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Upper Bound to Second Hankel Determinant for a family of Bi-Univalent Functions

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abstract: In the current investigation, we study a certain family of analytic and bi-univalent functions with respect to symmetric conjugate points defined in the open unit disk U and find an upper bounds for the second Hankel determinant $H_2(2)$ of the functions belongs to this class.

Key Words: Analytic functions, bi-univalent functions, symmetric conjugate, upper bounds, second Hankel determinant.

Contents

1. Introduction

Let $\mathcal{A}$ stand for the family of all analytic functions in the open unit disk $U = \{z \in C : |z| < 1\}$ and having the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let S indicate the family of all functions in $\mathcal{A}$ which are univalent in U.

One of the significant tools in the theory of univalent functions is Hankel determinant which are utility, for example, in showing that a function of bounded characteristic in U, i.e., a function which is a ratio of two bounded analytic functions, with its Laurent series around the origin having integral coefficients, is rational [8]. Also the Hankel determinant plays an important role in the study of singularities. Noonan and Thomas [19] defined the q^{th} Hankel determinant of $f \in \mathcal{A}$ for $n \geq 0$ and $q \geq 1$ as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

The Hankel determinants

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2$$

are well known as Fekete-Szegö and second Hankel determinant functionals, respectively. Fekete and Szegö [13] consider the further generalized functional $a_3 - \mu a_2^2$, where μ is real number. Recently, several authors established upper bounds for the Hankel determinant of functions belonging to various subclasses of univalent functions (see [1,3,9,16,17,18]).

According to the Koebe one-quarter theorem [11], every $f \in S$ has an inverse function f^{-1} which satisfies $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

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A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denote the family of bi-univalent functions in U satisfying (1.1). In fact, Srivastava et al. [23] have apparently revived the study of analytic and bi-univalent functions in recent years. Recently, many authors introduced various subclasses of the bi-univalent function class Σ and considered estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) (see [2,4,5,7,14,20,22,24,25]). The problem of finding the coefficient estimates on $|a_n|$ $(n = 3, 4, \cdots)$ for functions $f \in \Sigma$ is still an open problem.

On the other hand, Zaprawa [26,27] extended the study of the Fekete-Szegö problem for some classes of bi-univalent functions. Very recently, the upper bounds of $H_2(2)$ for some classes were discussed by Deniz et al. [10] (see also [6]).

El-Ashwah and Thomas [12] introduced the class S^*_{sc} of starlike functions with respect to symmetric conjugate points for $f \in S$ and satisfying the following condition:

$$Re\left\{\frac{zf'(z)}{f(z)-\overline{f(-\overline{z})}}\right\} > 0, \quad z \in U.$$

This class can be extended to other class in U, namely convex functions with respect to symmetric conjugate points. Let C_{sc} denote the class of convex functions with respect to symmetric conjugate points and satisfy the condition:

$$Re\left\{\frac{(zf'(z))'}{\left(f(z)-\overline{f(-\overline{z})}\right)'}\right\} > 0, \quad z \in U.$$

To prove our main results, we shall require the following lemmas.

Lemma 1.1. *[21] If the function $p \in \mathcal{P}$ is given by the series $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, then the sharp estimate $|p_k| \le 2$ $(k = 1, 2, 3, \cdots)$ holds.*

Lemma 1.2. *[15] If the function $p \in \mathcal{P}$, then*

$$2p_2 = p_1^2 + \left(4 - p_1^2\right) x$$

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right) x - p_1\left(4 - p_1^2\right) x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right) z,$$

fore some x, z with $|x| \le 1$ and $|z| \le 1$.

2. Main Results

We begin this section by defining the function family $\mathcal{N}^{sc}_{\Sigma}(\lambda, \gamma)$ as follows:

Definition 2.1. *A function $f \in \Sigma$ is said to be in the family $\mathcal{N}^{sc}_{\Sigma}(\lambda, \gamma)$ $(0 \le \lambda \le 1,\ 0 \le \gamma < 1)$ if it satisfies the conditions:*

$$Re\left\{\frac{\lambda z^2 f''(z) + zf'(z)}{\lambda z\left(f(z)-\overline{f(-\overline{z})}\right)' + (1-\lambda)\left(f(z)-\overline{f(-\overline{z})}\right)}\right\} > \frac{\gamma}{2}$$

and

$$Re\left\{\frac{\lambda w^2 g''(w) + wg'(w)}{\lambda w\left(g(w)-\overline{g(-\overline{w})}\right)' + (1-\lambda)\left(g(w)-\overline{g(-\overline{w})}\right)}\right\} > \frac{\gamma}{2},$$

where $g = f^{-1}$ is given by (1.2).

Theorem 2.2. *Let $f \in \mathcal{A}$ belongs to the family $\mathcal{N}_{\Sigma}^{sc}(\lambda,\gamma)$. Then*

$$\left|a_2a_4 - a_3^2\right| \leq \begin{cases} \frac{(1-\gamma)^2}{2(3\lambda+1)(\lambda+1)^3}\left((1-\gamma)^2\left(\lambda(2\lambda-3)-1\right)+(\lambda+1)^2\right), \\ \gamma \in \left[0, 1-\frac{(\lambda+1)\left((3\lambda+1)+\sqrt{(3\lambda+1)^2+8\lambda^2(\lambda(2\lambda-3)-1)}\right)}{2(2\lambda+1)(\lambda(2\lambda-3)-1)}\right]; \\ \frac{(1-\gamma)^2}{8(\lambda(3\lambda+4)+1)}\frac{(1-\gamma)^2(3\lambda+1)\left(8\lambda^2(2\lambda-1)-35\lambda-9\right)-6(1-\gamma)(\lambda+1)(6\lambda^2+5\lambda+1)-(\lambda+1)^2(4\lambda(3\lambda+1)+1)}{(1-\gamma)^2(2\lambda+1)^2(\lambda(2\lambda-3)-1)-(1-\gamma)(\lambda+1)(6\lambda^2+5\lambda+1)-2\lambda^2(\lambda+1)^2}, \\ \gamma \in \left(1-\frac{(\lambda+1)\left((3\lambda+1)+\sqrt{(3\lambda+1)^2+8(2\lambda+1)^2(\lambda(2\lambda-3)-1)}\right)}{4(2\lambda+1)(\lambda(2\lambda-3)-1)}, 1\right). \end{cases}$$

Proof. Suppose that $f \in \mathcal{N}_{\Sigma}^{sc}(\lambda,\gamma)$. Then there exists $p, q \in \mathcal{P}$ such that

$$\frac{\lambda z^2 f''(z) + zf'(z)}{\lambda z\left(f(z)-\overline{f(-\overline{z})}\right)' + (1-\lambda)\left(f(z)-\overline{f(-\overline{z})}\right)} = \frac{\gamma}{2} + \frac{1-\gamma}{2}p(z) \tag{2.1}$$

and

$$\frac{\lambda w^2 g''(w) + wg'(w)}{\lambda w\left(g(w)-\overline{g(-\overline{w})}\right)' + (1-\lambda)\left(g(w)-\overline{g(-\overline{w})}\right)} = \frac{\gamma}{2} + \frac{1-\gamma}{2}q(w), \tag{2.2}$$

where $g = f^{-1}$ and p, q have the following series representations:

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$$

and

$$q(w) = 1 + q_1w + q_2w^2 + q_3w^3 + \cdots .$$

By equating the coefficients in (2.1) and (2.2), we have

$$(\lambda+1)\, a_2 = \frac{1-\gamma}{2}p_1, \tag{2.3}$$

$$(2\lambda+1)\, a_3 = \frac{1-\gamma}{2}p_2, \tag{2.4}$$

$$2\,(3\lambda+1)\, a_4 - (\lambda(2\lambda+3)+1)\, a_2a_3 = \frac{1-\gamma}{2}p_3, \tag{2.5}$$

$$-(\lambda+1)\, a_2 = \frac{1-\gamma}{2}q_1, \tag{2.6}$$

$$(2\lambda+1)\left(2a_2^2 - a_3\right) = \frac{1-\gamma}{2}q_2 \tag{2.7}$$

and

$$-2\,(3\lambda+1)\,(5a_2^3 - 5a_2a_3 + a_4) + (\lambda(2\lambda+3)+1)\,(2a_2^2 - a_3)a_2 = \frac{1-\gamma}{2}q_3, \tag{2.8}$$

In view of (2.3) and (2.6), it easy to see that

$$p_1 = -q_1 \tag{2.9}$$

and

$$a_2 = \frac{1-\gamma}{2\,(\lambda+1)}p_1. \tag{2.10}$$

By subtracting (2.4) from (2.7) and using (2.10), we get

$$a_3 = \frac{(1-\gamma)^2}{4(\lambda+1)^2} p_1^2 + \frac{1-\gamma}{4(2\lambda+1)}(p_2 - q_2). \tag{2.11}$$

Also, subtracting (2.5) from (2.8), further computations using (2.10) and (2.11) lead to

$$a_4 = \frac{(2\lambda+1)(1-\gamma)^3}{16(3\lambda+1)(\lambda+1)^2} p_1^3 + \frac{5(1-\gamma)^2}{16(\lambda(2\lambda+3)+1)} p_1(p_2-q_2) + \frac{1-\gamma}{8(3\lambda+1)}(p_3-q_3). \tag{2.12}$$

Thus, using (2.10), (2.11) and (2.12), we deduce that

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = \Bigg| & \frac{(\lambda(2\lambda-3)-1)(1-\gamma)^4}{32(3\lambda+1)(\lambda+1)^3} p_1^4 + \frac{(1-\gamma)^3}{32(2\lambda+1)(\lambda+1)^2} p_1^2(p_2-q_2) \\ & + \frac{(1-\gamma)^2}{16(\lambda(3\lambda+4)+1)} p_1(p_3-q_3) - \frac{(1-\gamma)^2}{16(2\lambda+1)^2}(p_2-q_2)^2 \Bigg|. \end{aligned} \tag{2.13}$$

According to Lemma 1.2 and (2.9), we write

$$p_2 - q_2 = \frac{(4-p_1^2)(x-y)}{2} \tag{2.14}$$

and

$$\begin{aligned} p_3 - q_3 = & \frac{p_1^3}{2} + \frac{p_1(4-p_1^2)(x+y)}{2} - \frac{p_1(4-p_1^2)(x^2+y^2)}{4} \\ & + \frac{(4-p_1^2)\left[\left(1-|x|^2\right)z - \left(1-|y|^2\right)w\right]}{2}, \end{aligned} \tag{2.15}$$

fore some x, y, z and w with $|x| \le 1$, $|y| \le 1$, $|z| \le 1$ and $|w| \le 1$.
Substituting the calculated values from (2.14) and (2.15) in the right hand side of (2.13), it follows that

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| = & \Bigg| \frac{p_1^4(1-\gamma)^4(\lambda(2\lambda-3)-1)}{32(3\lambda+1)(\lambda+1)^3} + \frac{p_1^2(4-p_1^2)(1-\gamma)^3(x-y)}{64(2\lambda+1)(\lambda+1)^2} \\ & + \frac{p_1^4(1-\gamma)^2 + p_1^2(4-p_1^2)(1-\gamma)^2(x+y)}{32(\lambda(3\lambda+4)+1)} - \frac{p_1^2(4-p_1^2)(1-\gamma)^2(x^2+y^2)}{64(\lambda(3\lambda+4)+1)} \\ & + \frac{p_1(4-p_1^2)(1-\gamma)^2\left[\left(1-|x|^2\right)z - \left(1-|y|^2\right)w\right]}{32(\lambda(3\lambda+4)+1)} - \frac{(4-p_1^2)^2(1-\gamma)^2(x-y)^2}{64(2\lambda+1)^2} \Bigg| \\ \le & \frac{p_1^4(1-\gamma)^4(\lambda(2\lambda-3)-1)}{32(3\lambda+1)(\lambda+1)^3} + \frac{p_1^4(1-\gamma)^2 + p_1(4-p_1^2)(1-\gamma)^2}{32(\lambda(3\lambda+4)+1)} \\ & + \left[\frac{p_1^2(4-p_1^2)(1-\gamma)^3}{64(2\lambda+1)(\lambda+1)^2} + \frac{p_1^2(4-p_1^2)(1-\gamma)^2}{32(\lambda(3\lambda+4)+1)}\right](|x|+|y|) \\ & + \left[\frac{p_1^2(4-p_1^2)(1-\gamma)^2}{64(\lambda(3\lambda+4)+1)} - \frac{p_1(4-p_1^2)(1-\gamma)^2}{32(\lambda(3\lambda+4)+1)}\right](|x|^2+|y|^2) \\ & + \frac{(4-p_1^2)^2(1-\gamma)^2}{64(2\lambda+1)^2}(|x|+|y|)^2. \end{aligned}$$

Since the function p is in the class $\mathcal{P}$, so $|p_1| \le 2$. Choosing $p_1 = p$, we can assume without loss of generality that $p \in [0,2]$. Then, for $\eta_1 = |x| \le 1$ and $\eta_2 = |y| \le 1$, we have

$$\left|a_2a_4 - a_3^2\right| \le L_1 + L_2(\eta_1+\eta_2) + L_3(\eta_1^2+\eta_2^2) + L_4(\eta_1+\eta_2)^2 = M(\eta_1,\eta_2),$$

where

$$L_1 = L_1(p) = \frac{(1-\gamma)^2}{32\left(\lambda(3\lambda+4)+1\right)}\left[\left(\frac{(1-\gamma)^2\left(\lambda(2\lambda-3)-1\right)}{(\lambda+1)^2}+1\right)p^4-2p^3+8p\right] \geq 0,$$

$$L_2 = L_2(p) = \frac{p^2\left(4-p^2\right)(1-\gamma)^2}{64\left(\lambda+1\right)}\left(\frac{1-\gamma}{\lambda(2\lambda+3)+1}+\frac{2}{3\lambda+1}\right) \geq 0,$$

$$L_3 = L_3(p) = \frac{p(p-2)\left(4-p^2\right)(1-\gamma)^2}{64\left(\lambda(3\lambda+4)+1\right)} \leq 0,$$

$$L_4 = L_4(p) = \frac{\left(4-p^2\right)^2(1-\gamma)^2}{64\left(2\lambda+1\right)^2} \geq 0.$$

We next maximize the function $M(\eta_1,\eta_2)$ on the closed square $[0,1]\times[0,1]$. We must investigate the maximum of $M(\eta_1,\eta_2)$ according to $p\in(0,2)$, $p=0$ and $p=2$ taking into account the sign of $M_{\eta_1\eta_1}\cdot M_{\eta_2\eta_2}-\left(M_{\eta_1\eta_2}\right)^2$.
Since $L_3<0$ and $L_3+2L_4>0$ for $p\in(0,2)$, we conclude that

$$M_{\eta_1\eta_1}\cdot M_{\eta_2\eta_2}-\left(M_{\eta_1\eta_2}\right)^2<0.$$

Therefore the function M cannot have a local maximum in the interior of the closed square $[0,1]\times[0,1]$. Now, we investigate the maximum M on the boundary of the closed square $[0,1]\times[0,1]$.

When $\eta_1=0$ and $0\leq\eta_2\leq1$ (similarly $\eta_2=0$ and $0\leq\eta_1\leq1$), we have

$$M(0,\eta_2)=E(\eta_2)=L_1+L_2\eta_2+(L_3+L_4)\eta_2^2.$$

(1) The case $L_3+L_4\geq0$:
In this case for $0<\eta_2<1$ and any fixed p with $0\leq p\leq2$, it is easily observed that $E'(\eta_2)=L_2+2(L_3+L_4)\eta_2>0$. Therefore $E(\eta_2)$ is increasing function and hence, for fixed $p\in[0,2)$, the maximum of $E(\eta_2)$ occurs at $\eta_2=1$ and

$$\max E(\eta_2)=E(1)=L_1+L_2+L_3+L_4.$$

(2) The case $L_3+L_4<0$:
Since $L_2+2(L_3+L_4)\geq0$ for $0<\eta_2<1$ and any fixed p with $0\leq p\leq2$, it is easily observed that $L_2+2(L_3+L_4)<L_2+2(L_3+L_4)\eta_2<L_2$. Therefore $E'(\eta_2)>0$ and hence, for fixed $p\in[0,2)$, the maximum of $E(\eta_2)$ occurs at $\eta_2=1$.

Also, for $p=2$, we find

$$M(\eta_1,\eta_2)=\frac{(1-\gamma)^2}{2\left(\lambda(3\lambda+4)+1\right)}\left(\frac{(1-\gamma)^2\left(\lambda(2\lambda-3)-1\right)}{(\lambda+1)^2}+1\right). \tag{2.16}$$

Taking into account the value (2.16) and the cases 1 and 2, for $0\leq\eta_2\leq1$ and any fixed p with $0\leq p\leq2$,

$$\max E(\eta_2)=E(1)=L_1+L_2+L_3+L_4.$$

When $\eta_1=1$ and $0\leq\eta_2\leq1$ (similarly $\eta_2=1$ and $0\leq\eta_1\leq1$), we have

$$M(1,\eta_2)=K(\eta_2)=L_1+L_2+L_3+L_4+(L_2+2L_4)\eta_2+(L_3+L_4)\eta_2^2.$$

Similarly to the above cases of L_3+L_4, we find that

$$\max K(\eta_2)=K(1)=L_1+2L_2+2L_3+4L_4.$$

Since $E(1) \leq K(1)$ for $p \in [0,2]$, $\max M(\eta_1, \eta_2) = M(1,1)$ on the boundary of the closed square $[0,1] \times [0,1]$. Hence, the maximum of M occurs at $\eta_1 = 1$ and $\eta_2 = 1$ in the closed square $[0,1] \times [0,1]$.
Assume that $T : [0,2] \longrightarrow \mathbb{R}$ be defined by

$$T(p) = \max M(\eta_1, \eta_2) = M(1,1) = L_1 + 2L_2 + 2L_3 + 4L_4. \tag{2.17}$$

Now, substituting the values of L_1, L_2, L_3 and L_4 in (2.17), we conclude that

$$\begin{aligned} T(p) = {} & \frac{(1-\gamma)^2}{32\,(3\lambda+1)\,(2\lambda+1)^2\,(\lambda+1)^3} \Big\{ \Big[(1-\gamma)^2\,(2\lambda+1)^2\,(\lambda(2\lambda-3)-1) \\ & - (1-\gamma)\,(\lambda+1)\left(6\lambda^2+5\lambda+1\right) - 2\,(2\lambda+1)^2\,(\lambda+1)^2 + 2\,(3\lambda+1)\,(\lambda+1)^3 \Big] p^4 \\ & + 4\,(\lambda+1) \Big[(1-\gamma)\,(3\lambda+1)\,(2\lambda+1) + 3\,(2\lambda+1)^2\,(\lambda+1) - 4\,(3\lambda+1)\,(\lambda+1)^2 \Big] p^2 \\ & + 32\,(3\lambda+1)\,(\lambda+1)^3 \Big\}. \end{aligned}$$

Suppose that $T(p)$ has a maximum value in an interior of $p \in [0,2]$, then

$$\begin{aligned} T'(p) = {} & \frac{(1-\gamma)^2}{8\,(3\lambda+1)\,(2\lambda+1)^2\,(\lambda+1)^3} \Big\{ \Big[(1-\gamma)^2\,(2\lambda+1)^2\,(\lambda(2\lambda-3)-1) \\ & - (1-\gamma)\,(\lambda+1)\left(6\lambda^2+5\lambda+1\right) - 2\,(2\lambda+1)^2\,(\lambda+1)^2 + 2\,(3\lambda+1)\,(\lambda+1)^3 \Big] p^3 \\ & + 8\,(\lambda+1) \Big[(1-\gamma)\,(3\lambda+1)\,(2\lambda+1) + 3\,(2\lambda+1)^2\,(\lambda+1) - 4\,(3\lambda+1)\,(\lambda+1)^2 \Big] p \Big\}. \end{aligned}$$

After some calculations, we consider the following cases:
Case1: Assume that

$$(1-\gamma)^2\,(2\lambda+1)^2\,(\lambda(2\lambda-3)-1) - (1-\gamma)\,(\lambda+1)\left(6\lambda^2+5\lambda+1\right) - 2\lambda^2\,(\lambda+1)^2 \geq 0.$$

Thus

$$\gamma \in \left[0, 1 - \frac{(\lambda+1)\left((3\lambda+1) + \sqrt{(3\lambda+1)^2 + 8\lambda^2\,(\lambda(2\lambda-3)-1)} \right)}{2\,(2\lambda+1)\,(\lambda(2\lambda-3)-1)} \right]$$

and so $T'(p) > 0$ for $p \in (0,2)$. Since T is an increasing function in the interval $(0,2)$, hence the maximum point of T must be on the boundary of $p \in [0,2]$. Then, we have

$$\max_{0 \leq p \leq 2} T(p) = T(2) = \frac{(1-\gamma)^2}{2\,(3\lambda+1)\,(\lambda+1)^3} \left((1-\gamma)^2\,(\lambda(2\lambda-3)-1) + (\lambda+1)^2 \right).$$

Case2: Assume that

$$(1-\gamma)^2\,(2\lambda+1)^2\,(\lambda(2\lambda-3)-1) - (1-\gamma)\,(\lambda+1)\left(6\lambda^2+5\lambda+1\right) - 2\lambda^2\,(\lambda+1)^2 < 0.$$

that is,

$$\gamma \in \left(1 - \frac{(\lambda+1)\left((3\lambda+1) + \sqrt{(3\lambda+1)^2 + 8\lambda^2\,(\lambda(2\lambda-3)-1)} \right)}{2\,(2\lambda+1)\,(\lambda(2\lambda-3)-1)}, 1 \right).$$

Therefore $T'(p) = 0$ implies the real critical point $p_{0_1} = 0$ or

$$p_{0_2} = \sqrt{\frac{-2(1-\gamma)\,(\lambda+1)\left(6\lambda^2+5\lambda+1\right) + 2\,(\lambda+1)^2 \left[4\,(\lambda(3\lambda+4)+1) - 3\,(2\lambda+1)^2 \right]}{(1-\gamma)^2\,(2\lambda+1)^2\,(\lambda(2\lambda-3)-1) - (1-\gamma)\,(\lambda+1)\left(6\lambda^2+5\lambda+1\right) - 2\lambda^2\,(\lambda+1)^2}}.$$

When

$$\gamma \in \left(1-\frac{(\lambda+1)\left((3\lambda+1)+\sqrt{(3\lambda+1)^2+8\lambda^2(\lambda(2\lambda-3)-1)}\right)}{2(2\lambda+1)(\lambda(2\lambda-3)-1)},\right.$$
$$\left.1-\frac{(\lambda+1)\left((3\lambda+1)+\sqrt{(3\lambda+1)^2+8(2\lambda+1)^2(\lambda(2\lambda-3)-1)}\right)}{4(2\lambda+1)(\lambda(2\lambda-3)-1)}\right],$$

we observe that $p_{0_2} \geq 2$, that is, p_{0_2} is out of the interval $(0,2)$. Hence the maximum value of $T(p)$ occurs at $p_{0_1}=0$ or p_{0_2} which contradicts our assumption of having the maximum value at the interior point of $p \in [0,2]$. Since T is an increasing function in the interval $(0,2)$, so the maximum point of T must be on the boundary of $p \in [0,2]$, that is, $p=2$. Therefore, we obtain

$$\max_{0\leq p\leq 2} T(p)=T(2)=\frac{(1-\gamma)^2}{2(3\lambda+1)(\lambda+1)^3}\left((1-\gamma)^2(\lambda(2\lambda-3)-1)+(\lambda+1)^2\right).$$

When

$$\gamma \in \left(1-\frac{(\lambda+1)\left((3\lambda+1)+\sqrt{(3\lambda+1)^2+8(2\lambda+1)^2(\lambda(2\lambda-3)-1)}\right)}{4(2\lambda+1)(\lambda(2\lambda-3)-1)},1\right),$$

we observe that that $p_{0_2}<2$, that is, p_{0_2} is an interior of the interval $[0,2]$. Since $T''(p_{0_2})<0$, so the maximum value of $T(p)$ occurs at $p=p_{0_2}$. Therefore, we obtain

$$\max_{0\leq p\leq 2} T(p)=T(p_{0_2})=\frac{(1-\gamma)^2}{8(\lambda(3\lambda+4)+1)}\times$$
$$\times\frac{(1-\gamma)^2(3\lambda+1)\left(8\lambda^2(2\lambda-1)-35\lambda-9\right)-6(1-\gamma)(\lambda+1)(6\lambda^2+5\lambda+1)-(\lambda+1)^2(4\lambda(3\lambda+1)+1)}{(1-\gamma)^2(2\lambda+1)^2(\lambda(2\lambda-3)-1)-(1-\gamma)(\lambda+1)(6\lambda^2+5\lambda+1)-2\lambda^2(\lambda+1)^2}.$$

This completes the proof of our Theorem. □

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