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On $S_{\alpha}^{\beta}(\theta, A, F)-$Convergence and Strong $N_{\alpha}^{\beta}(\theta, A, F)-$Convergence

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abstract: In this paper, we introduce strong $N_{\alpha}^{\beta}(\theta, A, F)-$convergence and $S_{\alpha}^{\beta}(\theta, A, F)-$convergence with respect to a sequence of modulus functions and give some connections between strongly $N_{\alpha}^{\beta}(\theta, A, F)-$convergent sequences and $S_{\alpha}^{\beta}(\theta, A, F)-$convergent sequences for $0 < \alpha \le \beta \le 1$.

Key Words: $A-$statistical convergence, lacunary sequence, modulus function.

Contents

1. Introduction

In 1951, Steinhaus [34] and Fast [16] introduced the concept of statistical convergence and later in 1959, Schoenberg [26] reintroduced independently. Bhardwaj and Dhawan [1], Çakallı ([5], [6]), Caserta et al. [7], Çınar et al. [8], Connor [10], Çolak [9], Demirci [11], Di Maio and Kočinac [12], et al. ([13], [14], [15], [25]), Fridy [18], Işık et al. ([20], [21]), Salat [24], Şengul et al. ([27], [28], [29], [30]), Srivastava and Et [33], Taylan [35] and many authors investigated some arguments related to this notion.

A modulus f is a function from $[0, \infty)$ to $[0, \infty)$ such that

i) $f(x) = 0$ if and only if $x = 0$,

ii) $f(x + y) \le f(x) + f(y)$ for $x, y \ge 0$,

iii) f is increasing,

iv) f is continuous from the right at 0.

It follows that f must be continuous in everywhere on $[0, \infty)$. A modulus may be unbounded or bounded.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ of non-negative integers such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1 = k_1$ for convenience.

In [19], Fridy and Orhan introduced the concept of lacunary statistical convergence in the sense that a sequence (x_k) of real numbers is called lacunary statistically convergent to a real number ℓ, if

$$\lim_{r\to\infty} \frac{1}{h_r} \left|\{k \in I_r : |x_k - \ell| \ge \varepsilon\}\right| = 0$$

for every positive real number ε.

Recently, the set of all strong $N_{\theta}^{\alpha}(A, F)-$convergent sequences with respect to a sequence of modulus functions was defined by Şengül and Arıca [31] as below

$$N_{\theta}^{\alpha}(A, F) = \left\{ x = (x_i) : \lim_{r\to\infty} \frac{1}{h_r^{\alpha}} \sum_{i\in I_r} f_i(|A_i(x) - \ell|) = 0, \text{ for some } \ell \right\}.$$

Lacunary sequence spaces have been studied in ([2], [3], [4], [17], [19], [20], [22], [23], [29], [31], [36]).

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2. Main Results

In this section, we will give definitions of lacunary strong $N_{\alpha}^{\beta}(\theta,A,F)-$convergence and $S_{\alpha}^{\beta}(\theta,A,F)-$convergence, where $A=(a_{ik})$ is an infinite matrix of complex numbers, $F=(f_i)$ is a sequence of modulus functions and $0<\alpha\leq\beta\leq 1$, and give some results related to these concepts.

Definition 2.1. *Let $\theta=(k_r)$ be a lacunary sequence, $A=(a_{ik})$ be an infinite matrix of complex numbers, $F=(f_i)$ be a sequence of modulus functions and $0<\alpha\leq\beta\leq 1$. We say that the sequence $x=(x_k)$ is lacunary strong $A-$convergent of order (α,β) to a number ℓ with respect to a sequence of modulus functions (or $N_{\alpha}^{\beta}(\theta,A,F)-$convergent to ℓ) if*

$$\lim_{r}\frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r}f_i(|A_i(x)-\ell|)\right)^{\beta}=0,\ \textit{for some } \ell.$$

In this case, we write $x_i\rightarrow\ell(N_{\alpha}^{\beta}(\theta,A,F))$ or $N_{\alpha}^{\beta}(\theta,A,F)-\lim x_i=\ell$. The set of all lacunary strong $A-$convergent sequences of order (α,β) to a number ℓ with respect to a sequence of modulus functions will be denoted by $N_{\alpha}^{\beta}(\theta,A,F)$. If we get $f_i=f$ for all $i\in\mathbb{N}$, then $N_{\alpha}^{\beta}(\theta,A,F)=N_{\alpha}^{\beta}(\theta,A,f)$ which was studied by Şengül et al. [32]. If $A=I$ unit matrix, we write $N_{\alpha}^{\beta}(\theta,F)$ for $N_{\alpha}^{\beta}(\theta,A,F)$. In case of $\theta=(2^r)$, we write $w_{\alpha}^{\beta}(A,F)$ instead of $N_{\alpha}^{\beta}(\theta,A,F)$. If we get $f_i=f$ for all $i\in\mathbb{N}$, then $w_{\alpha}^{\beta}(A,F)=w_{\alpha}^{\beta}(A,f)$. If $A=I$ unit matrix, we write $w_{\alpha}^{\beta}(F)$ for $w_{\alpha}^{\beta}(A,F)$.

It can be shown that $N_{\alpha}^{\beta}(\theta,A,F)$ is a linear space.

Definition 2.2. *Let $\theta=(k_r)$ be a lacunary sequence, $A=(a_{ik})$ be an infinite matrix of complex numbers, $F=(f_i)$ be a sequence of modulus functions and $0<\alpha\leq\beta\leq 1$ be given. Then a sequence $x=(x_k)$ is said to be $S_{\alpha}^{\beta}(\theta,A,F)-$statistically convergent with respect to a sequence of modulus functions (or lacunary $A-$statistically convergent sequence of order (α,β) with respect to a sequence of modulus functions), if there is a real number ℓ such that*

$$\lim_{r\rightarrow\infty}\frac{1}{h_r^{\alpha}}\left|\{i\in I_r: f_i\left(|A_i(x)-\ell|\right)\geq\varepsilon\}\right|^{\beta}=0.$$

The set of all lacunary $A-$statistical convergent sequences of order (α,β) with respect to a sequence of modulus functions will be denoted by $S_{\alpha}^{\beta}(\theta,A,F)$. If we get $f_i(x)=x$ for all $i\in\mathbb{N}$, then $S_{\alpha}^{\beta}(\theta,A,F)=S_{\alpha}^{\beta}(\theta,A)$ which was studied by Şengül et al. [32]. If $\theta=(2^r)$, we write $S_{\alpha}^{\beta}(A,F)$ instead of $S_{\alpha}^{\beta}(\theta,A,F)$. If $\theta=(2^r)$ and $f_i(x)=x$ for all $i\in\mathbb{N}$, we write $S_{\alpha}^{\beta}(A)$ instead of $S_{\alpha}^{\beta}(\theta,A,F)$.

Theorem 2.3. *If $N_{\alpha}^{\beta}(\theta,A,F)-\lim x_i=\ell_1$ and $N_{\alpha}^{\beta}(\theta,A,F)-\lim x_i=\ell_2$, then $\ell_1=\ell_2$.*

Proof. Since $N_{\alpha}^{\beta}(\theta,A,F)-\lim x_i=\ell_1$ and $N_{\alpha}^{\beta}(\theta,A,F)-\lim x_i=\ell_2$, we can write

$$\lim_{r\rightarrow\infty}\frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r}f_i(|A_i(x)-\ell_1|)\right)^{\beta}=0$$

and

$$\lim_{r\rightarrow\infty}\frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r}f_i(|A_i(x)-\ell_2|)\right)^{\beta}=0$$

for $0<\alpha\leq\beta\leq 1$. Since

$$\begin{aligned}|\ell_1-\ell_2| &= |\ell_1-\ell_2+A_i(x)-A_i(x)|\\ &\leq |A_i(x)-\ell_1|+|A_i(x)-\ell_2|,\end{aligned}$$

we get

$$\begin{aligned}\frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r}f_i(|\ell_1-\ell_2|)\right)^{\beta} &= \frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r}f_i(|\ell_1-\ell_2+A_i(x)-A_i(x)|)\right)^{\beta}\\ &\leq \frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r}f_i(|A_i(x)-\ell_1|)\right)^{\beta}+\frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r}f_i(|A_i(x)-\ell_2|)\right)^{\beta}.\end{aligned}$$

This is possible with $\ell_1=\ell_2$. □

Theorem 2.4. *Let $0<\alpha\leq\beta\leq1$. If $\lim\limits_{u\to\infty}\inf_i \frac{f_i(u)}{u}>0$, then $N_{\alpha}^{\beta}(\theta,A,F)\subseteq N_{\alpha}^{\beta}(\theta,A)$.*

Proof. If $\lim\limits_{u\to\infty}\inf_i \frac{f_i(u)}{u}>0$, then there exist a number $v>0$ such that $f_i(u)\geq vu$ for all $u>0$ and $i\in\mathbb{N}$. Let $x\in N_{\alpha}^{\beta}(\theta,A,F)$. It is clear that

$$\frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r} f_i(|A_i(x)-\ell|)\right)^{\beta}\geq\frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r} v\,|A_i(x)-\ell|\right)^{\beta}=v^{\beta}\frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r}|A_i(x)-\ell|\right)^{\beta}.$$

Therefore $x_i\to\ell(N_{\alpha}^{\beta}(\theta,A))$.

If $v=0$, then $N_{\alpha}^{\beta}(\theta,A,F)\subseteq N_{\alpha}^{\beta}(\theta,A)$ may not be provided. Consider $A=I$, $\ell=0$ and $f_i(x)=x^{\frac{3}{i}}(i\geq1,x>0)$. Define $x=(x_i)$ by for $r=1,2,3,...$

$$x_i=\begin{cases}\sqrt[3]{h_r}, & \text{if } i=\frac{k_r}{2}\\ 0, & \text{otherwise}\end{cases}.$$

We can write

$$\frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r} f_i(|A_i(x)|)\right)^{\beta}=\frac{1}{h_r^{\alpha}}\left(f_{\frac{k_r}{2}}(\sqrt[3]{h_r})\right)^{\beta}=\frac{1}{h_r^{\alpha}}\left(h_r^{\frac{2}{k_r}}\right)^{\beta}\to0,\quad(\text{as } r\to\infty)$$

for $\alpha>\frac{2}{k_r}$, $\beta=1$ and so $x\in N_{\alpha}^{\beta}(\theta,A,F)$. But

$$\frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r}|A_i(x)|\right)^{\beta}=\frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r}|x_i|\right)^{\beta}=\frac{1}{h_r^{\alpha}}\left(\sqrt[3]{h_r}\right)^{\beta}\to1,\quad(\text{as } r\to\infty)$$

for $\alpha=\frac{1}{3}$, $\beta=1$ and

$$\frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r}|A_i(x)|\right)^{\beta}=\frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r}|x_i|\right)^{\beta}=\frac{1}{h_r^{\alpha}}\left(\sqrt[3]{h_r}\right)^{\beta}\to\infty,\ (\text{as } r\to\infty)$$

for $\alpha<\frac{1}{3}$ and $\beta=1$. $x\notin N_{\alpha}^{\beta}(\theta,A)$ is obtained. As a result $v>0$ must be. □

Theorem 2.5. *Let (f_i) be pointwise convergent. If $\lim\limits_{i} f_i(u)>0$ for $u>0$, then $N_{\alpha}^{\beta}(\theta,A,F)\subseteq S_{\alpha}^{\beta}(\theta,A)$ for $0<\alpha\leq\beta\leq1$.*

Proof. Let $\varepsilon>0$ and $x_i\to\ell(N_{\alpha}^{\beta}(\theta,A,F))$. If $\lim\limits_{i} f_i(u)>0$, then there exist a number $\rho>0$ such that $f_i(\varepsilon)>\rho$ for $u>\varepsilon$ and $i\in\mathbb{N}$. We have

$$\begin{aligned}\frac{1}{h_r^{\alpha}}\left(\sum_{i\in I_r} f_i(|A_i(x)-\ell|)\right)^{\beta} &\geq \frac{1}{h_r^{\alpha}}\left(\sum_{\substack{i\in I_r\\ |A_i(x)-\ell|\geq\varepsilon}} f_i(|A_i(x)-\ell|)\right)^{\beta}\\ &\geq \frac{1}{h_r^{\alpha}}\left|\{i\in I_r:|A_i(x)-\ell|\geq\varepsilon\}\right|^{\beta}(f_i(\varepsilon))^{\beta}\\ &\geq \rho^{\beta}\frac{1}{h_r^{\alpha}}\left|\{i\in I_r:|A_i(x)-\ell|\geq\varepsilon\}\right|^{\beta}\end{aligned}$$

for $0<\alpha\leq\beta\leq1$. It follows that $x_i\to\ell(S_{\alpha}^{\beta}(\theta,A))$.

The following result is a consequence of Theorem 2.5. □

Corollary 2.6. *Let* $0<\alpha\leq\beta\leq 1$. *If* $\lim_i f_i(u)>0$ *for* $u>0$, *then* $w_\alpha^\beta(A,F)\subseteq S_\alpha^\beta(A)$.

Theorem 2.7. *Let* $0<\alpha\leq\beta\leq 1$.

i) If $\liminf q_r>1$, *then* $w_\alpha^\beta(A,F)\subseteq N_\alpha^\beta(\theta,A,F)$,

ii) If $\limsup \frac{k_r}{k_{r-1}^\alpha}<\infty$, *then* $N(\theta,A,F)\subseteq w_\alpha^\beta(A,F)$.

Proof. $i)$ Let $x_i\to\ell(w_\alpha^\beta(A,F))$ and $\liminf q_r>1$. There exist a $\delta>0$ such that $q_r=\frac{k_r}{k_{r-1}}\geq 1+\delta$. We have

$$\left(\frac{h_r}{k_r}\right)\geq\frac{\delta}{\delta+1}\Rightarrow\left(\frac{h_r}{k_r}\right)^\alpha\geq\left(\frac{\delta}{\delta+1}\right)^\alpha$$

for $0<\alpha\leq\beta\leq 1$. We can write

$$\begin{aligned}\frac{1}{k_r^\alpha}\left(\sum_{i=1}^{k_r}f_i(|A_i(x)-\ell|)\right)^\beta &\geq \frac{1}{k_r^\alpha}\left(\sum_{i\in I_r}f_i(|A_i(x)-\ell|)\right)^\beta\\ &= \left(\frac{h_r^\alpha}{k_r^\alpha}\right)\frac{1}{h_r^\alpha}\left(\sum_{i\in I_r}f_i(|A_i(x)-\ell|)\right)^\beta\\ &\geq \left(\frac{\delta}{\delta+1}\right)^\alpha\frac{1}{h_r^\alpha}\left(\sum_{i\in I_r}f_i(|A_i(x)-\ell|)\right)^\beta.\end{aligned}$$

So $x_i\to\ell(N_\alpha^\beta(\theta,A,F))$ is obtained.

$ii)$ Omitted.

□

Theorem 2.8. *Let* $\theta=(k_r)$ *and* $\theta'=(s_r)$ *be two lacunary sequences such that* $I_r\subset J_r$ *for all* $r\in\mathbb{N}$ *and let* $\alpha_1,\alpha_2,\beta_1$ *and* β_2 *be such that* $0<\alpha_1\leq\alpha_2\leq\beta_1\leq\beta_2\leq 1$,

(i) If

$$\lim_{r\to\infty}\inf\frac{h_r^{\alpha_1}}{\ell_r^{\alpha_2}}>0 \tag{1}$$

then $S_{\alpha_2}^{\beta_2}\left(\theta',A,F\right)\subset S_{\alpha_1}^{\beta_1}(\theta,A,F)$,

(ii) If

$$\lim_{r\to\infty}\frac{\ell_r}{h_r^{\alpha_2}}=1 \tag{2}$$

then $S_{\alpha_1}^{\beta_2}(\theta,A,F)\subset S_{\alpha_2}^{\beta_1}\left(\theta',A,F\right)$.

Proof. (i) Let $x\in S_{\alpha_2}^{\beta_2}\left(\theta',A,F\right)$. For given $\varepsilon>0$ we have

$$\{i\in J_r: f_i(|A_i(x)-\ell|)\geq\varepsilon\}\supseteq\{i\in I_r: f_i(|A_i(x)-\ell|)\geq\varepsilon\}$$

and so

$$\frac{1}{\ell_r^{\alpha_2}}\left|\{i\in J_r: f_i(|A_i(x)-\ell|)\geq\varepsilon\}\right|^{\beta_2}\geq\frac{h_r^{\alpha_1}}{\ell_r^{\alpha_2}}\frac{1}{h_r^{\alpha_1}}\left|\{i\in I_r: f_i(|A_i(x)-\ell|)\geq\varepsilon\}\right|^{\beta_1}.$$

Thus if $x\in S_{\alpha_2}^{\beta_2}\left(\theta',A,F\right)$, then $x\in S_{\alpha_1}^{\beta_1}(\theta,A,F)$.

(ii) Let $x \in S_{\alpha_1}^{\beta_2}(\theta, A, F)$ and (2) holds. We can write

$$
\begin{aligned}
\frac{1}{\ell_r^{\alpha_2}}\left|\{i \in J_r : f_i(|A_i(x)-\ell|) \geq \varepsilon\}\right|^{\beta_1} &= \frac{1}{\ell_r^{\alpha_2}}\left|\{s_{r-1} < i \leq k_{r-1} : f_i(|A_i(x)-\ell|) \geq \varepsilon\}\right|^{\beta_1} \\
&\quad + \frac{1}{\ell_r^{\alpha_2}}\left|\{k_r < i \leq s_r : f_i(|A_i(x)-\ell|) \geq \varepsilon\}\right|^{\beta_1} \\
&\quad + \frac{1}{\ell_r^{\alpha_2}}\left|\{k_{r-1} < i \leq k_r : f_i(|A_i(x)-\ell|) \geq \varepsilon\}\right|^{\beta_1} \\
&\leq \frac{(k_{r-1}-s_{r-1})^{\beta_1}}{\ell_r^{\alpha_2}} + \frac{(s_r-k_r)^{\beta_1}}{\ell_r^{\alpha_2}} \\
&\quad + \frac{1}{\ell_r^{\alpha_2}}\left|\{i \in I_r : f_i(|A_i(x)-\ell|) \geq \varepsilon\}\right|^{\beta_1} \\
&\leq \frac{k_{r-1}-s_{r-1}}{\ell_r^{\alpha_2}} + \frac{s_r-k_r}{\ell_r^{\alpha_2}} + \frac{1}{\ell_r^{\alpha_2}}\left|\{i \in I_r : f_i(|A_i(x)-\ell|) \geq \varepsilon\}\right|^{\beta_1} \\
&\leq \frac{\ell_r-h_r}{\ell_r^{\alpha_2}} + \frac{1}{\ell_r^{\alpha_2}}\left|\{i \in I_r : f_i(|A_i(x)-\ell|) \geq \varepsilon\}\right|^{\beta_1} \\
&\leq \frac{\ell_r-h_r^{\alpha_2}}{h_r^{\alpha_2}} + \frac{1}{h_r^{\alpha_1}}\left|\{i \in I_r : f_i(|A_i(x)-\ell|) \geq \varepsilon\}\right|^{\beta_2} \\
&\leq \left(\frac{\ell_r}{h_r^{\alpha_2}} - 1\right) + \frac{1}{h_r^{\alpha_1}}\left|\{i \in I_r : f_i(|A_i(x)-\ell|) \geq \varepsilon\}\right|^{\beta_2}
\end{aligned}
$$

for every $r \in \mathbb{N}$. Therefore $S_{\alpha_1}^{\beta_2}(\theta, A, F) \subset S_{\alpha_2}^{\beta_1}\left(\theta^{'}, A, F\right)$. □

The following result is consequence of Theorem 2.8.

Corollary 2.9. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$.*

(i) If (1) holds then, $S\left(\theta^{'}, A, F\right) \subset S(\theta, A, F)$ for $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 1$,

(ii) If (2) holds then, $S(\theta, A, F) \subset S\left(\theta^{'}, A, F\right)$ for $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 1$.

Theorem 2.10. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$ and let $\alpha_1, \alpha_2, \beta_1$ and β_2 be such that $0 < \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \leq 1$.*

(i) If (1) holds then, $N_{\alpha_2}^{\beta_2}\left(\theta^{'}, A, F\right) \subset S_{\alpha_1}^{\beta_1}(\theta, A, F)$,

(ii) If (2) holds and a sequence of modulus functions $F = (f_i)$ be bounded then, $S_{\alpha_1}(\theta, A, F) \subset N_{\alpha_2}\left(\theta^{'}, A, F\right)$.

Proof. (i) For any $\varepsilon > 0$, we have

$$\begin{aligned}
\left(\sum_{i\in J_r} f_i(|A_i(x)-\ell|)\right)^{\beta_2} &= \left(\sum_{\substack{i\in J_r\\ f_i(|A_i(x)-\ell|)\geq\varepsilon}} f_i(|A_i(x)-\ell|) + \sum_{\substack{i\in J_r\\ f_i(|A_i(x)-\ell|)<\varepsilon}} f_i(|A_i(x)-\ell|)\right)^{\beta_2} \\
&\geq \left(\sum_{\substack{i\in I_r\\ f_i(|A_i(x)-\ell|)\geq\varepsilon}} f_i(|A_i(x)-\ell|) + \sum_{\substack{i\in I_r\\ f_i(|A_i(x)-\ell|)<\varepsilon}} f_i(|A_i(x)-\ell|)\right)^{\beta_1} \\
&\geq \left(\sum_{\substack{i\in I_r\\ f_i(|A_i(x)-\ell|)\geq\varepsilon}} f_i(|A_i(x)-\ell|)\right)^{\beta_1} \\
&\geq |\{i\in I_r : f_i(|A_i(x)-\ell|)\geq\varepsilon\}|^{\beta_1}\varepsilon^{\beta_1}
\end{aligned}$$

and so that

$$\begin{aligned}
\frac{1}{\ell_r^{\alpha_2}}\left(\sum_{i\in J_r} f_i(|A_i(x)-\ell|)\right)^{\beta_2} &\geq \frac{1}{\ell_r^{\alpha_2}}|\{i\in I_r : f_i(|A_i(x)-\ell|)\geq\varepsilon\}|^{\beta_1}\varepsilon^{\beta_1} \\
&\geq \frac{h_r^{\alpha_1}}{\ell_r^{\alpha_2}}\frac{1}{h_r^{\alpha_1}}|\{i\in I_r : f_i(|A_i(x)-\ell|)\geq\varepsilon\}|^{\beta_1}\varepsilon^{\beta_1}.
\end{aligned}$$

Therefore $N_{\alpha_2}^{\beta_2}\left(\theta', A, F\right) \subset S_{\alpha_1}^{\beta_1}(\theta, A, F)$.

(ii) Let $x \in S_{\alpha_1}(\theta, A, F)$ and (2) holds. Assume that $F = (f_i)$ is bounded. Therefore $f_i(x) \leq K$, for a positive integer K and all $x \geq 0$. Now, since $I_r \subseteq J_r$ and $h_r \leq \ell_r$ for all $r \in N$, we can write

$$\begin{aligned}
\frac{1}{\ell_r^{\alpha_2}}\sum_{i\in J_r} f_i(|A_i(x)-\ell|) &= \frac{1}{\ell_r^{\alpha_2}}\sum_{i\in J_r - I_r} f_i(|A_i(x)-\ell|) + \frac{1}{\ell_r^{\alpha_2}}\sum_{i\in I_r} f_i(|A_i(x)-\ell|) \\
&\leq \frac{\ell_r - h_r}{\ell_r^{\alpha_2}}K + \frac{1}{\ell_r^{\alpha_2}}\sum_{i\in I_r} f_i(|A_i(x)-\ell|) \\
&\leq \frac{\ell_r - h_r^{\alpha_2}}{\ell_r^{\alpha_2}}K + \frac{1}{\ell_r^{\alpha_2}}\sum_{i\in I_r} f_i(|A_i(x)-\ell|) \\
&\leq \left(\frac{\ell_r}{h_r^{\alpha_2}} - 1\right)K + \frac{1}{h_r^{\alpha_2}}\sum_{\substack{i\in I_r\\ f_i(|A_i(x)-\ell|)\geq\varepsilon}} f_i(|A_i(x)-\ell|) \\
&\quad + \frac{1}{h_r^{\alpha_2}}\sum_{\substack{i\in I_r\\ f_i(|A_i(x)-\ell|)<\varepsilon}} f_i(|A_i(x)-\ell|) \\
&\leq \left(\frac{\ell_r}{h_r^{\alpha_2}} - 1\right)K + \frac{K}{h_r^{\alpha_1}}|\{i\in I_r : f_i(|A_i(x)-\ell|)\geq\varepsilon\}| + \frac{h_r}{h_r^{\alpha_2}}\varepsilon.
\end{aligned}$$

Therefore $S_{\alpha_1}(\theta, A, F) \subset N_{\alpha_2}\left(\theta', A, F\right)$.

□

The following result is consequence of Theorem 2.10.

Corollary 2.11. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$.*

(i) If (1) *holds then, $N\left(\theta', A, F\right) \subset S\left(\theta, A, F\right)$ for $\alpha_1 = \alpha_2 = 1$ and $\beta_1 = \beta_2 = 1$.*

(ii) If (2) *holds and a sequence of modulus functions $F = (f_i)$ be bounded then, $S\left(\theta, A, F\right) \subset N\left(\theta', A, F\right)$ for $\alpha_1 = \alpha_2 = 1$.*

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