}}$ $(\bigvee_{i\in I}^{K!}$).

Some more difficult questions about these order-theoretic relations are based on the *lattice of all the closure systems*, with a fixed support X (see [14]).

Proposition 3.1. Given a fixed set $X \neq \emptyset$, the pair $\mathbb{CSP}(X)$:=$(CSP(X), \subseteq)$ (considering here $CSP(X)$ $= \{K : K$ is a c.s. of $X\})$ is a complete lattice. In this case, $\bigwedge^{\mathbb{CSP}(X)}\{K_i : i \in I\} = \bigcap_{i\in I} K_i$, meanwhile that $\bigvee^{\mathbb{CSP}(X)}\{K_i : i \in I\} = \bigwedge^{\mathbb{CSP}(X)}\{K : K_i \subseteq K,$ for every $i \in I\}$ [1].

So, given $(X, \overline{K})$, $T \in \overline{K}$ and $REL_{\overline{K}}(T) := \{K : K$ is a c.s. of X satisfying $(*)$ of Problem 1$\}$, we have that $(\mathbb{REL}_{\overline{K}}(T), \subseteq)$ is a bounded subposet of $(\mathbb{CSP}(X), \subseteq)$ with first element $\overline{K}^*_T$ and greatest element $M(T)$. Moreover:

Proposition 3.2. $\mathbb{REL}_{\overline{K}}(T)$ is a complete subsemilattice of $\mathbb{CSP}(X)$.

Proof. Straightforward. □

Another subposet of $\mathbb{CSP}(X)$ is the system $\mathbb{M}(\overline{K})$:=$(M(\overline{K}), \subseteq)$, being $M(\overline{K})$ defined as follows: $M(\overline{K}) := \{K : K = M(T),$ for some $T \in \overline{K}\}$. Again, it is natural here to ask the following question: it is $(\mathbb{M}(\overline{K}), \subseteq)$ a sublattice of $(\mathbb{CSP}(X), \subseteq)$? Unfortunately, in this case we have:

Proposition 3.3. $(\mathbb{M}(\overline{K}), \subseteq)$ is not a sublattice of $(\mathbb{CSP}(X), \subseteq)$, in general.

Proof. The following counterexample proofs our claim: Consider X:=$[1, 7]_{\mathbb{N}} = \{x \in \mathbb{N} : 1 \leq x \leq 7\}$. It is easy to see that the family $\overline{K}^{\star}$:= $\{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4, 5\}, \{1, 2, 3, 7\}, \{1, 2, 6\}, X\}$ is a CSP. In addition, $M(\{1\}) = M(\{1, 2\}) = M(\{1, 3\}) = \{W \subseteq X : 1 \in W\}$. Consider now T_1:=$\{1, 4, 5\}$ and T_2:=$\{1, 2, 3, 7\}$. Even when $M(T_1)$, $M(T_2) \in \mathbb{M}(\overline{K}^{\star})$, we have that $M(T_1) \bigwedge^{\mathbb{CSP}(X)} M(T_2) = M(T_1) \cap M(T_2) \notin \mathbb{M}(\overline{K}^{\star})$. To prove our claim, let us show $(*)$: for every $T \in \overline{K}^{\star}$, $M(T) \neq M(T_1) \cap M(T_2)$. First of all, $M(\{1\}) \neq M(T_1) \cap M(T_2)$, because $\{1, 4\} \in M(\{1\}) \setminus M(T_1)$. Thus, $M(T_1) \cap M(T_2) \neq M(\{1, 2\})$, and therefore $M(T_1) \cap M(T_2) \neq M(\{1, 3\})$. In addition, $\{1, 3, 6\} \in M(T_1) \cap M(T_2) \setminus M(\{1, 2, 6\})$, $\{1, 7\} \in M(\{1, 4, 5\}) \setminus M(T_1) \cap M(T_2)$, $\{1, 5\} \in M(\{1, 2, 3, 7\}) \setminus M(T_1) \cap M(T_2)$. Finally, $M(X) = \overline{K}^{\star} \neq M(T_1) \cap M(T_2)$, obviously. From all these facts together, $(*)$ is valid. □

Note here that $\mathbb{M}(\overline{K}^{\star})$, in the previous example, is a *lattice itself* (indeed, it is isomorphic to the non-distributive lattice M_5: see [3]). So, an interesting open problem motivated by this is the following: it is the system $(\mathbb{M}(\overline{K}), \subseteq)$ (given *any closure system* $\overline{K}$) a lattice itself, independently of $(\mathbb{CLS}(X), \subseteq)$? We will return to this problem in the last section.

4. Continuity in the $M(T)$-spaces

In this section we will study some properties of continuous functions, when applied to CSP of the form $(X, M(T))$. For that, recall first the following basic notions, to unify notation:

Definition 4.1. Let (X_i, K_i) $(i = 1, 2)$ be two CSP. The map $f : X_1 \longrightarrow X_2$ is **a** (X_1, K_2)**-**(X_2, K_2) **- continuous function** iff, for every $E \in K_2$, $f^{-1}(E) \in K_1$. When $(X_1, K_1) = (X_2, K_2)$ we will say that f is (X_1, K_1)-continuous, to simplify notation.

Some well-known equivalent formulations to Definition 4.1 that will be used without explicit mentions are:

Theorem 4.2. Let (X_i, K_i) $(i = 1, 2)$ be two closure spaces, being Cl_i $(\vdash_i)$ their corresponding closure operators (relations). For every function $f : X_1 \longrightarrow X_2$, the following affirmations are equivalent:
(a) f is (X_1, K_1)-(X_2, K_2)-continuous.
(b) If $A \cup \{x\} \subseteq X_1$ and $A \vdash_1 x$, then $f(A) \vdash_2 f(x)$.
(c) $f(Cl_1(A)) \subseteq Cl_2(f(A))$ for every $A \subseteq X_1$.
(d) $Cl_1(f^{-1}(B)) \subseteq f^{-1}(Cl_2(B))$ for every $B \subseteq X_2$.

[1] Of course, it is usual to adapt this result to the *dual lattice to* $\mathbb{CSP}(X)$, $\mathbb{CSP}^* = (CSP^*(X), \leq)$, where $CSP^*(X) = \{Cl : Cl$ is a c.o. of $X\}$, and $Cl_1 \leq Cl_2$ iff, for every $A \subseteq X$, $Cl_1(A) \subseteq Cl_2(A)$. Anyway, in this paper we will only work with $\mathbb{CSP}(X)$.

We will proceed now to analyze continuity in closure spaces of the form $(X, M(T))$. For that, we must distinguish when a given CSP of this kind constitutes the *domain* or the *codomain* of the functions to be analyzed. Let us begin with this obvious fact:

Proposition 4.3. If f is $(X_1, K_1) - (X_2, K_2)$-continuous, then it is $(X_1, M(T)) - (X_2, K_2)$-continuous, for every $T \in K_1$.

In addition, $M(T)$-continuity can be characterized by means of the "original" c.s. $\overline{K}$ (and therefore by $Cl_{\overline{K}}$ and $\vdash_{\overline{K}}$), taking into account Lemma 2.7.

Proposition 4.4. Let $(X, \overline{K})$ be a CSP and $T \in \overline{K}$ as before. Then:
(a) For every CSP $(X^{'}, K^{'})$, $f : X \to X^{'}$ is a $(X, M(T)) - (X^{'}, K^{'})$-continuous map if and only if, for every $A \subseteq X$, $f(Cl_{\overline{K}}(A \cap T) \cup A) \subseteq Cl_{K'}(f(A))$.
(b) For every CSP (X^{*}, K^{*}), $f : X^{*} \to X$ is a $(X^{*}, K^{*}) - (X, M(T))$-continuous map if and only if, for every $A \subseteq X$, $f(Cl_{K^{*}}(A)) \subseteq Cl_{\overline{K}}(f(A) \cap T) \cup f(A)$.

By the way, in the previous result, (a) can be simplified:

Proposition 4.5. Let (X_i, K_i) $i = 1, 2$ closure spaces, being Cl_i $(\vdash_i)$ their respective c.o. (c.r.). For every map $f : X_1 \longrightarrow X_2$, every $T \in K_1$, are equivalent:
(a) f is $(X_1, M(T))$-(X_2, K_2)-continuous.
(b) For every $A \cup \{x\} \subseteq X_1$, $A \cap T \vdash_1 x$ implies $f(A \cap T) \vdash_2 f(x)$.
(c) $f(Cl_1(A \cap T)) \subseteq Cl_2(f(A \cap T))$, for every $A \subseteq X_1$.
(d) $Cl_1(f^{-1}(B) \cap T) \subseteq f^{-1}(Cl_2(B)) \cap T$, for every $B \subseteq X_2$.

Proof. (a) implies (b): suppose (a) valid and consider $A \cup \{x\} \in X_1$, with $A \cap T \vdash_1 x$. We define $V := Cl_2(f(A \cap T)) \in K_2$. Note that $A \cap T \subseteq f^{-1}(V)$, and so $A \cap T \subseteq f^{-1}(V) \cap T$. Hence, $f^{-1}(V) \cap T \vdash_1 x$. Since $f^{-1}(V) \cap T \in K_1$, we have $f(x) \in f(f^{-1}(V)) \subseteq V = Cl_2(f(A \cap T))$. That is, $f(A \cap T) \vdash_2 f(x)$.
(b) implies (c): straightforward.
(c) implies (d): Consider $B \subseteq X_2$. By (c), it is valid that $f(Cl_1(f^{-1}(B) \cap T)) \subseteq Cl_2(f(f^{-1}(B) \cap T)) \subseteq Cl_2(B \cap f(T)) \subseteq Cl_2(B)$. Hence, $Cl_1(f^{-1}(B) \cap T) \subseteq f^{-1}(Cl_2(B))$. Moreover, $Cl_1(f^{-1}(B) \cap T) \subseteq T$ (since $T \in K_1$). Hence $Cl_1(f^{-1}(B) \cap T) \subseteq f^{-1}(Cl_2(B)) \cap T$.
(d) implies (a): $Cl_1(f^{-1}(B) \cap T) \subseteq f^{-1}(Cl_2(B)) \cap T$ for every $B \subseteq X_2$.
Let $E \in K_2$, by hypothesis $Cl_1(f^{-1}(E) \cap T) \subseteq f^{-1}(Cl_2(E)) \cap T = f^{-1}(E) \cap T$, so $f^{-1}(E) \cap T \in K_1$. That is, $E \in M(T)$, as it was desired. □

Corollary 4.6. For every CSP $(X, \overline{K})$, for every $T \in \overline{K}$, the following conditions are equivalent (for every function $f : X \to X$):
(a) f is $(X, M(T))$-continuous.
(b) For every $A \subseteq X$, $f(Cl_{\overline{K}}(A \cap T) \cup A) \subseteq Cl_{\overline{K}}(f(A) \cap T) \cup f(A)$.
(c) For every $A \subseteq X$, $f(Cl_{\overline{K}}(A \cap T)) \subseteq Cl_{\overline{K}}(f(A \cap T) \cap T) \cup f(A \cap T)$.

Proof. Conditions (a) and (b) are equivalent by Lemma 2.7 and Theorem 4.2. Conditions (a) and (c) are equivalent by Proposition 4.5 (c) and Lemma 2.7, again. □

5. Transference of Finitariness and Structurality

The preservation of continuity in the $M(T)$-spaces (Proposition 4.3) suggests the study of transference of other properties. In this section we will focused on two special ones, often studied when dealing with Abstract Logic (see [2]): finitariness and structurality. For that, we will fix some notation. First:

Definition 5.1. A CSP (X, K) is **finitary** iff, for every $Y \subseteq X$, $Y \cup \{a\} \subseteq X$ such that $a \in Cl_K(Y)$, there exists $Y_0 \subseteq Y$, Y_0 finite, such that $a \in Cl_K(Y_0)$.

Proposition 5.2. If $(X, \overline{K})$ is finitary, then $(X, M(T))$ is finitary too, for every $T \in \overline{K}$.

Proof. Suppose $(X, \overline{K})$ finitary, and let $a \in Cl_{M(T)}(Y) = Cl_{\overline{K}}(Y \cap T) \cup Y$. If $a \in Y$, then $Y_0 = \{a\} \subseteq Y$, and $a \in Cl_{M(T)}(Y_0)$. If $a \in Cl_{\overline{K}}(Y \cap T)$, there is $Z_0 \subseteq Y \cap T$, Z_0 finite, such that $a \in Cl_{\overline{K}}(Z_0)$. Let $Y_0 := Z_0$ in this case: so, $a \in Cl_{\overline{K}}(Y_0) = Cl_{\overline{K}}(Y_0 \cap T) \subseteq Cl_{\overline{K}}(Y_0 \cap T) \cup Y_0 = Cl_{M(T)}(Y_0)$, with $Y_0 \subseteq Y$, Y_0 finite. □

That is, finitariness is preserved under applications of $M(T)$. As we said, it is an important property within the context of Abstract Logic (even when it can be studied far away from the scope of Logic, indeed). However, since the next property to be analyzed (structurality) is fully intrinsic to Abstract Logic, it is worth to define the basic notions relative to this area. For that, we are based on [1], [9] and [14]. Also, we will use some notions on [3], with some notational changes:

Definition 5.3. A **similarity type** is a sequence $S = (s_1, s_2, \ldots, s_k)$, with $s_i \in \omega$ $(1 \leq i \leq k)$ [2]. An **abstract algebra of type** S (or, briefly, an S-algebra) is a pair $\mathbf{A} = (A, F)$, being A a non-void set (the **support of A**) and $F = (f_1, \ldots, f_k)$ is the set of **operations of A** which are maps such that, for every $1 \leq i \leq k$, $f_i : A^{s_i} \to A$ (or, as it is usually said, f_i is of **arity** s_i). We say that the algebras $\mathbf{A_1}$ and $\mathbf{A_2}$ are **similar** if both have the same type S.

Once the idea of abstract algebra has been formalized, the other useful notion in this section is the following:

Definition 5.4. Let $\mathbf{A_i} = (A_i, F_i)$ $(i = 1, 2)$ be two similar algebras. We say that $h : A_1 \to A_2$ is an **homomorphism** iff, for every $1 \leq i \leq k$, $h(f_i^1(x_1, \ldots, x_{c_k})) = f_i^2(h(x_1), \ldots, h(x_{c_k}))$ (for every $x_1, \ldots, x_{c_k} \in A_1$). If $\mathbf{A_1} = \mathbf{A_2}$ we say that h is an **endomorphism**.

The previous notions allow to enrich the basic idea of closure spaces, in the following sense:

Definition 5.5. An **abstract logic** is, simply, a pair $\mathcal{L} = (\mathbf{A}_{\mathcal{L}}, K_{\mathcal{L}})$, being $\mathbf{A}_{\mathcal{L}} = (A_{\mathcal{L}}, F_{\mathcal{L}})$ an abstract algebra, and being $K_{\mathcal{L}}$ a c.s. over $A_{\mathcal{L}}$.

So, an abstract logic is a "very specific space closure", since its support is not "merely any set", but is related to some algebra. This allows to study some new properties that relate the closure-theoretic notions with the algebraic ones. One of this kind of properties is actually structurality.

Definition 5.6. We say that an abstract logic $\mathcal{L} = (\mathbf{A}_{\mathcal{L}}, K_{\mathcal{L}})$ is **structural** iff every endomorphism defined of $\mathbf{A}_{\mathcal{L}}$ is a continuous function (recall here Definition 4.1.). That is (by Theorem 4.2), if for every endomorphism $h : A_{\mathcal{L}} \to A_{\mathcal{L}}$, for every $B \subseteq A_{\mathcal{L}}$, $h(Cl_{K_{\mathcal{L}}}(B)) \subseteq Cl_{K_{\mathcal{L}}}(h(B))$.

It is worth to comment here the following point about notation: the expression "abstract logic" given in Definition 5.5 is indebted to the first researchers in this area (for that, see [1] or [11]). This name is motivated because such kind of CSP generalizes the more specific notion of *sentential logic*, which will be given in the sequel (because some results/examples about structurality on $M(T)$-spaces that we will show later deal with sentential logics, indeed).

Definition 5.7. Let $\mathcal{V}$ be a fixed, countable set, whose elements (denoted by $p_1, p_2, p_3 \ldots$) will be called **atomic formulas**. Given a similarity type $S = (s_1, \ldots, s_k)$ consider any couple $F = (c_1, \ldots, c_k)$ (with $F \cap \mathcal{V} = \emptyset$), (wherein F is called **the set of connectives of** $L(S)$) and define the **arity of** c_i as the number s_i. The **sentential language** $L(S)$**, generated by** S is the smallest set that verifies:
1) $\mathcal{V} \subseteq L(S)$, $c_i \in L(S)$ for every c_i of arity 0.
2) For every $c_i \in F$ with arity s_i, for every $\alpha_1, \ldots, \alpha_{s_i} \in L(S)$, the "string" $c_i(\alpha_1, \ldots, \alpha_{s_i}) \in L(S)$.
The elements of $L(S)$ will be simply called as the **formulas of** $L(S)$ [3].

[2] In the standard representation of similarity type, $s_i \geq s_j$ for every $1 \leq i \leq j \leq k$ but, obviously, this is not necessary.
[3] The definition of any formula $c_i(\alpha_1, \ldots, \alpha_{s_i})$ uses, implicitly, another set (of punctuation symbols), whose element are "(", ")" and ",". Anyway it is well-known that this set is not essential.

Note here that a sentential language (cf. Definition 5.7) *can be understood as an S-algebra* of the form $\mathbf{L}(S) = (L(S), F)$, indeed. More technically, $\mathbf{L}(S)$ is the *absolutely free algebra, generated by S over $\mathcal{V}$*, identifying any connective c_i of arity s_i as an s_i-ary operation on $L(S)$ [4] (see [1], [14]) . This essential idea motivates the following:

Definition 5.8. A **sentential (abstract) logic** is an abstract logic of the form $\mathcal{L} = (\mathbf{L}(S_\mathcal{L}), K_\mathcal{L})$, being $\mathbf{L}(S_\mathcal{L})$ a sentential language.

The previous definition works in an efficent way to deal with the more intuitive notions of logic, with a convenient formalism. So, for instance, Classical Sentential Logic is $CL = (\mathbf{L}(S_{CL}), K_{\models_{CL}})$, where $F_{CL} = \{\neg, \vee, \wedge, \rightarrow\}$ having similarity $S_{CL} = (1, 2, 2, 2)$, and $\models_{CL}$ is the closure relation defined by means of the well known (classical) two-valued truth-tables (by the way, $\models_{CL}$ determines K_{CL} and Cl_{CL} as usual).

In the standard literature, the terminology for sentential logics is often different from the one applied to a general CSP: given a sentential logic $\mathcal{L} = (\mathbf{L}(S_\mathcal{L}), K_\mathcal{L})$, the elements of $K_\mathcal{L}$ are usually called $\mathcal{L}$-*theories*. In addition, the c. o. $Cl_{K_\mathcal{L}}$ is called *the consequence operator of* $\mathcal{L}$, and the relation $\vdash_{K_\mathcal{L}}$ is called the *consequence relation* associated to $\mathcal{L}$. Besides that, every endomorphism $h : L(S) \rightarrow L(S)$ is called a *substitution*. Finally, continuous functions are sometimes called *translations* [5]. With this notation, a sentential logic $\mathcal{L}$ is structural iff all its substitutions are translations from $\mathcal{L}$ to itself. For instance, every matrix logic (such as any many-valued ones, including the classical logic) is structural. At a more abstract level, given any boolean algebra $\mathbf{A}$, if we consider the family $K := \{F \subseteq A : F \text{ is a filter of } A\}$, it holds that $(\mathbf{A}, K)$ is a structural logic. Some examples of non-structural (sentential) logics are Annotated Paraconsistent Logics (see [5], or [13]), and Halpern Epistemic Logic HKB' (see [10]).

Remark 5.9.

• Note the following property of sentential languages: for every homomorphism $h : L(S) \rightarrow L(S)$, $comp(\widehat{\sigma}(\alpha)) \geq comp(\alpha)$ (being $comp(\alpha)$ the number of connectives appearing in α). This fact will be useful in the sequel.

• The notion of sentential logic allows to notice in a nice way why, in the definition of structurality, it is not needed that (given a logic $\mathcal{L} = (L(S_\mathcal{L}), K_\mathcal{L})$), $Cl_{K_\mathcal{L}}(h(B)) = h(Cl_{K_\mathcal{L}}(B))$. For instance, consider CL the classical logic (which is structural), and the map $h : \mathcal{V} \rightarrow L(S_{CL})$ defined by: $h(\alpha) = \alpha \vee \alpha$ (for $\alpha \in \mathcal{V}$). Since $\mathbf{L}(S_{CL})$ is an absolutely free algebra, h can be extended to an endomorphism $\widehat{h} : L(S_{CL}) \rightarrow L(S_{CL})$ in an unique way. Consider now $B = \{p_1 \rightarrow p_2\} \subseteq L(S_{CL})$: for every $\gamma \in \widehat{h}(Cl_{CL}(B))$, $comp(\gamma) \geq 2$, since $\mathcal{V} \cap Cl_{CL}(B) = \emptyset$. Now, let us define $\gamma_0 := p_1 \rightarrow p_1 \in L(S_{CL})$. Obviously $\gamma_0 \in Cl_{CL}(\emptyset) \subseteq Cl_{CL}(\widehat{h}(B))$. In addition, since $comp(\gamma_0) = 1$, we have $\gamma_0 \notin \widehat{h}(Cl_{CL}(B))$. Hence, $Cl_{CL}(\widehat{h}(B)) \nsubseteq \widehat{h}(K_{CL}(B))$.

• However, there is a special case of abstract logic $\mathcal{L} = (\mathbf{A}_\mathcal{L}, K_\mathcal{L})$ where, for every endomorphism h, $h(Cl_{K_\mathcal{L}}(B)) = Cl_{K_\mathcal{L}}(h(B))$, for every $B \subseteq A$: consider *any algebra* $\mathbf{A} = (A, F)$, and $K = \{B \subseteq A : B \text{ is a subuniverse of } A\}$. Then $Cl_K(B) = Sg(B)$, the *subuniverse generated by* B, and it is possible to prove that for every endomorphism $h : A \rightarrow A$, $h(Sg(B)) = Sg(h(B))$ [6].

We can analyze the transference of structurality to $M(T)$-spaces now. Unfortunately, as a first result, we have:

Proposition 5.10. Structurality is not preserved by means of $M(T)$-spaces. That is, there are structural abstract logics $(\mathbf{A}, \overline{K})$ and sets $T \in \overline{K}$ such that the logic $(A, M(T))$ is not structural.

Proof. The following counterexample (dealing with classical sentential logic CL again) proves our claim: first, CL is a structural logic, as it was previously commented. Let $T := Cl_{CL}(\{p_1\}) \in K_{CL}$. In additon,

[4] By the way, for every pair of absolutely free algebras $\mathbf{A_1}$, $\mathbf{A_2}$ with the same type of similarity and the same number of generators (in this case, $|\mathcal{V}|$), $\mathbf{A_1}$ is isomorphic to $\mathbf{A_2}$ (see [3]). This fact justifies the absence of restrictions for the set F.

[5] In some literature, the notion of translation is actually more general than continuity. See [4], for instance.

[6] This result is a particular case of a stronger one, as it is well known: this property is valid for every *homomorphism of algebras* $h : \mathbf{A_1} \rightarrow \mathbf{A_2}$. See [3], Theorem 6.6.

consider $\alpha{:=}p_1 \vee p_3$, and let $\widehat{h}$ the endomorphism univocally determined by $h : \mathcal{V} \to L(S_{CL})$ such that: $h(p_1) = h(p_2) = p_2$, and $h(p_i) = p_4$ for every p_i $(i \geq 3)$. Consider now $B := \{p_1, p_2\} \in L(S_{CL})$. Noting that $\alpha \in Cl_{CL}(B \cap T) \subseteq Cl_{M(T)}(B)$ (from Lemma 2.7), we have $\beta = \widehat{h}(\alpha) = p_2 \vee p_4 \in \widehat{h}(Cl_{M(T)}(B))$. On the other hand, $Cl_{M(T)}(\widehat{h}(B)) = Cl_{M(T)}(\{p_2\}) = Cl_{CL}(\{p_2\} \cap Cl_{CL}(\{p_1\})) \cup \{p_2\} = Cl_{CL}(\emptyset) \cup \{p_2\}$ So, $\beta \notin Cl_{M(T)}(\widehat{h}(B))$. Thus, $\widehat{h}(Cl_{M(T)}(B)) \nsubseteq Cl_{M(T)}(\widehat{h}(B))$. That is, $(L(S_{CL}), M(T))$ is not structural. □

This result suggests very interesting problems among which we find: what conditions must satisfy a structural abstract logic $\mathcal{L} = (\mathbf{A}, \overline{K})$ to transfer such property from $\overline{K}$ to $M(T)$? A (partial) answer is given in the following result:

Proposition 5.11. Let $\mathcal{L} = (A, \overline{K})$ be a logic, and $T \in K$. If $T \subseteq h^{-1}(T)$ *for every endomorphism* $h : A \to A$, then, $(A, M(T))$ is structural.

Proof. Suppose $B \in M(T)$, and let $h : A \to A$ be any endomorphism. Since $B \cap T \in \overline{K}$, then $h^{-1}(B \cap T)$ $= h^{-1}(B) \cap h^{-1}(T) \in \overline{K}$. Now, since $T \in \overline{K}$, we have $h^{-1}(B) \cap h^{-1}(T) \cap T \in \overline{K}$. Since (by Hypothesis), $h^{-1}(T) \cap T = T$, we have $h^{-1}(B) \cap T \in \overline{K}$, and so $h^{-1}(B) \in M(T)$, as it is desired. □

Corollary 5.12. Let $(\mathbf{A}, \overline{K})$ be a structural logic with $\mathbf{A} = (A, F)$, and $F_0{:=}\{f \in F : \ f \text{ has arity } 0\}$. Then, for every $T \in \overline{K}$ such that $T \subseteq F_0$, it holds that $(A, M(T))$ is structural, too.

Proof. Obvious, because $T \subseteq F_0$ implies $h(T) = T$ for every endomorphism $h : A \to A$. □

This result seems a little strict, but there are cases wherein such a kind of logics appear in a natural way. For instance:

Example 5.13. Let $\mathcal{L} = (\mathbf{A}, \overline{K})$ be the abstract logic such that $\mathbf{A}$ is a Boolean algebra with support A, and consider $F_0 := \{\overline{0}, \overline{1}\}$ be the set of the standard 0-ary operations of $\mathbf{A}$, and $\overline{K} := \{F :$ F is a filter of $\mathbf{A}\}$. If $T = \{1\}$, then $(A, M(T))$ is a structural logic, since $(A, \overline{K})$ is it a structural one, too. By the way, $M(T) = \{D \subseteq A : 1 \in D\}$.

6. Final Conclusions

This note begins the study of a reasonable generalization of the relative closure spaces, as it was previously mentioned. The main motivation of this is based on the treatment and application of this kind of CSPs to Abstract (Sentential) Logics, as it is shown in the last section. It is maybe with this purpose that all the technical results presented in the rest of the sections have been developed. Anyway, some results here shown suggest different lines of research in the future, even outside of the scope of Abstract Logic. Among them, we mention:

- Recovering: in which way a given CSP (X, K) can be *recovered* once a CSP $(X, M(T))$ is known? That is: which properties should have a given closure space $(X, M(T))$ to obtain the "original" space (X, K)?
- About $(\mathbb{M}(K), \subseteq)$ (see Section 3): despite Proposition 3.3, a very interesting open problem here is to know if such poset is a lattice itself.
- Continuity: in Section 4 we have obtained a characterization of the continuous maps possessing $M(T)$-spaces in its *domains* (Proposition 4.5). It would be desirable a similar result for continuous maps with $M(T)$-spaces in the codomain of such functions.
- $M(T)$-spaces in the context of Leibniz Hierarchy: continuing with Abstract Logic in a deeper level, it would be reasonable to study the preservation of the properties that define the so-called *Leibniz Hierarchy*, essential in the the subarea known as *Abstract Algebraic Logic* (see [8] or [9]). That is, for instance: is protoalgebraizability (equivalentiality, algebraizability) preserved in the $M(T)$-spaces? It is worth to note that these properties are usually referred to structural logics. This justify this first approximation to the preservation of structurality exposed in Section 5. This and all the previous problems exposed here justify this seminal study of $M(T)$-spaces under our point of view.

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