Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–8.
ISSN-0037-8712 in press
doi:10.5269/bspm.51691

On Spectral Polynomial of Splices and Links of Graphs*

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ABSTRACT: The spectral polynomial of a graph is the characteristic polynomial of its adjacency matrix. Spectral polynomial of the splice and links of complete graph and star have been obtained recently in the literature. In this paper we generalize these results using the concept of equitable partition.

Key Words: Spectral polynomial, splices, links, equitable partition.

Contents

1. Introduction

A graph $G = (V(G), E(G))$, where $V(G) = \{v_1, v_2, \dots, v_n\}$ is a finite non-empty set of elements called vertices and $E(G)$ is a set of unordered pairs of distinct vertices called edges. Graphs considered here are simple and undirected. We follow the book [2] for terminology and definitions. The adjacency matrix $A(G)$ of a graph G of order n is the $n \times n$ matrix indexed by $V(G)$, whose (i, j)-th entry is defined as $a_{ij} = 1$ if $v_i v_j \in E(G)$ and 0, otherwise. The spectral polynomial of G is defined by $|xI_n - A(G)|$ and is denoted by $\phi(G : x) = \phi(A(G))$, where I_n is the identity matrix of order n. The eigenvalues of G are the roots of the spectral polynomial, denoted by $\lambda_1^{m_1}, \lambda_2^{m_2}, \dots, \lambda_k^{m_k}$, where m_i denotes the multiplicity of λ_i for $i = 1, 2, \dots, k$ such that $\sum_{i=1}^{k} m_i = n$.

Definition 1.1. [1] *If G is labeled graph of order n with vertices $v_1, v_2, \dots, v_n$, the graph $G[G_1, G_2, \dots, G_n]$ called* **generalized composition** *is formed by taking the disjoint graphs $G_1, G_2, \dots, G_n$ and then joining every vertex of G_i to every vertex of G_j whenever v_i adjacent to v_j in G.*

Definition 1.2. [1] *A partition $V_1 \bigcup V_2 \bigcup \dots \bigcup V_m$ of $V(G)$ is* **equitable** *if for each i and for all $v_1, v_2 \in V_i$, $| N(v_1) \bigcap V_j | = | N(v_2) \bigcap V_j |$ for all j, where $N(v_i)$ is the open neighbourhood of v_i. The partition of $V(G)$ into singletons is always equitable. In generalized composition, if a graph G is regular, then $V(G)$ can be taken as partite set in an equitable partition.*

If $P = V_1 \bigcup V_2 \bigcup \dots \bigcup V_m$ is an equitable partition, we associate with it an $m \times m$ matrix $Q = [q_{ij}]$, where $q_{ij} = | N(v) \bigcap V_j |$ for any $v \in V_i$. Such a matrix is called a **quotient matrix**.

Let $\phi(M) = \phi(M : x)$ denotes the characteristic polynomial of matrix M.

Theorem 1.3. [1] *If $V_1, V_2, \dots, V_m$ is an equitable partion of a graph G then $\phi(Q)$ divides $\phi(G : x)$.*

Theorem 1.4. [1] *If $G_1, G_2, \dots, G_n$ are all regular, then $V(G_1) \bigcup V(G_2) \bigcup \dots \bigcup V(G_n)$ is an equitable partition of $G[G_1, G_2, \dots, G_n]$ and*

$$\phi(G[G_1, G_2, \dots, G_n] : x) = \phi(Q) \prod_{i=1}^{n} \frac{\phi(G_i : x)}{x - r_i}.$$

* H. S. Ramane thanks University Grants Commission (UGC), New Delhi for support through grant under UGC-SAP DRS-III, 2016-2021: F.510/3/DRS-III/2016 (SAP-I). D. Patil thanks Karnataka Science and Technology Promotion Society, Bengaluru for fellowship No. DST/KSTePS/Ph.D Fellowship/OTH-01:2018-19. K. Ashoka thanks Karnatak University, Dharwad for URS fellowship No. URS/2019-344.
2010 *Mathematics Subject Classification:* 05C50.
Submitted January 03, 2020. Published June 10, 2021

Došlić (2005) [4], defined splice and link of two graphs as follows:

Definition 1.5. [4] *Let G_1 and G_2 be two graphs and let us label two vertices, one in $V(G_1)$ and the other in $V(G_2)$, by v. The* **vertex joining graph at** *v or* **the splice** *of these two graphs is denoted by $G_1 \vee_v G_2$ and is obtained by identifying the vertices v of the two graphs.*

Definition 1.6. [4] *Let G_1 and G_2 be two graphs and let us label two vertices, one in $V(G_1)$ and the other in $V(G_2)$, by v. The* **edge joining graph at** *v or* **the link** *of these two graphs is denoted by $G_1 \vee_v^e G_2$ and obtained by adding a new edge between the labeled vertices v of two graphs.*

Proposition 1.7. *(Schur Complement [2]) Suppose that the order of all four matrices M, N, P and Q satisfy the rules of operations on matrices. Then we have*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = \begin{cases} |Q|\ |M - NQ^{-1}P|, & \text{if } Q \text{ is a non-singular matrix} \\ |M|\ |Q - PM^{-1}N|, & \text{if } M \text{ is a non-singular matrix.} \end{cases}$$

Let K_n denote the complete graph on n vertices, $K_{r,s}$ denote the complete bipartite graph on $r+s$ vertices and $S_n = K_{1,n-1}$ denote the star on n vertices. In [3], the spectral polynomial of splices and links of complete graph and star has been obtained. In this paper, we generalize these results by using equitable partition.

2. Spectral polynomial of splice of graphs

Here we define splice for p simple, connected graphs $G_1, G_2, \ldots, G_p$ which is the generalization of the concept of splice of two graphs.

Definition 2.1. *Let $G_1, G_2, \ldots, G_p$ be p disjoint graphs and let us label p vertices, one in each $V(G_i)$ for $i = 1, 2, \ldots, p$, by v. The* **vertex joining graph at** *v or* **the splice** *of these graphs, denoted as $\vee_v[G_1, G_2, \ldots, G_p]$, is obtained by identifying the vertices v of the p graphs (see Figure 1).*

If $|V(G_i)| = n_i$ and $|E(G_i)| = m_i$, for $i = 1, 2, \ldots, p$, then $|V(\vee_v[G_1, G_2, \ldots, G_p])| = n_1 + n_2 + \cdots + n_p - (p-1)$ and $|E(\vee_v[G_1, G_2, \ldots, G_p])| = m_1 + m_2 + \cdots + m_p$.

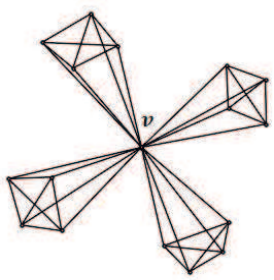

Figure 1: $\vee_v[K_5, K_5, K_5, K_5]$

Theorem 2.2. *The spectral polynomial of $\vee_v[\underbrace{K_n, K_n, \ldots, K_n}_{p\ \text{copies}}]$ is*

$$(x - n + 2)^{p-1}(x+1)^{p(n-2)}\big(x^2 - (n-2)x - p(n-1)\big).$$

Proof. Making the vertices of $\vee_v[\underbrace{K_n, K_n, \ldots, K_n}_{p\ \text{copies}}]$ as two partite sets: $V_1 = \{v\}$ and $V_2 = \{u : u$ is adjacent to $v\}$, these two partite sets lead to the quotient matrix

$$Q = \begin{pmatrix} 0 & p(n-1) \\ 1 & n-2 \end{pmatrix}.$$

The spectral polynomial of Q is $\phi(Q) = (x^2 - (n-2)x - p(n-1))$ and the partition V_2 accounts part of the spectra of $\vee_v[K_n, K_n, \ldots, K_n]$, which is $\{(n-2)^{(p-1)}, (-1)^{p(n-2)}\}$.
Hence, by Theorem 1.4, the result follows. □

For $p = 2$, the above theorem leads to Theorem 3.3 and Theorem 3.4 of [3].

Theorem 2.3. *The spectral polynomial of* $G = \vee_v[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{p \text{ copies}}]$ *is*

$$\phi(G:x) = \begin{cases} x^{rp+sp-3p+1}(x^2 - (p+r-1)s)(x^2 - (r-1)s)^{p-1}, & \\ \qquad\qquad \text{if } v \text{ is selected among the } r \text{ vertices of } K_{r,s} \\ x^{rp+sp-3p+1}(x^2 - (p+s-1)r)(x^2 - (s-1)r)^{p-1}, & \\ \qquad\qquad \text{if } v \text{ is selected among the } s \text{ vertices of } K_{r,s}, \end{cases}$$

with $\phi(G:x) = x^{2rp-3p+1}(x^2 - (p+r-1)r)(x^2 - (r-1)r)^{p-1}$ *when* r=s.

Proof. (i) When v is selected among r vertices of $K_{r,s}$, making the vertices of $\vee_v[K_{r,s}, K_{r,s}, \ldots, K_{r,s}]$ into $2p+1$ partite sets as: $V_1 = \{v\}$, $V_i = \{u : u$ is not adjacent to v in a copy of $K_{r,s}\}$ for $i = 2, 3, \ldots, p+1$ and $V_j = \{w : w$ is adjacent to v in a copy of $K_{r,s}\}$ for $j = p+2, p+3, \ldots, 2p+1$. These partite sets lead to the quotient matrix

$$Q = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & s & s & s & \ldots & s \\ 0 & 0 & 0 & \ldots & 0 & s & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & s & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & s \\ 1 & r-1 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ 1 & 0 & r-1 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \ldots & r-1 & 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Using Proposition 1.7, we have, $\phi(Q) = x(x^2 - (p+r-1)s)(x^2 - (r-1)s)^{p-1}$ and the partitions V_i and V_j accounts part of the spectra of $\vee_v[K_{r,s}, K_{r,s}, \ldots, K_{r,s}]$, which is $\{0^{(rp+sp-3p)}\}$.
Hence, by Theorem 1.4, the result follows.
(ii) Interchanging r and s in the proof of above Case (i), the result follows.
When $r = s$, $\phi(G:x) = x^{2rp-3p+1}(x^2 - (p+r-1)r)(x^2 - (r-1)r)^{p-1}$. □

Corollary 2.4. *The spectral polynomial of* $G = \vee_v[\underbrace{S_n, S_n, \ldots, S_n}_{p \text{ copies}}]$ *is*

$$\phi(G:x) = \begin{cases} x^{np-p-1}(x^2 - (n-1)p), & \\ \qquad\qquad \text{if } v \text{ is the central vertex of the star } S_n \\ x^{np-3p+1}(x^2 - (p+n-2))(x^2 - (n-2))^{p-1}, & \\ \qquad\qquad \text{if } v \text{ is a non-central vertex of the star } S_n. \end{cases}$$

Proof. Taking $r = 1$ and $s = n-1$ in Case (i) of Theorem 2.2, we get $\phi(G:x) = x^{np-p-1}(x^2 - (n-1)p)$. Taking $r = n-1$ and $s = 1$ in Case (i) of Theorem 2.2, we get

$$\phi(G:x) = x^{np-3p+1}(x^2 - (p+n-2))(x^2 - (n-2))^{p-1}.$$

□

For $p = 2$, the above corollary leads to Theorem 3.5 and Corollary 3.6 of [3].

3. Spectral polynomial of link of graphs

Now, we define link for $2p$ simple, connected graphs $G_1, G_2, \ldots, G_{2p}$ which is the generalization of the concept of link of two graphs.

Definition 3.1. *Let $G_1, G_2, \ldots, G_{2p}$ be $2p$ graphs and let us label p vertices, one in each $V(G_i)$ for $i = 1, 2, \ldots, p$, by v and other p vertices, one in each $V(G_i)$ for $i = p+1, p+2, \ldots, 2p$, by v'. The* **edge joining graph at** *vv' or* **the link** *of these graphs be denoted as $\vee^e_{vv'}[G_1, G_2, \ldots, G_{2p}]$ which is obtained by adding a new edge between the identified vertices v and v' of $2p$ graphs (for the purpose of symmetry in the structure, p copies are taken at v and other p copies at v') (see Figure 2).*

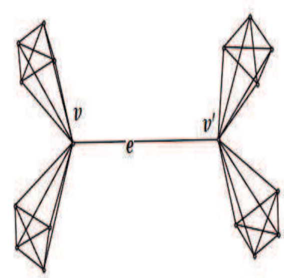

Figure 2: $\vee^e_{vv'}[K_5, K_5, K_5, K_5]$

Theorem 3.2. *The spectral polynomial of $\vee^e_{vv'}[\underbrace{K_n, K_n, \ldots, K_n}_{2p \ \ copies}]$ is*

$$(x+1)^{2p(n-2)}\Big(x-(n-2)\Big)^{(2p-2)}\Bigg(x^4-(2n-4)x^3+(n^2-2np-4n+2p+3)x^2$$
$$+2(n-2)\Big[1+p(n-1)\Big]x+\Big[p^2(n-1)^2-(n-2)^2\Big]\Bigg).$$

Proof. From $\vee^e_{vv'}[K_n, K_n, \ldots, K_n]$ we have, $e = vv'$ and p copies of K_n identified at v and other p copies of K_n identified at v'. Making $(2np - 2p + 2)$ vertices of $\vee^e_{vv'}[K_n, K_n, \ldots, K_n]$ into four partite sets as $V_1 = \{v\}$, $V_2 = \{v'\}$, $V_3 = \{u : u \text{ is adjacent to } v\}$ and $V_4 = \{u' : u' \text{ is adjacent to } v'\}$, these four partite sets lead to the quotient matrix

$$Q = \begin{pmatrix} 0 & 1 & p(n-1) & 0 \\ 1 & 0 & 0 & p(n-1) \\ 1 & 0 & n-2 & 0 \\ 0 & 1 & 0 & n-2 \end{pmatrix}.$$

The spectral polynomial of Q is,

$$\phi(Q) = (x^2-(n-3)x-(2n-3))(x^2-(n-1)x-1)$$

and the partition V_3 and V_4 accounts part of the spectra of $\vee^e_{vv'}[K_n, K_n, \ldots, K_n]$, which is $\{(-1)^{2p(n-2)}, (n-2)^{(2p-2)}\}$.
Hence, by Theorem 1.4, the result follows. □

For $p = 1$, the above theorem leads to Theorem 3.11 of [3].

Theorem 3.3. *The spectral polynomial of* $G = \vee_{vv'}^{e}[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{2p \ \ copies}]$ *is*

$$\phi(G:x) = \begin{cases} x^{2rp+2sp-2p-4}\Big(x^6 - (2rs+2sp-2s+1)x^4 + (r^2s^2+2rs^2p+s^2p^2-2rs^2 \\ \qquad -2s^2p+2rs+s^2-2s)x^2 - s^2(r-1)^2\Big), \\ \qquad\qquad \text{if } v \text{ ad } v' \text{ are selected among the } r \text{ vertices of } K_{r,s} \\ x^{2rp+2sp-2p-4}(x^6 - (2rs+2rp-2r+1)x^4 + (r^2s^2+2r^2sp+r^2p^2-2r^2s \\ \qquad -2r^2p+2rs+r^2-2r)x^2 - r^2(s-1)^2), \\ \qquad\qquad \text{if } v \text{ ad } v' \text{ are selected among the } s \text{ vertices of } K_{r,s}, \end{cases}$$

with

$$\phi(G:x) = x^{4rp-2p-4}\Big(x^6 - (2r^2+2rp-2r+1)x^4 + (r^4+2r^3p+r^2p^2-2r^3 -2r^2p+3r^2-2r)x^2 - r^2(r-1)^2\Big), \text{ when } r = s.$$

Proof. (i) When v and v' are selected among r vertices of two copies of $K_{r,s}$ respectively, then from $\vee_{vv'}^{e}[K_{r,s}, K_{r,s}, \ldots, K_{r,s}]$, we have, $e = vv'$ and p copies of $K_{r,s}$ identified at v and other p copies of $K_{r,s}$ identified at v'. Making $(2rp+2sp-2p+2)$ vertices of $\vee_{vv'}^{e}[K_{r,s}, K_{r,s}, \ldots, K_{r,s}]$ into six partite sets as: $V_1 = \{v\}$, $V_2 = \{u' : u'$ is adjacent to v' among p copies of $K_{r,s}$ which are identified at $v'\}$, $V_3 = \{w : w$ is not adjacent to v among p copies of $K_{r,s}$ which are identified at $v\}$, $V_4 = \{v'\}$, $V_5 = \{u : u$ is adjacent to v among p copies of $K_{r,s}$ which are identified at $v\}$ and $V_6 = \{w' : w'$ is not adjacent to v' among p copies of $K_{r,s}$ which are identified at $v'\}$ these six partite sets lead to the quotient matrix

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 & sp & 0 \\ 0 & 0 & 0 & 1 & 0 & r-1 \\ 0 & 0 & 0 & 0 & s & 0 \\ 1 & sp & 0 & 0 & 0 & 0 \\ 1 & 0 & r-1 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Using Proposition 1.7, we have

$$\begin{aligned} \phi(Q) &= x^6 - (2rs+2sp-2s+1)x^4 \\ &\quad +(r^2s^2+2rs^2p+s^2p^2-2rs^2-2s^2p+2rs+s^2-2s)x^2 - s^2(r-1)^2 \end{aligned}$$

and the partitions V_i and V_j accounts part of the spectra of $\vee_{vv'}^{e}[K_{r,s}, K_{r,s}, \ldots, K_{r,s}]$, which is $\{0^{(2rp+2sp-2p-4)}\}$.
Hence, by Theorem 1.4, the result follows.

(ii) Interchanging r and s in the proof of above Case (i), the result follows.
When $r = s$, $\phi(G:x) = x^{4rp-2p-4}(x^6 - (2r^2+2rp-2r+1)x^4 + (r^4+2r^3p+r^2p^2-2r^3-2r^2p+3r^2-2r)x^2 - r^2(r-1)^2)$. □

Theorem 3.4. *The spectral polynomial of* $G = \vee_{vv'}^{e}[S_n, S_n]$ *is*

$$\phi(G:x) = \begin{cases} x^{2n-4}(x^4 - (2n-1)x^2 - (n-1)x + (n-1)^2), \\ \qquad\qquad \text{when } v \text{ and } v' \text{ are the central vertices} \\ x^{2n-6}(x^6 - (2n-1)x^4 + (n^2-3)x^2 - (n^2-4n+4)), \\ \qquad\qquad \text{when } v \text{ and } v' \text{ are the non-central vertices.} \end{cases}$$

Proof. (i) When v and v' are central vertices in two copies of S_n respectively, making $2n$ vertices of $\vee^e_{vv'}[S_n, S_n]$ which includes two copies of S_n into four partite sets as: $V_1 = \{v\}$, $V_2 = \{v'\}$, $V_3 = \{u : u$ is adjacent to v such that $u \neq v'\}$ and $V_4 = \{u' : u'$ is adjacent to v' such that $u' \neq v\}$, these six partite sets lead to the quotient matrix

$$Q = \begin{pmatrix} 0 & 1 & n-1 & 0 \\ 1 & 0 & 0 & n-1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Using Proposition 1.7, we have, $\phi(Q) = (x^4 - (2n-1)x^2 - (n-1)x + (n-1)^2)$ and the partitions V_3 and V_4 accounts part of the spectra of $\vee^e_{vv'}[S_n, S_n]$, which is $\{0^{(2n-4)}\}$.
Hence, by Theorem 1.4, the result follows.

(ii) When v and v' are pendent vertices in two copies of S_n, denote the two central vertices as u and u'. Making $2n$ vertices of $\vee^e_{vv'}[S_n, S_n]$ which includes two copies of S_n into six partite sets as: $V_1 = \{u\}$, $V_2 = \{v'\}$, $V_3 = \{w' : w'$ is adjacent to u' such that $w' \neq v'$ $\}$, $V_4 = \{v\}$, $V_5 = \{w : w$ is adjacent to u such that $w \neq v\}$ and $V_6 = \{u'\}$, these six partite sets lead to the quotient matrix

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 & n-2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & n-2 & 0 & 0 & 0 \end{pmatrix}.$$

Using Proposition 1.7, we have, $\phi(Q) = (x^6 - (2n-1)x^4 + (n^2-3)x^2 - (n^2 - 4n + 4))$ and the partitions V_3 and V_5 accounts part of the spectra of $\vee^e_{vv'}[S_n, S_n]$, which is $\{0^{(2n-6)}\}$.
Hence, by Theorem 1.4, the result follows. □

For $p = 1$, Case (ii) of the above theorem leads to Theorem 3.12 of [3].
Here we define link for p copies of simple, connected graphs $G_1, G_2, \ldots, G_p$ which is a special kind of the concept of link of two graphs.

Definition 3.5. *Let $G_1, G_2, \ldots, G_p$ be p graphs and let us label p vertices, one in each of $V(G_i)$ for $i = 1, 2, \ldots, p$, by v. The* **edge joining graphs at** *v or* **the link** *of these graphs be denoted as $\vee^e_v[G_1, G_2, \ldots, G_p]$ which is obtained by adding new edges between the vertices labeled by v of p graphs (see Figure 3).*

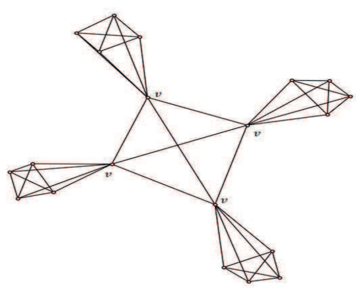

Figure 3: $\vee^e_v[K_5, K_5, K_5, K_5]$

Theorem 3.6. *The spectral polynomial of* $\vee^e_v[\underbrace{K_n, K_n, \ldots, K_n}_{p \text{ copies}}]$ *is*

$$(x+1)^{p(n-2)} \left(x^2 - (n+p-3)x + (np - 2n - 2p + 3)\right) \left(x^2 - (n-3)x - (2n-3)\right)^{p-1}.$$

Proof. From $\vee_v^e[K_n, K_n, \ldots, K_n]$, making np vertices of $\vee_v^e[K_n, K_n, \ldots, K_n]$ into $2p$ partite sets as: $V_i = \{v\}$ and $V_j = \{u : u \text{ is adjacent to } v\}$ for $i = 1, 2, 3, \ldots, p$ and $j = p+1, p+2, p+3, \ldots, 2p$ (here V_i is a singleton set for the vertex v in K_n). These $2p$ partite sets lead to the quotient matrix

$$Q = \begin{pmatrix} (J-I)_{p\times p} & (n-1)I_{p\times p} \\ I_{p\times p} & (n-2)I_{p\times p} \end{pmatrix}$$

where J is the matrix with each entry one.
Using Proposition 1.7, we have,
$\phi(Q) = \left(x^2 - (n+p-3)x + (np-2n-2p+3)\right)\left(x^2-(n-3)x-(2n-3)\right)^{p-1}$ and the p partitions V_j for $j = p+1, p+2, p+3, \ldots, 2p$ accounts to part of the spectra of $\vee_v^e[K_n, K_n, \ldots, K_n]$, which is $\{(-1)^{p(n-2)}\}$.
Hence, by Theorem 1.4, the result follows. □

Theorem 3.7. *The spectral polynomial of* $G = \vee_v^e[\underbrace{K_{r,s}, K_{r,s}, \ldots, K_{r,s}}_{p\ \ copies}]$ *is*

$$\phi(G:x) = \begin{cases} x^{rp+sp-3p}\Big(x^3-(p-1)x^2-srx+s(r-1)(p-1)\Big) \\ \qquad\qquad \Big(x^3+x^2-srx-s(r-1)\Big)^{(p-1)}, \\ \qquad\qquad \text{if } v \text{ is selected among the } r \text{ vertices of } K_{r,s} \\ x^{rp+sp-3p}\Big(x^3-(p-1)x^2-srx+r(s-1)(p-1)\Big) \\ \qquad\qquad \Big(x^3+x^2-srx-r(s-1)\Big)^{(p-1)}, \\ \qquad\qquad \text{if } v \text{ is selected among the } s \text{ vertices of } K_{r,s}, \end{cases}$$

with $\phi(G:x) = x^{2rp-3p}\Big(x^3-(p-1)x^2-r^2x+r(r-1)(p-1)\Big)\Big(x^3+x^2-r^2x-r(r-1)\Big)^{(p-1)}$ *when r=s.*

Proof. (i) When v is selected among r vertices of $K_{r,s}$, making the vertices of $\vee_v^e[K_{r,s}, K_{r,s}, \ldots, K_{r,s}]$ which includes p copies of $K_{r,s}$ into $3p$ partite sets as: $V_i = \{u : u \text{ is not adjacent to } v \text{ in a copy of } K_{r,s}\}$, $V_j = \{v\}$, $V_t = \{w : w \text{ is adjacent to } v \text{ in a copy of } K_{r,s}\ \}$ for $i = 1, 2, 3, \ldots, p$, $j = p+1, p+2, \ldots, 2p$ and $t = 2p+1, 2p+2, \ldots, 3p$. These partite sets lead to the quotient matrix

$$Q = \begin{pmatrix} O_{p\times p} & O_{p\times p} & sI_{p\times p} \\ O_{p\times p} & (J-I)_{p\times p} & sI_{p\times p} \\ (r-1)I_{p\times p} & I_{p\times p} & O_{p\times p} \end{pmatrix}$$

where O denotes the zero matrix. Using Proposition 1.7, we have,

$$\phi(Q) = (x^3-(p-1)x^2-srx+s(r-1)(p-1))(x^3+x^2-srx-s(r-1))^{(p-1)}$$

and the partitions V_i and V_t accounts part of the spectra of $\vee_v^e[K_{r,s}, K_{r,s}, \ldots, K_{r,s}]$, which is $\{0^{(rp+sp-3p)}\}$. Hence, by Theorem 1.4, the result follows.

(ii) Interchanging r and s in the proof of the case (i), the result follows.
When $r = s$,

$$\phi(G:x) = x^{2rp-3p}(x^3-(p-1)x^2-r^2x+r(r-1)(p-1))(x^3+x^2-r^2x-r(r-1))^{(p-1)}.$$

□

Remark 3.8. *When we put,* $r = 1$ *and* $s = n-1$ *in the above theorem, we get* $G = \vee_v^e[\underbrace{K_{1,n-1}, K_{1,n-1}, \ldots, K_{1,n-1}}_{p\ \ copies}]$ *which is same as,* $\vee_v^e[\underbrace{S_n, S_n, \ldots, S_n}_{p\ \ copies}]$, *where* v *is a central vertex.*

Hence, $\phi(G:x) = x^{np-2p}\left(x^2-(p-1)x-(n-1)\right)\left(x^2+x-(n-1)\right)^{p-1}$.

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