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Fixed Point Results for Almost Nonexpansive Mappings in b-metric Spaces

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ABSTRACT: The aim of this paper is to introduce a new class of mappings called *almost nonexpansive mappings* in a b-metric space. Some characteristics of this class of mappings are discussed. Fixed point and common fixed point results for such mappings are obtained. An application to the Cauchy problem in a Banach space is also shown in this paper.

Key Words: b-metric space, almost nonexpansive mappings, asymptotic regularity.

Contents

1. Introduction

One of the most prominent theorems in fixed point theory for nonexpansive mappings is due to Browder [3] in 1965. Similar fixed point results for nonexpansive mappings were also developed by Kirk [13] and Gohde [9] the same year, independently. Browder's fixed point theorem asserts the following.

Theorem 1.1. *[3] Let Y be a uniformly convex Banach space, U a nonexpansive mapping of the bounded closed convex subset C of Y into C. Then U has a fixed point in C.*

One may note that the assumption of a closed, bounded convex subset of Banach space is crucial for the existence result of the above mentioned papers. And we recall that a nonexpansive mapping on a complete b-metric space need not have a fixed point. In this regard, the self mapping $fy = y + 1$ defined on the set of real numbers with the usual metric is an example. There are several results on fixed point of nonexpansive mappings in the literature. One may refer to [14] and the references therein.

In 1972, Nussbaum [17] introduced the class of *locally almost nonexpansive mappings* (One may refer to [23]) as follows.

Definition 1.2. *[17] Let Y be a Banach space and C be a nonempty closed bounded convex subset of Y. Then a mapping $f : Y \longrightarrow C$ is called* locally almost nonexpansive *if for all $x \in C$ and $\varepsilon > 0$, there exists a weak neighbourhood $N_x = N(x, \varepsilon)$ of Y in C such that*

$$\|Tu - Tv\| \leq \|u - v\| + \varepsilon \qquad \forall \quad u, v \in N_x.$$

The notion of b-metric spaces, introduced in 1989 by Bakhtin [1] and formally defined by Czerwik [5] in 1993 is the following. One may refer to [6], [7], [11], [20] and the references therein in this regard.

Definition 1.3. *[5] Let Y be a non empty set and $s \geq 1$ be a given real number. A function $\rho : Y \times Y \longrightarrow [0, \infty)$ is called b-metric if it satisfies the following properties.*

1. $\rho(u, v) = 0$ if and only if $u = v$;

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2. $\rho(u,v) = \rho(v,u);$ *and*

3. $\rho(u,v) \le s[\rho(u,w) + \rho(w,v)],$ *for all* $u, v, w \in X$.

The triplet (Y, ρ, s) *is called a b-metric space with coefficient* s.

As noted in various papers in the literature, a b-metric may not be continuous. For instance, one may refer Example 2.6 of [16]. Many examples of b-metric spaces can be found in the literature. Some classical examples are included in [2] and [22]. Here, we give a new example of a b-metric space.

Example 1.4. *Let* $Y = \{x_1, x_2, x_3, x_4\}$ *and define a function* $\rho : Y \times Y \longrightarrow [0, \infty)$ *as follows.*

$$\begin{aligned}
&\rho(x_i, x_i) = 0 \qquad \text{for } i = 1,2,3,4;\\
&\rho(x_1, x_2) = \rho(x_2, x_1) = \frac{3}{2}; \qquad \rho(x_2, x_3) = \rho(x_3, x_4) = \frac{1}{4};\\
&\rho(x_1, x_3) = \rho(x_3, x_1) = \frac{1}{2}; \qquad \rho(x_1, x_4) = \rho(x_4, x_1) = \frac{3}{4};\\
&\rho(x_2, x_4) = \rho(x_4, x_2) = \rho(x_3, x_4) = \rho(x_4, x_3) = 2.
\end{aligned}$$

The function ρ *then defines a b-metric on* Y *with coefficient* $s = 2$*, and* (Y, ρ, s) *is a b-metric space.*

The following lemma is useful in overcoming some of the problems due to the possible discontinuity of the b-metric ρ.

Lemma 1.5. *[15] Let* (Y, ρ, s) *be a b-metric space with coefficient* $s \ge 1$ *and* $\{y_n\}$ *be a convergent sequence in* Y *with* $\lim_{n\to\infty} y_n = y$*. Then for all* $z \in Y$

$$s^{-1}\rho(y,z) \le \lim_{n\to\infty} \inf \rho(y_n, z) \le \lim_{n\to\infty} \sup \rho(y_n, z) \le s\rho(y,z).$$

From the above lemma, it follows that if $\{y_n\} \subseteq Y$ is a sequence such that $\lim_{n\to\infty} y_n = y$ for some $y \in Y$, then

$$\lim_{n\to\infty} \rho(y_n, y) = 0.$$

In 2014, Roshan et al. [21] obtained some common fixed point theorems using the following lemma.

Lemma 1.6. *[21] Let* $\{y_n\}$ *be a sequence in a b-metric space* (Y, ρ, s) *such that*

$$\lim_{n\to\infty} \rho(y_n, y_{n+1}) = 0.$$

If $\{y_n\}$ *is not a Cauchy sequence, then there exists* $\varepsilon > 0$ *and two sequences* $\{m(k)\}$ *and* $\{n(k)\}$ *of positive integers such that*

$$\begin{aligned}
\varepsilon &\le \liminf_{k\to\infty} \rho\big(y_{m(k)}, y_{n(k)}\big) \le \limsup_{k\to\infty} \rho\big(y_{m(k)}, y_{n(k)}\big) \le s\varepsilon,\\
\frac{\varepsilon}{s} &\le \liminf_{k\to\infty} \rho\big(y_{m(k)}, y_{n(k)+1}\big) \le \limsup_{k\to\infty} \rho\big(y_{m(k)}, y_{n(k)+1}\big) \le s^2\varepsilon,\\
\frac{\varepsilon}{s} &\le \liminf_{k\to\infty} \rho\big(y_{m(k)+1}, y_{n(k)}\big) \le \limsup_{k\to\infty} \rho\big(y_{m(k)+1}, y_{n(k)}\big) \le s^2\varepsilon,\\
\frac{\varepsilon}{s^2} &\le \liminf_{k\to\infty} \rho\big(y_{m(k)+1}, y_{n(k)+1}\big) \le \limsup_{k\to\infty} \rho\big(y_{m(k)+1}, y_{n(k)+1}\big) \le s^3\varepsilon.
\end{aligned}$$

Recently, Górnicki [10] proved the following fixed point result assuming asymptotic regularity of the mapping on the metric space (Y, d).

Theorem 1.7. *[10] Let* (Y, d) *be a complete metric space and* $f : Y \longrightarrow Y$ *be a continuous mapping asymptotically regular on* Y *and if there exists* $0 \le M < 1$ *and* $0 \le K < \infty$ *satisfying*

$$d(fx, fy) \le M d(x,y) + K\big\{d(x, fx) + d(y, fy)\big\},$$

for all $x, y \in Y$ *then* f *has a unique fixed point* $z \in Y$ *and* $f^n y \longrightarrow z$ *for all* $y \in Y$.

Here, a mapping f of a b-metric space (Y, ρ, s) into itself is said to be *asymptotically regular on* Y [4] if

$$\lim_{n\to\infty} \rho(f^n x, f^{n+1}x) = 0 \qquad \text{for all} \;\; x \in Y.$$

In this paper, we say that a self mapping f on a b-metric space is *asymptotically regular at a point* $x' \in Y$ if

$$\lim_{n\to\infty} \rho\big(f^n x', f^{n+1}x'\big) = 0.$$

Definition 1.8. *[12] A function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is said to be a sub-additive altering distance function if*

(i) φ is an altering distance function (i.e., φ is continuous, strictly increasing and $\varphi(t) = 0$ if and only if $t = 0$),

(ii) $\varphi(x + y) \leq \varphi(x) + \varphi(y) \qquad \forall \;\; x, y \in [0,\infty)$.

As mentioned in [12], $\varphi_1(x) = kx$ for some $k \geq 1$, $\varphi_2(x) = \sqrt[n]{x}$, $n \in \mathbb{N}$, $\varphi_3(x) = \log(1 + x)$, $x \geq 0$ and $\varphi_4(x) = \tan^{-1} x$ are some examples of sub-additive altering distance functions.

2. Main results

Inspired by the definition of locally almost nonexpansive mappings [17], we introduce a class of mappings called almost nonexpansive mappings in a b-metric space and obtain a result on the existence of fixed point in a complete b-metric space. Considering φ as a sub-additive altering distance function which is also sub-homogeneous, we define the following class of mappings.

Definition 2.1. *Let (Y, ρ, s) be a b-metric space. A mapping $f : Y \longrightarrow Y$ is said to be an almost nonexpansive mapping on Y if there exists a nonnegative real number $p < 1$ such that for all $x, y \in Y$,*

$$s^q \varphi\big(\rho(fx, fy)\big) \leq pQ(x, y) + \varphi\big(\rho(x, y)\big),$$

for some $q \geq 5$, where $Q(x, y) = \big|\varphi\big(\rho(fx, y)\big) - \varphi\big(\rho(x, fy)\big)\big|$ or $\varphi\big(\rho(x, fx)\big) + \varphi\big(\rho(y, fy)\big)$.

Example 2.2. *From the definition, it is clear that every nonexpansive mapping on a metric space (Y, d) is an almost nonexpansive mapping.*

We recall that a mapping f on a subset C of a b-metric space (Y, ρ, s) is quasi-nonexpansive [8] if $\rho(fx, z) \leq \rho(x, z)$ for all $x \in C$ and $z \in F(f)$, the set of fixed points of f.

Almost nonexpansive mappings with $Q(x, y) = \big|\varphi\big(\rho(fx, y)\big) - \varphi\big(\rho(x, fy)\big)\big|$ is a quasi-nonexpansive mapping. For, if $z \in F(f)$ then

$$\begin{aligned}
\varphi\big(\rho(fx, fz)\big) &\leq s^q \varphi\big(\rho(fx, fz)\big) \leq p\big|\varphi\big(\rho(fx, z)\big) - \varphi\big(\rho(x, fz)\big)\big| + \varphi\big(\rho(x, z)\big)\\
&= \begin{cases} p\Big\{\varphi\big(\rho(fx, z)\big) - \varphi\big(\rho(x, fz)\big)\Big\} + \varphi\big(\rho(x, z)\big), \text{ if } \rho(fx, z) > \rho(x, fz) \\ p\Big\{\varphi\big(\rho(x, fz)\big) - \varphi\big(\rho(fx, z)\big)\Big\} + \varphi\big(\rho(x, z)\big), \text{ if } \rho(fx, z) < \rho(x, fz), \end{cases}
\end{aligned}$$

and in either of the cases, we get,

$$\varphi\big(\rho(fx, fz)\big) \leq \varphi\big(\rho(x, z)\big),$$

that is,

$$\rho(fx, z) \leq \rho(x, z).$$

However, with $Q(x, y) = \varphi\big(\rho(x, fx)\big) + \varphi\big(\rho(y, fy)\big)$, an almost nonexpansive mapping may not be a quasi-nonexpansive mapping as can be seen in the following example.

Example 2.3. *Consider the b-metric space (X, d, s), where $X = [0, \infty)$, $s = 1$ and $\rho(x, y) = x + y$ for all x and y in X with $x \neq y$, and $\rho(x, y) = 0$ if $x = y$. Let φ be the identity mapping and $T : X \longrightarrow X$ be defined by*

$$Tx = \begin{cases} 0, & 0 \leq x < 1; \\ 2, & x = 1; \\ \frac{1}{x}, & x > 1. \end{cases}$$

Then it can be easily checked that T is an almost nonexpansive mapping with $Q(x, y) = \varphi\big(\rho(x, fx)\big) + \varphi\big(\rho(y, fy)\big)$. Further, $0 \in F(T)$ and

$$\rho(T1, 0) = \rho(2, 0) = 2 > 1 = \rho(1, 0),$$

so that T is not a quasi-nonexpansive mapping.

In the following theorem, we prove that an almost nonexpansive mappings f has a fixed point if it is *asymptotically regular* at a point $y_0 \in Y$. The implication of which is that we can determine a class of nonexpansive mappings for which a fixed point exists in a complete b-metric space.

Theorem 2.4. *Let (Y, ρ, s) be a complete b-metric space with $s = 1$ or $s \geq 2$ and $f : Y \longrightarrow Y$ be an almost nonexpansive mapping with*

$$s^q \varphi\big(\rho(fx, fy)\big) \leq p\Big|\varphi\big(\rho(y, fx)\big) - \varphi\big(\rho(x, fy)\big)\Big| + \varphi\big(\rho(x, y)\big), \tag{2.1}$$

for some $q \geq 5$ and $0 \leq p < 1$ with $sp < 1$. If f is asymptotically regular at $y_0 \in Y$, then f has a fixed point.

Proof. Since f is asymptotically regular at $y_0 \in Y$, we have,

$$\lim_{n\to\infty} \rho(f^n y_0, f^{n+1} y_0) = 0.$$

We first show that the sequence $\{y_n\} \subseteq Y$ defined by $y_n = f^n y_0$ is a Cauchy sequence, that is, for every $\varepsilon > 0$ one can find $k \in \mathbb{N}$ with $\forall\ m, n \geq k$, $\rho(y_m, y_n) < \varepsilon$.

Suppose not, then by Lemma 1.6, there exists $\varepsilon > 0$ for which one can find subsequence $\big\{y_{n(i)}\big\}$ and $\big\{y_{m(i)}\big\}$ of $\{y_n\}$ with $n(i) > m(i) \geq i$ and

(a) $m(i) = 2t$ and $n(i) = 2t' + 1$, where t and t' are nonnegative integers,

(b) $\rho\big(y_{m(i)}, y_{n(i)}\big) \geq \varepsilon$, and

(c) $n(i)$ is the smallest number such that condition (b) holds, that is, $\rho\big(y_{m(i)}, y_{n(i)-1}\big) < \varepsilon$.

Then by the continuity, sub-homogeneity and increasing property of φ, we have,

$$\varphi(\varepsilon) \leq \varphi\left(\limsup_{i\to\infty} \rho\big(y_{m(i)}, y_{n(i)}\big)\right) \leq s\varphi(\varepsilon), \tag{2.2}$$

$$\frac{\varphi(\varepsilon)}{s} \leq \varphi\left(\limsup_{i\to\infty} \rho\big(y_{m(i)}, y_{n(i)+1}\big)\right) \leq s^2\varphi(\varepsilon), \tag{2.3}$$

$$\frac{\varphi(\varepsilon)}{s} \leq \varphi\left(\limsup_{i\to\infty} \rho\big(y_{m(i)+1}, y_{n(i)}\big)\right) \leq s^2\varphi(\varepsilon), \tag{2.4}$$

$$\frac{\varphi(\varepsilon)}{s^2} \leq \varphi\left(\limsup_{i\to\infty} \rho\big(y_{m(i)+1}, y_{n(i)+1}\big)\right) \leq s^3\varphi(\varepsilon). \tag{2.5}$$

From (2.1), we have,

$$s^q\varphi\left(\rho\big(y_{m(i)+1}, y_{n(i)+1}\big)\right) \leq p\Big|\varphi\left(\rho\big(y_{n(i)+1}, y_{m(i)}\big)\right) - \varphi\left(\rho\big(y_{m(i)}, y_{n(i)+1}\big)\right)\Big| + \varphi\left(\rho\big(y_{m(i)}, y_{n(i)}\big)\right).$$

Taking the upper limit as $i \to \infty$, using (2.2), (2.3) and (2.4), we get,

$$s^q \varphi\left(\limsup_{i\to\infty} \rho\big(y_{m(i)+1}, y_{n(i)+1}\big)\right) \leq p\left|s^2\varphi(\varepsilon) - \frac{\varphi(\varepsilon)}{s}\right| + s\varphi(\varepsilon) < 2s^2\varphi(\varepsilon),$$

or,

$$\varphi\left(\limsup_{i\to\infty} \rho\big(y_{m(i)+1}, y_{n(i)+1}\big)\right) < \frac{2\varphi(\varepsilon)}{s^{q-2}} < \frac{2\varphi(\varepsilon)}{s^3},$$

since $q \geq 5$, contradicting (2.5) for $s \geq 2$. We note here that, if $s = 1$, then f is asymptotically regular at $y_0 \in Y$ implies $\{y_n\}$ is a Cauchy sequence, which can be easily seen.

Hence $\{f^n y_0\}$ is a Cauchy sequence and since (Y, ρ, s) is complete, there exists $z \in Y$ such that

$$\lim_{n\to\infty} f^n y_0 = z.$$

Now, applying (2.1),

$$\begin{aligned}\varphi\big(\rho(z, fz)\big) &\leq s\varphi\big(\rho(z, f^{n+1}x_0)\big) + s^q\varphi\big(\rho(f^{n+1}x_0, fz)\big)\\ &\leq s\varphi\big(\rho(z, f^{n+1}x_0)\big) + p\Big|\varphi\big(\rho(z, f^{n+1}x_0)\big) - \varphi\big(\rho(f^n x_0, fz)\big)\Big| + \varphi\big(\rho(f^n x_0, z)\big).\end{aligned}$$

Taking the limit as $n \to \infty$, using Lemma 1.5 (in lieu of continuity of ρ), we get,

$$(1 - sp)\varphi\big(\rho(z, fz)\big) \leq 0,$$

which proves that $\varphi\big(\rho(z, fz)\big) = 0$ or $\rho(z, fz) = 0$, i.e., $fz = z$, as required. □

Remark 2.5. *In Theorem 2.4, f has a unique fixed point if (Y, ρ, s) is not a metric space (i.e., $s \neq 1$). For, if $w \in Y$ is another fixed point of f, then from (2.1)*

$$s^q\rho(z, w) \leq p\big|\rho(z, w) - \rho(z, w)\big| + \rho(z, w),$$

which implies $\rho(z, w) = 0$ (since $s \geq 2$), as claimed.

It is clear from the proof of Theorem 2.4 that the condition $0 \leq p < 1$ in the definition of almost nonexpansive mappings is essential for the existence of a fixed point. The following example shows such a mapping with $p = 1$ having no fixed point.

Example 2.6. *Consider the b-metric space (Y, ρ, s) where $Y = \mathbb{R} \cup \{\infty\}$ is the projectively extended real line (with $s = 1$) and $\rho : Y \times Y \longrightarrow [0, \infty)$ is defined by*

$$\rho(x, y) = \begin{cases} 0, & \text{if } \ x = y;\\ |x - y|, & \text{if } \ x, y \in \mathbb{R};\\ \rho(x, \infty) = \rho(\infty, x) = |x + 1|, & \text{if } \ x \in \mathbb{R}.\end{cases}$$

Define $f : Y \longrightarrow Y$ as follows

$$fx = \begin{cases} x + 1, & x > 0;\\ 1, & x = 0 \ \text{ or } \ \infty\ .\end{cases}$$

Clearly, f is asymptotically regular at $x = 0 \in Y$. Let φ be the identity mapping.

If $x = y = 0$ or ∞, then $\rho(fx, fy) = 0$ and (2.1) is trivially satisfied.

Now, for $x, y \in \mathbb{R} \setminus \{0\}$, we have,

$$\rho(fx, fy) = \big|x + 1 - y - 1\big| = |x - y| \leq p\big|\rho(fx, y) - \rho(x, fy)\big| + \rho(x, y).$$

On the other hand, if $x = 0$ and $y \in \mathbb{R} \setminus \{0\}$ (or vice versa), then

$$\rho(fx, fy) = \big|1 - y - 1\big| = |x - y| \le p\big|\rho(fx, y) - \rho(x, fy)\big| + \rho(x, y).$$

If $x = \infty$ and $y = 0$ or vice versa, then

$$\rho(f\infty, fy) = d(1, 1) = 0,$$

and (2.1) is trivially satisfied.

On the other hand, if $x = \infty$ and $y \in \mathbb{R} \setminus \{0\}$ (or vice versa), then

$$\rho(f\infty, fy) = \rho(1, y + 1) = |y| < |y + 1| = \rho(\infty, y) \le p\big|\rho(f\infty, y) - \rho(\infty, fy)\big| + \rho(\infty, y).$$

Thus f is an almost nonexpansive mapping (with $p = 1$), asymptotically regular at the point $x = 0$, but f does not have a fixed point.

Corollary 2.7. *A nonexpansive mapping on a complete metric space (Y, d) has a fixed point if it is asymptotically regular at a point in Y.*

Proof. The proof follows from Theorem 2.4 by taking φ as the identity mapping, $p = 0$ and $s = 1$. □

Example 2.8. *Consider the complete b-metric space (Y, ρ, s) with $s = 1$, where $Y = [0, 1]$ and for all $x, y \in Y$, $\rho(x, y) = x + y$ if $x \neq y$, and $\rho(x, y) = 0$ if $x = y$. Let φ be the identity mapping and $f : Y \longrightarrow Y$ be given by $fx = x^2$ for all $x \in Y$. Then f is asymptotically regular on Y.*

Now, for $p \ge 0$ and $x \neq y$,

$$\varphi\big(\rho(Tx, Ty)\big) = |x^2 + y^2| \le |x + y| \le p\Big|\varphi\big(\rho(y, fx)\big) - \varphi\big(\rho(x, fy)\big)\Big| + \varphi\big(\rho(x, y)\big).$$

The case $x = y$ being trivial.

Therefore, by Theorem 2.4, f has a fixed point.

Example 2.9. *Consider the complete b-metric space (Y, ρ, s) where $Y = [0, \infty)$ and $\rho(x, y) = |x - y|$ with $s = 1$. Define $f : Y \longrightarrow Y$ as follows*

$$fx = \begin{cases} x - 1, & x > 1; \\ 0, & 0 \le x \le 1. \end{cases}$$

Let φ be the identity mapping. Then f is asymptotically regular at each point $x \in [0, 1]$.

Now, for $x, y > 1$, we have,

$$\rho(fx, fy) = \big|x - 1 - y + 1\big| = |x - y| \le p\Big|\rho(fx, y) - \rho(x, fy)\Big| + \rho(x, y),$$

i.e., (2.1) is satisfied for any $0 \le p < 1$.

If $x, y \in [0, 1]$, then $\rho(fx, fy) = 0$ and (2.1) is trivially satisfied.

On the other hand, if $x \in [0, 1]$ and $y > 1$ (or vice versa), then

$$\rho(fx, fy) = \big|y - 1\big| \le |y - x| \le p\big|\rho(fx, y) - \rho(x, fy)\big| + \rho(x, y),$$

i.e., (2.1) is satisfied for any $0 \le p < 1$.

Thus f is an almost nonexpansive mapping, asymptotically regular at some point in Y and by Theorem 2.4, f has a fixed point, which is $x = 0$ here.

In the following we prove a result similar to Theorem 1.7 replacing the asymptotic regularity of f on Y by asymptotic regularity at a point of Y with $M = 1$ and $0 \le K < 1$.

Theorem 2.10. *Let (Y, ρ, s) be a complete b-metric space and $f : Y \longrightarrow Y$ be a mapping such that for all $x, y \in Y$,*

$$s^q\varphi\big(\rho(fx, fy)\big) \leq p\Big\{\varphi\big(\rho(x, fx)\big) + \varphi\big(\rho(y, fy)\big)\Big\} + \varphi\big(\rho(x, y)\big), \tag{2.6}$$

for some nonnegative real number $p < 1$ with $sp < 1$ and $q \geq 5$. If f is asymptotically regular at a point x_0 of Y, then f have a fixed point.

Proof. Since f is asymptotically regular at $x_0 \in Y$, $\lim_{n\to\infty} \rho\big(f^n x_0, f^{n+1}x_0\big) = 0$. Thus the sequence $\{x_n\}$ defined by $x_n = f^n x_0$, $n \in \mathbb{N}$ is a Cauchy sequence (as in the proof of Theorem 2.4). Since (Y, ρ, s) is complete, there exists $z \in Y$ such that

$$\lim_{n\to\infty} x_n = z.$$

Now, since φ is sub-additive and sub-homogeneous altering distance function and $q \geq 5$,

$$\varphi\big(\rho(z, fz)\big) \leq s\varphi\big(\rho(z, f^{n+1}x_0)\big) + s\varphi\big(\rho(f^{n+1}x_0, fz)\big) \leq s\varphi\big(\rho(z, f^{n+1}x_0)\big) + s^q\varphi\big(\rho(f^{n+1}x_0, fz)\big).$$

Using (2.6) in the above expression, we get,

$$\varphi\big(\rho(z, fz)\big) \leq s\varphi\big(\rho(z, f^{n+1}x_0)\big) + p\Big\{\varphi\big(\rho(f^n x_0, f^{n+1}x_0)\big) + \varphi\big(\rho(z, fz)\big)\Big\} + \varphi\big(\rho(f^n x_0, z)\big).$$

Taking the limit as $n \to \infty$ and using Lemma 1.5, we get,

$$(1 - sp)\varphi\big(\rho(z, fz)\big) \leq 0,$$

which proves that $fz = z$, as required. □

Example 2.11. *Consider the complete b-metric space (Y, ρ, s) where $Y = [0, \infty)$, $s = 1$ and $\rho : Y \times Y \longrightarrow [0, \infty)$ is given by*

$$\rho(x, y) = \begin{cases} x + y, & x \neq y; \\ 0, & x = y. \end{cases}$$

Let φ be the identity mapping and $f : Y \longrightarrow Y$ be defined as follows.

$$fx = \begin{cases} 0, & x < 1; \\ \frac{1}{x}, & x \geq 1. \end{cases}$$

We first note that f is asymptotically regular at each point $x \leq 1$ of Y.
Now, if $x < 1$ and $y < 1$, then $\rho(fx, fy) = 0$ and (2.6) is trivially satisfied.
Also, if $x \geq 1$ and $y \geq 1$ with $x \neq y$, then

$$\rho(fx, fy) = \frac{1}{x} + \frac{1}{y} \leq x + y \leq p\big\{\rho(x, fx) + \rho(y, fy)\big\} + \rho(x, y),$$

i.e., (2.6) holds.
If $x \geq 1$ and $y \geq 1$ with $x = y$, then $\rho(fx, fy) = 0$ and (2.6) is trivially satisfied.
Finally, if $x < 1$ and $y \geq 1$ (or vice versa), then

$$\rho(fx, fy) = \frac{1}{y} \leq x + y \leq p\big\{\rho(x, fx) + \rho(y, fy)\big\} + \rho(x, y),$$

i.e., (2.6) holds.
Hence by Theorem 2.10, f has a fixed point. We see that $x = 0$ and $x = 1$ are the fixed points of f.

In [18], Öztürk and Başarir defined *property P* as follows.

Definition 2.12. *[18] Let Y be a cone metric space. A self mapping f on Y is said to have the property P if $F(f) = F\big(f^n\big)$ for all $n \in \mathbb{N}$.*

We give a characterization of this in b-metric space in terms of almost nonexpansive mapping.

Theorem 2.13. *Let (Y, ρ, s) be a complete b-metric space and $f : Y \longrightarrow Y$ be an almost nonexpansive mapping with $sp < 1$ which is asymptotically regular on Y. Then f has property P.*

Proof. Since $fu = u$ implies $f^n u = u$ for all $n \in \mathbb{N}$, it suffices to show that $F(f^n) \subseteq F(f)$. Let $u \in F(f^n)$. We note that f^n is asymptotically regular on Y. Also, since u is a fixed point of f^n, there exists a sequence $\{x_k\}$, $x_k = f^{nk}x_0$ for some $x_0 \in Y$ with $\lim_{k\to\infty} x_k = u$.

Now, since $q \geq 5$,

$$\varphi\big(\rho(u, fu)\big) \leq s\varphi\big(\rho(u, f^{nk}x_0)\big) + s^q\varphi\big(\rho(f^{nk}x_0, fu)\big).$$

If $Q(x, y) = \Big|\varphi\big(\rho(y, fx)\big) - \varphi\big(\rho(x, fy)\big)\Big|$, then

$$\varphi\big(\rho(u, fu)\big) \leq p\Big|\varphi\big(\rho(f^{nk}x_0, u)\big) - \varphi\big(\rho(f^{nk-1}x_0, fu)\big)\Big| + \varphi\big(\rho(f^{nk-1}x_0, u)\big) + s\varphi\big(\rho(u, f^{nk}x_0)\big)$$

Since this is true for all $k \in \mathbb{N}$, taking the limit as $k \to \infty$, using Lemma 1.5, we get,

$$(1 - sp)\varphi\big(\rho(u, fu)\big) \leq 0,$$

which implies $\varphi\big(\rho(u, fu)\big) = 0$.

This shows that $fu = u$, i.e., $u \in F(f)$, as required.

If $Q(x, y) = \varphi\big(\rho(x, fx)\big) + \varphi\big(f(y, fy)\big)$, then

$$\varphi\big(\rho(u, fu)\big) \leq p\Big\{\varphi\big(\rho(f^{nk-1}x_0, f^{nk}x_0)\big) + \varphi\big(\rho(u, fu)\big)\Big\} + \varphi\big(\rho(f^{nk-1}x_0, u)\big) + s\varphi\big(\rho(u, f^{nk}x_0)\big).$$

Since this is true for all $k \in \mathbb{N}$, taking the limit as $k \to \infty$, we get,

$$(1 - p)\varphi\big(\rho(u, fu)\big) \leq 0,$$

which implies $\varphi\big(\rho(u, fu)\big) = 0$. It proves that $fu = u$, or $u \in F(f)$, as required. □

Common fixed point

In the next two theorems, we establish some common fixed point results using almost nonexpansive property for a pair of mappings. Before that, we state the following terminology.

Two self mappings f and g on a b-metric space (Y, ρ, s) are *pairwise asymptotically regular at a point* $y_0 \in Y$ if

$$\lim_{n\to\infty} \rho\big((gf)^n y_0, f(gf)^n y_0\big) = 0 \quad \text{and} \quad \lim_{n\to\infty} \rho\big(f(gf)^n y_0, (gf)^{n+1}y_0\big) = 0.$$

For example, consider the mappings f and g defined on $\mathbb{R}$ (with the usual metric) by $fx = x$ and $gx = x^2$. Then it is easy to see that f and g are pairwise asymptotically regular at the point $x = 1$.

Theorem 2.14. *Let (Y, ρ, s) be a complete b-metric space with $s = 1$ or $s \geq 2$ and f, g be two self mappings on Y, pairwise asymptotically regular at a point $y_0 \in Y$ such that there exists a nonnegative real number $p < 1$ with $sp < 1$, satisfying*

$$s^q\varphi\big(\rho(fx, gy)\big) \leq p\Big|\varphi\big(\rho(fx, y) - \varphi\big(\rho(x, gy)\big)\Big| + \varphi\big(\rho(x, y)\big), \tag{2.7}$$

for all $x, y \in Y$ and for some $q \geq 5$. Then f and g have a common fixed point.

Proof. For $y_0 \in Y$, let $\{y_n\}$ be the sequence generated by the the iteration,

$$y_{2n-1} = fy_{2n-2}, \qquad \text{and} \qquad y_{2n} = gy_{2n-1}, \qquad n = 1, 2, 3, \dots .$$

We assume, without loss of generality, that $y_n \neq y_{n+1}$ for every nonnegative integer n. For, if $y_{n_0} = y_{n_0+1}$, then by our choice of the sequence $\{y_n\}$, $z = y_{n_0}$ is a common fixed point of f and g. To see this, we consider the following cases.

If n_0 is even, say, $n_0 = 2k$ for some nonnegative integer k, then we have, $y_{2k} = y_{2k+1} = fy_{2k}$, which shows that $z = y_{2k}$ is a fixed point of f.

Moreover, from (2.7), we get,

$$\varphi\big(\rho(fy_{2k}, gy_{2k+1})\big) \leq p\Big|0 - \varphi\big(\rho(y_{2k}, gy_{2k+1})\big)\Big| + 0 = p\varphi\big(\rho(fy_{2k}, gy_{2k+1})\big),$$

which implies $\rho(fy_{2k}, gy_{2k+1}) = \rho(y_{2k+1}, gy_{2k+1}) = 0$ and hence $y_{2k+1} = gy_{2k+1}$. Thus, $z = y_{2k} = y_{2k+1}$ is also a fixed point of g.

If n_0 is odd, we get analogous result and hence $z = y_{n_0}$ is a common fixed point of f and g if $y_{n_0} = y_{n_0+1}$ for some nonnegative integer n_0.

Thus we assume $y_n \neq y_{n+1}$ for all nonnegative integers n.

Since $y_n = f(gf)^{(n+1)/2}y_0$, if n is odd and $y_n = (gf)^{n/2}y_0$, if n is even, the pairwise asymptotic regularity of f and g implies

$$\lim_{n\to\infty} \rho(y_n, y_{n+1}) = 0.$$

As in the proof of Theorem 2.4, $\{y_n\}$ is a Cauchy sequence. The completeness of (Y, ρ, s) then ensures the existence of $z \in Y$ such that $\lim_{n\to\infty} y_n = z$.

Consequently,

$$\lim_{n\to\infty} fy_{2n} = z \qquad \text{and} \qquad \lim_{n\to\infty} gy_{2n+1} = z.$$

Now, from (2.7), we have,

$$\begin{aligned}\varphi\big(\rho(z, gz)\big) &\leq s\varphi\big(\rho(z, fy_{2n})\big) + s\varphi\big(\rho(fy_{2n}, gz)\big)\\ &\leq s\varphi\big(\rho(z, fy_{2n})\big) + p\Big|\varphi\big(\rho(fy_{2n}, z)\big) - \varphi\big(\rho(y_{2n}, gz)\big)\Big| + \varphi\big(\rho(y_{2n}, z)\big).\end{aligned}$$

In the limit as $n \to \infty$, using Lemma 1.5,

$$(1 - sp)\varphi\big(\rho(z, gz)\big) = 0 \quad \text{or,} \quad \rho(z, gz) = 0,$$

since $sp < 1$, that is, z is a fixed point of g.

Similarly, using (2.7) we have,

$$\begin{aligned}\varphi\big(\rho(fz, z)\big) &\leq s\varphi\big(\rho(fz, gy_{2n-1})\big) + s\varphi\big(\rho(gy_{2n-1}, z)\big)\\ &\leq p\Big|\varphi\big(\rho(fz, y_{2n-1})\big) - \varphi\big(\rho(z, gy_{2n-1})\big)\Big| + \varphi\big(\rho(y_{2n-1}, z)\big) + s\varphi\big(\rho(y_{2n}, z)\big).\end{aligned}$$

In the limit as $n \to \infty$, we get,

$$(1 - sp)\rho(fz, z) = 0 \qquad \text{or,} \qquad \rho(fz, z) = 0,$$

which proves that z is also a fixed point of f, as required. □

Theorem 2.15. *Let (Y, ρ, s) be a complete b-metric space and f, g be two self mappings on Y, pairwise asymptotically regular at a point $y_0 \in Y$ such that there exists a nonnegative real number $p < 1$ with $sp < 1$, satisfying*

$$s^q\varphi\big(\rho(fx, gy)\big) \leq p\Big\{\varphi\big(\rho(x, fx) + \varphi\big(\rho(y, gy)\big)\Big\} + \varphi\big(\rho(x, y)\big), \tag{2.8}$$

for all $x, y \in Y$ and for some $q \geq 5$. Then f and g have a common fixed point.

Proof. The proof is analogous to the proof of Theorem 2.14. □

Example 2.16. *Consider the b-metric space (Y,ρ,s) and the mapping f as defined in Example 2.11. Let $g: Y \longrightarrow Y$ be defined by*

$$gx = \begin{cases} 0, & x < 1; \\ \frac{1}{x^2} & x \geq 1. \end{cases}$$

We note here that f and g are pairwise asymptotically regular at each point $x \leq 1$. Let φ be the identity mapping.

Now, if $x < 1$ and $y < 1$, then since $\rho(fx, gy) = 0$, (2.8) is obviously satisfied.

And if $x \geq 1$ and $y \geq 1$, $x \neq y$, then

$$\rho(fx, gy) = \frac{1}{x} + \frac{1}{y^2} \leq x + y \leq p\Big\{\rho(x, fx) + \rho(y, gy)\Big\} + \rho(x, y).$$

If $x = y$ with $x \neq 1$ in the above case, then

$$2\rho(fx, gy) = 2\left(\frac{1}{x} + \frac{1}{x^2}\right) \leq \left\{x + \frac{1}{x} + x + \frac{1}{x^2}\right\} = \rho(x, fx) + \rho(x, gx),$$

or,

$$\rho(fx, gx) \leq p\big\{\rho(x, fx) + \rho(x, gx)\big\} + \rho(x, x),$$

for $p = \frac{1}{2}$.

Again, if $x < 1$ and $y \geq 1$, then

$$\rho(fx, gy) = \frac{1}{y} \leq x + y \leq p\Big\{\rho(x, fx) + \rho(y, gy)\Big\} + \rho(x, y),$$

and (2.8) holds.

Finally, if $x \geq 1$ and $y < 1$, then

$$\rho(fx, gy) = \rho\left(\frac{1}{x^2}, 0\right) = \frac{1}{x^2} \leq x + y \leq p\Big\{\rho(x, fx) + \rho(y, gy)\Big\} + \rho(x, y),$$

and (2.8) holds.

Thus, all the conditions of Theorem 2.15 are satisfied and hence f and g have a common fixed point. By inspection, $x = 0$ and $x = 1$ are the common fixed points of f and g here.

3. Application to the Cauchy problem

In this section, an application of Theorem 2.4 to the Cauchy problem in a Banach space is discussed following the method used in [19]. If Y is a Banach space and $I = [a, b] \subset \mathbb{R}$, then we know that the notion of Riemann integral and its related properties naturally extends from the case of real-valued functions to Y-valued functions on I. In particular, if $f \in C(I, Y)$ then f is Riemann integrable on I and

$$\left\| \int_a^b f(t)dt \right\|_Y \leq \int_a^b \|f(t)\|_Y dt.$$

The Cauchy problem [19]. Let Y be a Banach space, $U \subset \mathbb{R} \times Y$ be an open set and $u_0 = (t, x_0) \in U$. Let $f: U \longrightarrow Y$ be a continuous mapping. The problem involves finding a closed interval I with $t_0 \in I^o$ (here I^o is the interior of I) and a differentiable function $x: I \longrightarrow Y$ such that

$$\begin{cases} x'(t) &= f\big(t, x(t)\big), \quad t \in I; \\ x(t_0) &= x_0. \end{cases} \tag{3.1}$$

It can be easily seen that (3.1) is equivalent to the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f\big(\tau, x(\tau)\big) d\tau, \quad t \in I, \tag{3.2}$$

that is, x is a solution of (3.1) if and only if it is a solution of (3.2).

Theorem 3.1. *Consider the initial value problem (3.1) and suppose that f is continuous and*

(i) for all $(t,x),(t,y) \in U$, f satisfies the Lipschitz condition,

$$\big|f\big(t, x(t)\big) - f\big(t, y(t)\big)\big| \leq |x(t) - y(t)|\,; \tag{3.3}$$

(ii) there exists $\overline{B}_{\mathbb{R}\times Y}(u_0,\gamma) \subset U$ such that for all $(t,x) \in \overline{B}_{\mathbb{R}\times Y}(u_0,\gamma)$

$$\|f(t,x)\|_Y \leq m\ ,$$

for some $m \geq 0$.

Then there exists $\alpha_0 > 0$ such that for all $\alpha < \alpha_0$ the Cauchy problem (3.1) has a solution in the interval I_α, where $I_\alpha = [t_0 - \alpha, t_0 + \alpha]$.

Proof. Let $r = \min\{1,\gamma\}$ and $\alpha_0 = \min\big\{r, \frac{r}{m}\big\}$ and $\alpha < \alpha_0$. Consider the complete metric space $Z = \overline{B}_{C(I_\alpha,Y)}\big(x_0, r\big)$ with the metric induced by the norm of $C(I_\alpha, Y)$ and x_0 is the constant function with value x_0. Since $\alpha < r$, if $z \in Z$ then $\big(t, z(t)\big) \in \overline{B}_{\mathbb{R}\times Y}\big(u_0, r\big) \subset U$ for all $t \in I_\alpha$.

For $z \in Z$, define

$$Tz(t) = x_0 + \int_{t_0}^{t} f\big(\tau, z(\tau)\big) d\tau.$$

We see that

$$\sup_{t\in I_\alpha} \big\|Tz(t) - x_0\big\|_X \leq \sup_{t\in I_\alpha} \left| \int_{t_0}^{t} \|f\big(\tau, z(\tau)\big)\|_Y d\tau \right| \leq m\alpha \leq r,$$

so that T is a self mapping on Z.

Now,

$$\begin{aligned}
\big\|Tx(t) - Ty(t)\big\|_Z &\leq \left\| \int_{t_0}^{t} \Big(f\big(\tau, x(\tau)\big) - f\big(\tau, y(\tau)\big) \Big) d\tau \right\|_Y \\
&\leq \int_{t_0}^{t} \big\|f\big(\tau, x(\tau)\big) - f\big(\tau, y(\tau)\big)\big\|_Y d\tau \\
&\leq \int_{t_0}^{t} \big\|x(t) - y(t)\big\|_Y d\tau \\
&\leq \sup_{t\in I_\alpha} \big\|x(t) - y(t)\big\|_Z \int_{t_0}^{t} d\tau \\
&\leq \sup_{t\in I_\alpha} \big\|x(t) - y(t)\big\|_Z \big|t - t_0\big| \\
&\leq \alpha \sup_{t\in I_\alpha} \big\|x(t) - y(t)\big\|_Z \\
&\leq \sup_{t\in I_\alpha} \big\|x(t) - y(t)\big\|_Z.
\end{aligned}$$

Since this relation holds true for all $x, y \in C\big(I_\alpha, Y\big)$ and all $t \in I_\alpha$, we conclude that

$$\rho\big(Tx, Ty\big) \le \rho(x, y) \le p\big|\rho(Tx, y) - \rho(x, Ty)\big| + \rho(x, y).$$

Moreover, by the definition of T, it is clear that T is asymptotically regular on Y.

Taking $\varphi(x) = x$, we see that all the conditions of Theorem 2.4 are satisfied. Hence T has a fixed point, that is, (3.1) has a solution in the interval I. □

4. Conclusion

Fixed point existence results for almost nonexpansive mappings, asymptotically regular at a point are discussed in the setting of a b-metric space and generalizations of the same to the existence of common fixed point is derived in this paper. The fixed point existence result for almost nonexpansive mappings asymptotically regular at a point satisfying (2.1) is obtained for $s = 1$ or $s \ge 2$. The case $s \in (0, 1)$ may be investigated further.

There are several interesting discussion on fixed point theory considering different spaces with different practical applications. In [24], the authors discussed some fixed point theorems in generalized fuzzy metric spaces. In this context, the applicability of our results in case of generalized fuzzy metric spaces is a possible scope of further study.

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