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A Resonance Problem for p-Laplacian with Mixed Boundary Conditions

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ABSTRACT: In this work, we are interested at the existence of nontrivial solutions for a nonlinear elliptic problems with resonance part and mixed boundary conditions. Our approach is variational and is based on the well known Landesman-Laser type conditions.

Key Words: p-Laplacian, mixed boundary conditions, Landesman-Lazer type conditions.

Contents

1. Introduction and main results

In this work, we deal with the following problems with mixed boundary conditions

$$\begin{cases} -\Delta_p u &= \lambda_1 |u|^{p-2}u + f(x,u) - h(x) \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \sigma, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda_1 |u|^{p-2}u + g(x,u) \quad \text{on} \quad \Gamma, \end{cases} \tag{1.1}$$

where $p > 1$, Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with C^1 boundary $\partial\Omega$ such that $\partial\Omega = \sigma \cup \Gamma$ and $\sigma \cap \Gamma = \emptyset$, Γ is a sufficiently smooth $(N-1)$-dimensional, ν is the outward normal vector on $\partial\Omega$, $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \Gamma \times \mathbb{R} \longrightarrow \mathbb{R}$ are a bounded Carathéodory functions, $h \in L^{p'}(\Omega)$, $\left(p' = \frac{p}{p-1}\right)$ and λ_1 designates the first eigenvalue for the eigenvalue problem

$$\begin{cases} -\Delta_p u &= \lambda |u|^{p-2}u \quad \text{in} \quad \Omega, \\ u &= 0 \quad \text{on} \quad \sigma, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u \quad \text{on} \quad \Gamma. \end{cases} \tag{1.2}$$

The investigation of existence of solutions for problems at resonance has drawn the attention of many authors, see for example [1,2,3,4,5,6,7,10].

In the recent paper of G. Li et al [8], the authors obtained, by using the Ljusternik-Schnirelman principle, the existence of a nondecreasing sequence of nonnegative eigenvalues of problem (1.2), and showed that the first eigenvalue λ_1 is simple, isolated and given by

$$\lambda_1 = \inf_{u \in X} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx + \int_\Gamma |u|^p ds},$$

where $X := \left\{u \in W^{1,p}(\Omega) : u|_\sigma = 0\right\}$, is a closed subspace of $W^{1,p}(\Omega)$ endowed with the norm

$$||u|| = \left(\int_\Omega (|\nabla u|^p + |u|^p) dx\right)^{1/p}.$$

Let us denote by φ_1 the positive eigenfunction associated with λ_1, which can be chosen normalized. the authors characterized the seconde eigenvalue as follows

$$\lambda_2 = \inf\{\lambda : \lambda \text{ is an eigenvalue of (1.2), with } \lambda > \lambda_1\}.$$

We assume that f and g satisfy the following hypotheses:

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(F) For almost every $x \in \Omega$, there exist

$$\lim_{s \to \pm\infty} f(x,s) = f_{\pm}(x),$$

(G) For almost every $x \in \Gamma$, there exist

$$\lim_{\tau \to \pm\infty} g(x,\tau) = g_{\pm}(x).$$

We study the solvability of problem (1.1) under the well known Landesman-Laser type conditions for the resonance part. The following theorems (see [11]) is our main ingredient

Theorem 1.1. *Let X be a Banach space and $\Phi \in \mathcal{C}^1(X,\mathbb{R})$. Assume that Φ satisfies the Palais-Smale condition and bounded from below. Then $c = \inf_X \Phi$ is a critical point.*

Theorem 1.2. *Let X be a Banach space. Let $\Phi : X \to \mathbb{R}$ be a C^1 functional that satisfies the Palais-Smale condition, and suppose that $X = V \oplus W$, with V is a finite dimensional subspace of X. If there exists $R > 0$ such that*

$$\max_{v \in V, ||v||=R} \Phi(v) < \inf_{w \in W} \Phi(w),$$

then Φ has a least a critical point on X.

Now, we are ready to state our main results.

Theorem 1.3. *Assume that (F) and (G) hold. Suppose that $f(x,.)$ and $g(x,.)$ be strictly decreasing. Then problem (1.1) has at least one weak solution if and only if*

$$\int_\Omega f_+(x)\varphi_1 dx + \int_\Gamma g_+(x)\varphi_1 ds < \int_\Omega h(x)\varphi_1 dx < \int_\Omega f_-(x)\varphi_1 dx + \int_\Gamma g_-(x)\varphi_1 ds. \tag{1.3}$$

Theorem 1.4. *Assume that (F) and (G) hold. Suppose that $f(x,.)$ and $g(x,.)$ be increasing. Then problem (1.1) has at least one weak solution if and only if*

$$\int_\Omega f_-(x)\varphi_1 dx + \int_\Gamma g_-(x)\varphi_1 ds < \int_\Omega h(x)\varphi_1 dx < \int_\Omega f_+(x)\varphi_1 dx + \int_\Gamma g_+(x)\varphi_1 ds. \tag{1.4}$$

Theorem 1.5. *Assume that (F) and (G) hold. If $h \in L^{p'}(\Omega)$ satisfy (1.3) or (1.4), then problem*

$$\begin{cases} -\Delta_p u &= \lambda |u|^{p-2}u + f(x,u) - h(x) & \text{in} & \Omega, \\ u &= 0 & \text{on} & \sigma, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u + g(x,u) & \text{on} & \Gamma, \end{cases} \tag{1.5}$$

with $\lambda_1 < \lambda < \lambda_2$, has at least one solution.

2. Preliminaries

Denoting by $\Phi : X \to \mathbb{R}$ the variational functional corresponding to the problem (1.1)

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda_1}{p}\left(\int_\Omega |u|^p dx + \int_\Gamma |u|^p ds\right) - \int_\Omega F(x,u)dx$$
$$- \int_\Gamma G(x,u)ds + \int_\Omega hu dx,$$

where

$$F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for a.e. } x \in \Omega, \ \forall t \in \mathbb{R},$$
$$G(x,\tau) = \int_0^\tau g(x,\xi)d\xi \quad \text{for a.e. } x \in \Gamma, \ \forall \tau \in \mathbb{R}.$$

It is obvious that the functional $\Phi \in \mathcal{C}^1(X, \mathbb{R})$, with derivative at point $u \in X$ is given by

$$\begin{aligned} \langle \Phi'(u), v \rangle = & \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx - \lambda_1 \left(\int_\Omega |u|^{p-2} uv dx + \int_\Gamma |u|^{p-2} uv ds \right) \\ & - \int_\Omega f(x,u) v dx - \int_\Gamma g(x,u) v ds + \int_\Omega h v dx, \end{aligned} \tag{2.1}$$

for every $v \in X$, and its critical points correspond to solutions of (1.1).

Now, let denote $V = \langle \varphi_1 \rangle$ the linear spans of φ_1 and

$$W = \left\{ u \in X : \int_\Omega |\varphi_1|^{p-1} u dx + \int_\Gamma |\varphi_1|^{p-1} u ds = 0 \right\}. \tag{2.2}$$

We can decompose X as a direct sum of V and W. Indeed, for $u \in X$, writing $u = \alpha \varphi_1 + w$ where $w \in X$ and

$$\alpha = \lambda_1 \frac{\int_\Omega |\varphi_1|^{p-1} u dx + \int_\Gamma |\varphi_1|^{p-1} u ds}{\int_\Omega |\nabla \varphi_1|^p dx}.$$

Due to the fact that

$$\lambda_1 = \frac{\int_\Omega |\nabla \varphi_1|^p dx}{\int_\Omega |\varphi_1|^p dx + \int_\Gamma |\varphi_1|^p ds},$$

we get

$$\int_\Omega |\varphi_1|^{p-1} w dx + \int_\Gamma |\varphi_1|^{p-1} w ds = 0.$$

Therefore $w \in W$, (the uniqueness of w comes from the uniqueness of λ_1.) Hence

$$X = V \oplus W.$$

Recall that a functional Φ satisfies the Palais-Smale condition on X, if for any sequence such that $|\Phi(u_n)| \leq c$ and $\Phi'(u_n) \to 0$, we can show that there exists a convergent subsequence.

Lemma 2.1. *Assume that* (F), (G) *and* (1.3) *or* (1.4) *are verified. Then the functional* Φ *satisfies the Palais-Smale condition on* X.

Proof. Let (u_n) be a sequence in X, and c a real number such that:

$$|\Phi(u_n)| \leq c \ \ for \ all \ n, \tag{2.3}$$

$$\Phi'(u_n) \to 0. \tag{2.4}$$

We claim that (u_n) is bounded in X. Otherwise, suppose by contradiction that

$$||u_n|| \to +\infty, \ \ as \ n \to +\infty.$$

Put $v_n = u_n / ||u_n||$, thus (v_n) is bounded, for a subsequence still denoted (v_n), we can assume that $v_n \rightharpoonup v$ weakly in X, by Sobelev injection theorems we have $v_n \to v$ strongly in $L^p(\Omega)$ and $v_n \to v$ a.e in Ω, since the range of the trace mapping $X \hookrightarrow L^p(\Gamma)$ is continuous and compact (see [9]), $v_n \to v$ strongly in $L^p(\Gamma)$. Dividing (2.3) by $||u_n||^p$, we get

$$\begin{aligned} \lim_{n \to +\infty} \Big\{ & \frac{1}{p} \int_\Omega |\nabla v_n|^p dx - \frac{\lambda_1}{p} \Big(\int_\Omega |v_n|^p dx + \int_\Gamma |v_n|^p ds \Big) \\ & - \int_\Omega \frac{F(x,u_n)}{||u_n||^p} dx - \int_\Gamma \frac{G(x,u_n)}{||u_n||^p} ds + \int_\Omega h \frac{u_n}{||u_n||^p} \Big\} = 0. \end{aligned} \tag{2.5}$$

By the hypotheses on the functions f, g, h and (u_n), we obtain

$$\lim_{n \to +\infty} \left(\int_\Omega \frac{F(x,u_n)}{||u_n||^p} dx + \int_\Gamma \frac{G(x,u_n)}{||u_n||^p} ds - \int_\Omega h \frac{u_n}{||u_n||^p} dx \right) = 0,$$

while

$$\lim_{n\to+\infty}\int_\Omega |v_n|^p dx=\int_\Omega |v|^p dx, \text{ and } \lim_{n\to+\infty}\int_\Gamma |v_n|^p ds=\int_\Gamma |v|^p ds,$$

from (2.5) we deduce that

$$1=\lim_{n\to+\infty}\int_\Omega (|\nabla v_n|^p+|v_n|^p)dx=\lambda_1\left(\int_\Omega |v|^p dx+\int_\Gamma |v|^p ds\right)+\int_\Omega |v|^p dx.$$

Then $v\not\equiv 0$. According to the variational characterization of λ_1 and the weak lower semi continuity of norm yield

$$\lambda_1\left(\int_\Omega |v|^p dx+\int_\Gamma |v|^p ds\right)+\int_\Omega |v|^p dx\le \int_\Omega \left(|\nabla v|^p+|v|^p\right)dx$$
$$\le \liminf_{n\to+\infty}\int_\Omega \left(|\nabla v_n|^p+|v_n|^p\right)dx=\lambda_1\left(\int_\Omega |v|^p dx+\int_\Gamma |v|^p ds\right)+\int_\Omega |v|^p dx,$$

which implies that

$$v_n\to v \text{ strongly in } X, \quad \text{and} \quad \int_\Omega |\nabla v|^p dx=\lambda_1\left(\int_\Omega |v|^p dx+\int_\Gamma |v|^p ds\right).$$

Thus, by the simplicity of the eigenfunction φ_1, we deduce that $v=\pm\varphi_1$.

Now, from (2.3) we have

$$\begin{aligned}-cp\le \int_\Omega |\nabla u_n|^p dx-\lambda_1\left(\int_\Omega |u_n|^p dx+\int_\Gamma |u_n|^p ds\right)-p\int_\Omega F(x,u_n)dx\\ -p\int_\Gamma G(x,u_n)ds+p\int_\Omega hu_n dx\le cp.\end{aligned} \tag{2.6}$$

In view of (2.4), for all $\varepsilon>0$ and n large enough, one can also have

$$\begin{aligned}-\varepsilon\|u_n\|\le -\int_\Omega |\nabla u_n|^p dx+\lambda_1\left(\int_\Omega |u_n|^p dx+\int_\Gamma |u_n|^p ds\right)+\int_\Omega f(x,u_n)u_n dx\\ +\int_\Gamma g(x,u_n)u_n ds-\int_\Omega hu_n\le \varepsilon\|u_n\|.\end{aligned} \tag{2.7}$$

Let

$$\varphi(x,s)=\begin{cases}\frac{F(x,s)}{s} & \text{if } \ s\neq 0\\ f(x,0) & \text{if } \ s=0,\end{cases} \tag{2.8}$$

and

$$\psi(x,s)=\begin{cases}\frac{G(x,s)}{s} & \text{if } \ s\neq 0\\ g(x,0) & \text{if } \ s=0.\end{cases} \tag{2.9}$$

Suppose that $v_n\to-\varphi_1$ (for example), then $u_n(x)\to-\infty$ for a.e. $x\in\Omega$, it follows from hypotheses (F) and (G) that

$$\begin{cases} f(x,u_n)\to f_+(x) & \text{a.e } x\in\Omega,\\ \varphi(x,u_n)\to f_+(x) & \text{a.e } x\in\Omega,\\ g(x,u_n)\to g_+(x) & \text{a.e } x\in\Gamma,\\ \psi(x,u_n)\to g_+(x) & \text{a.e } x\in\Gamma.\end{cases}$$

Moreover, the Lebesgue's theorem imply

$$\lim_{n\to+\infty}\int_\Omega (f(x,u_n)v_n-p\varphi(x,u_n)v_n)dx=(p-1)\int_\Omega f_+(x)\varphi_1 dx, \tag{2.10}$$

$$\lim_{n\to+\infty}\int_\Gamma (g(x,u_n)v_n-p\psi(x,u_n)v_n)ds=(p-1)\int_\Gamma g_+(x)\varphi_1 ds. \tag{2.11}$$

Combining (2.6) and (2.7), we get

$$-cp-\varepsilon\|u_n\| \le \int_\Omega f(x,u_n)u_n dx - p\int_\Omega F(x,u_n)dx + \int_\Gamma g(x,u_n)u_n ds$$
$$-p\int_\Gamma G(x,u_n)ds + (p-1)\int_\Omega hu_n \le cp+\varepsilon\|u_n\|.$$

Dividing by $||u_n||$ the last inequalities, we obtain

$$\frac{-cp}{||u_n||}-\varepsilon \le \int_\Omega f(x,u_n)v_n dx - p\int_\Omega \varphi(x,u_n)v_n dx + \int_\Gamma g(x,u_n)v_n ds$$
$$-p\int_\Gamma \psi(x,u_n)v_n ds + (p-1)\int_\Omega hv_n \le \frac{cp}{||u_n||}+\varepsilon,$$

and passing to the limits, we deduce from (2.10) and (2.11) that

$$\int_\Omega f_+(x)\varphi_1 dx + \int_\Gamma g_+(x)\varphi_1 ds = \int_\Omega h(x)\varphi_1 dx,$$

which contradicts (1.3). Thus (u_n) is bounded in X, for a subsequence denoted also (u_n), there exists $u\in X$ such that $u_n \rightharpoonup u$ weakly in X, and strongly in $L^p(\Omega)$ and $L^p(\Gamma)$. Since

$$\lim_{n\to+\infty}\langle \Phi'(u_n),(u_n-u)\rangle = 0,$$

we have

$$\begin{aligned}\langle \Phi'(u_n),(u_n-u)\rangle &= \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n-u)dx\\ &\quad -\lambda_1\int_\Omega |u_n|^{p-2}u_n(u_n-u)dx\\ &\quad -\lambda_1\int_\Gamma |u_n|^{p-2}u_n(u_n-u)ds - \int_\Omega f(x,u_n)(u_n-u)dx\\ &\quad -\int_\Gamma g(x,u_n)(u_n-u)ds + \int_\Omega h(u_n-u)dx = o_n(1).\end{aligned} \tag{2.12}$$

It can be easily seen that

$$\lim_{n\to+\infty}\int_\Omega |u_n|^{p-2}u_n(u_n-u)dx = \lim_{n\to+\infty}\int_\Gamma |u_n|^{p-2}u_n(u_n-u)ds = 0,$$

and

$$\lim_{n\to+\infty}\int_\Omega f(x,u_n)(u_n-u)dx = \lim_{n\to+\infty}\int_\Gamma g(x,u_n)(u_n-u)ds = 0,$$
$$\lim_{n\to+\infty}\int_\Omega h(x)(u_n-u)dx = 0.$$

Consequently, from (2.12) it follows that

$$\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)dx = 0.$$

Thus by the $(S+)$ property, $u_n \to u$ strongly in X and Φ satisfies the (PS) condition. □

Lemma 2.2. *Assume that (F), (G) and (1.3) are satisfied. Then the functional Φ is coercive on X.*

Proof. Suppose by contadiction that Φ is not coercive, then there exists a sequence (u_n) such that $||u_n|| \to +\infty$, and $|\Phi(u_n)| \leq c$. In the proof of lemma 2.1, we have showed that $v_n = u_n/||u_n|| \to \pm\varphi_1$. Since

$$0 \leq \int_\Omega |\nabla u_n|^p dx - \lambda_1 \left(\int_\Omega |u_n|^p dx + \int_\Gamma |u_n|^p ds \right),$$

one has

$$-\int_\Omega F(x, u_n) dx - \int_\Gamma G(x, u_n) ds + \int_\Omega h u_n dx \leq \Phi(u_n) \leq c. \tag{2.13}$$

Assume $v_n \to +\varphi_1$ (for example). Dividing (2.13) by $||u_n||$, we get

$$-\int_\Omega \frac{F(x, u_n)}{||u_n||} dx - \int_\Gamma \frac{G(x, u_n)}{||u_n||} ds + \int_\Omega h \frac{u_n}{||u_n||} dx \leq \frac{c}{||u_n||}.$$

Passing to the limits, we have

$$\int_\Omega f_+(x)\varphi_1 dx + \int_\Gamma g_+(x)\varphi_1 ds \geq \int_\Omega h(x)\varphi_1 dx,$$

which contradicts (1.3). □

3. Proof of main results

Proof of Theorem 1.3. If (1.3) holds, the coerciveness of the functional Φ and the Palais-Smale condition entrain, from theorem 1.1, that Φ attains its minimum, so problem (1.1) admits at least a weak solution in X.

we show that (1.3) is a necessary condition. Let $u \in X$ be a weak solution of (1.1). Then taking $v = \varphi_1$ as a test function in (2.1), we obtain

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi_1 dx = \lambda_1 \left(\int_\Omega |u|^{p-2} u \varphi_1 dx + \int_\Gamma |u|^{p-2} u \varphi_1 ds \right)$$
$$+ \int_\Omega f(x, u)\varphi_1 dx + \int_\Gamma g(x, u)\varphi_1 ds - \int_\Omega h(x)\varphi_1 dx,$$

so

$$\int_\Omega f(x, u)\varphi_1 dx + \int_\Gamma g(x, u)\varphi_1 ds = \int_\Omega h(x)\varphi_1 dx.$$

Since $f(x, .)$ and $g(x, .)$ are strictly decreasing functions, we have

$$\int_\Omega f_+(x)\varphi_1 dx < \int_\Omega f(x, u)\varphi_1 dx < \int_\Omega f_-(x)\varphi_1 dx \text{ for a.a.} x \in \Omega, \tag{3.1}$$

and

$$\int_\Gamma g_+(x)\varphi_1 ds < \int_\Gamma g(x, u)\varphi_1 ds < \int_\Gamma g_-(x)\varphi_1 ds \text{ for a.a.} x \in \Gamma. \tag{3.2}$$

Summing (3.1) and (3.2), we obtain

$$\int_\Omega f_+(x)\varphi_1 dx + \int_\Gamma g_+(x)\varphi_1 ds < \int_\Omega h(x)\varphi_1 dx < \int_\Omega f_-(x)\varphi_1 dx + \int_\Gamma g_-(x)\varphi_1 ds.$$

□

Proof of Theorem 1.4. If (1.4) holds, then Φ has the geometry of the saddle point theorem 1.2. Indeed, splitting $X = V \oplus W$, it is well known that

$$\int_\Omega |\nabla u|^p dx \geq \lambda_2 \Big(\int_\Omega |u|^p dx + \int_\Gamma |u|^p ds \Big) \text{ for all } u \in W. \tag{3.3}$$

Thus for $u \in W$, using Hölder inequality, (3.3) and recalling the properties of the functions F and G, we obtain

$$\begin{aligned}
\Phi(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda_1}{p}\left(\int_\Omega |u|^p dx + \int_\Gamma |u|^p ds\right) \\
&\quad - \int_\Omega F(x,u)dx - \int_\Gamma G(x,u)ds + \int_\Omega h(x)udx \\
&\geq \frac{1}{p}\left(1-\frac{\lambda_1}{\lambda_2}\right)||\nabla u||^p_{L^p(\Omega)} - C_1\Big(|\Omega|^{1/p'} + |\Gamma|^{1/p'} + \|h\|_{p'}\Big)||u||, \qquad (3.4)
\end{aligned}$$

and

$$\Phi(u) \geq \frac{\lambda_2-\lambda_1}{p}\left(\int_\Omega |u|^p dx + \int_\Gamma |u|^p ds\right) - C_2\Big(|\Omega|^{1/p'} + |\Gamma|^{1/p'} + \|h\|_{p'}\Big)||u||, \qquad (3.5)$$

where C_1 and C_2 are positive constants, $\|.\|_{p'}$ denote the norm in $L^{p'}(\Omega)$. Summing (3.4) and (3.5), we get

$$\begin{aligned}
\Phi(u) &\geq \frac{\lambda_2-\lambda_1}{p(1+\lambda_2)}\Big(||\nabla u||^p_{L^p(\Omega)} + \int_\Omega |u|^p dx + \int_\Gamma |u|^p ds\Big) - C_3\Big(|\Omega|^{1/p'} + |\Gamma|^{1/p'} + \|h\|_{p'}\Big)||u|| \\
&\geq \frac{\lambda_2-\lambda_1}{p(1+\lambda_2)}||u||^p - C_3\Big(|\Omega|^{1/p'} + |\Gamma|^{1/p'} + \|h\|_{p'}\Big)||u||. \qquad (3.6)
\end{aligned}$$

Then Φ is coercive on W, so that

$$\inf_{w\in W} \Phi(w) > -\infty. \qquad (3.7)$$

On the other hand, for every $t \in \mathbb{R}$, one has

$$\begin{aligned}
\Phi(t\varphi_1) &= -\int_\Omega F(x,t\varphi_1)dx - \int_\Gamma G(x,t\varphi_1)ds + t\int_\Omega h(x)\varphi_1 dx \\
&= t\left(\int_\Omega h(x)\varphi_1 dx - \int_\Omega \varphi(x,t\varphi_1)\varphi_1 dx - \int_\Gamma \psi(x,t\varphi_1)\varphi_1 ds\right),
\end{aligned}$$

where φ and ψ has been defined by (2.8) and (2.9). From the Lebesgue theorem, it follows that $\int_\Omega \big(h(x)-\varphi(x,t\varphi_1)\big)\varphi_1 dx - \int_\Gamma \psi(x,t\varphi_1)\varphi_1 ds$ tends to $\int_\Omega \big(h(x)-f_+(x)\big)\varphi_1 - \int_\Gamma g_+(x)\varphi_1 ds$, as $t \to +\infty$ and the limit is negative by (1.4). Analogously, if t tends to $-\infty$, we have the same result with $f_-(x)$ and $g_-(x)$ exchanged with $f_+(x)$ and $g_+(x)$ respectively. In both cases we get

$$\lim_{t\to\pm\infty} \Phi(t\varphi_1) = -\infty. \qquad (3.8)$$

By (3.7) and (3.8), there exists $R > 0$ such that

$$\max_{v\in V, ||v||=R} \Phi(v) < \inf_{w\in W} \Phi(w).$$

Hence, Φ satisfies the hypotheses of Theorem 1.2, and there exists a critical point of Φ, that is a solution of (1.1).

For the necessary condition, we can take the same technique as in the proof of theorem 1.3. □

Proof of Theorem 1.5. The result of Lemma 2.1 holds true for the Euler functional associated to problem (1.5), that is

$$\Phi_\lambda(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\Big(\int_\Omega |u|^p dx + \int_\Gamma |u|^p ds\Big)$$
$$- \int_\Omega F(x,u)dx - \int_\Gamma G(x,u)ds + \int_\Omega hudx$$

for every $u \in X$. Indeed, Let (u_n) be a sequence satisfying (2.3) and (2.4), suppose that (u_n) is unbounded, and define $v_n = u_n/||u_n||$, so that, up to subsequence, (v_n) converges weakly to a function v in X. Dividing (2.4) by $||u_n||^{p-1}$, and then taking $\langle \Phi'_\lambda(u_n), v_n - v\rangle = o_n(1)$, we get

$$\lim_{n\to+\infty} \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \nabla(v_n - v)dx = 0$$

this fact implies (as in proof of Lemma 2.1) that $v_n \to v$ strongly in X. since $\langle \Phi'_\lambda(u_n), \psi/||u_n||^{p-1}\rangle = o_n(1)$, with $\psi \in X$,

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla \psi dx = \lambda\Big(\int_\Omega |v|^{p-2} v\psi dx + \int_\Gamma |v|^{p-2} v\psi ds\Big),$$

so that v solve the problem $-\Delta_p u = \lambda |u|^{p-2}u$ with mixed boundary condition on $\partial\Omega$. But this equation, being $\lambda \in (\lambda_1, \lambda_2)$, has zero as the only solution by definition of λ_2. Thus $v = 0$, a contradiction with the strong convergence of v_n to v. Hence (u_n) is bounded. This implies, by same argument in proof of Lemma 2.1, that (u_n) is strongly convergent.

On the other hand, as in the second part of the proof of Theorem 1.3, rewrite everything with λ instead of λ_1 and use the fact that $\lambda < \lambda_2$, we get the coerciveness of Φ_λ on W.

Now, recalling that

$$\int_\Omega |\nabla t\varphi_1|^p dx = \lambda_1\Big(\int_\Omega |t\varphi_1|^p dx + \int_\Gamma |t\varphi_1|^p ds\Big), \quad \text{for every } t \in \mathbb{R}$$

thus

$$\begin{aligned}\Phi_\lambda(t\varphi_1) \quad &= \quad \frac{\lambda_1 - \lambda}{p}|t|^p\Big(\int_\Omega |\varphi_1|^p dx + \int_\Gamma |\varphi_1|^p ds\Big) \\ &\quad + t\left(\int_\Omega h(x)\varphi_1 dx - \int_\Omega \varphi(x, t\varphi_1)\varphi_1 dx - \int_\Gamma \psi(x, t\varphi_1)\varphi_1 dx\right),\end{aligned}$$

since $\lambda > \lambda_1$ and $p > 1$, we have, as before

$$\lim_{t\to\pm\infty} \Phi_\lambda(t\varphi_1) = -\infty.$$

Using again the saddle point theorem, the desired result follows. □

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