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Convergence Theorem for Split Feasibility Problem, Equilibrium Problem and Zeroes of Sum of Monotone Operators *

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ABSTRACT: The main purpose of this paper is to introduce a parallel iterative algorithm for approximating the solution of a split feasibility problem on the zero of monotone operators, generalized mixed equilibrium problem and fixed point problem. Using our algorithm, we state and prove a strong convergence theorem for approximating a common element in the set of solutions of a problem of finding zeroes of sum of two monotone operators, generalized mixed equilibrium problem and fixed point problem for a finite family of η-demimetric mappings in the frame work of a reflexive, strictly convex and smooth Banach spaces. We also give a numerical experiment applying our main result. Our result improves, extends and unifies other results in this direction in the literature.

Key Words: Split generalized mixed equilibrium problem, monotone mapping, strong convergence, Banach space, quasi-ϕ-nonexpansive mapping, parallel algorithm.

Contents

1. Introduction

The problem of finding a point in the nonempty intersection of convex sets formulated as $x \in H$ such that

$$x \in \cap_{i=1}^{M} C_i \neq \emptyset,$$

where C_i, $i = 1, \cdots, M$ are nonempty, closed and convex subsets of a Hilbert space H is referred to as the Convex Feasibility Problem (CFP). There are numerous areas of applications of the (CFP) in many applied disciplines such as applied mathematics, engineering, approximation theory, image recovery, signal processing. It also finds applications in control theory, biomedical engineering, communications and geophysics and so on (see [9] and the references cited therein).

On the other hand, let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, the Split Feasibility Problem (SFP) consists of finding a point $x \in C$ such that $Sx \in Q$ where $S : H_1 \to H_2$ is a bounded linear operator. Observe that, by defining, $S^{-1}(Q) = \{x : Sx \in Q\}$, then the SFP reduces to the CFP which is to find $x \in C \cap S^{-1}(Q)$. Despite this close relations, the methodologies employed for finding the solution of each of the problems are different, (see [17,36,35,45] and the references cited therein).

Furthermore, there are some generalizations of the CFP which can be formulated in various forms; finding a common fixed point of nonexpansive mappings, finding a common minimum of convex functionals, finding a common zero of maximal monotone operators, solving variational inequalities, solving a system of equilibrium problems. For surveys of methods for solving such problems, (see [3,13,25,26,27,28,29,30, 31,32] and the references cited therein).

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The monotone operator theory closely related to the fixed point theory has appeared as an effective and powerful tool for studying a wide range of problems arising in different branches of knowledge, from social, engineering to pure sciences in a unified and general framework. In recent years, monotone operators have received a lot of attention by many authors in the area of approximating the zero points of monotone operators and its relation to finding fixed point of mappings which have Lipschitz uniform continuity, (see [15,24]). There are various number of applications of the problem of finding zero points of the sum of two operators; (see [22,23] and the references cited therein). For more on monotone operators and generalized sums of two monotone operators, (see [22,23] and the references cited therein).
Recently, some authors have considered the problems of obtaining common solutions of inclusion problems and fixed point problems in the framework of Hilbert spaces; (see [46] and the references cited therein). Very recently, Petrot et al. [33], presented an alternative algorithm for finding a solution of split feasibility problem for a point in zeros of finite sum of α_i-inverse strongly monotone operators and maximal monotone operators and fixed points of nonexpansive mappings. They proposed two parallel type algorithm for approximating the solution of this problem in the framework of two real Hilbert spaces. However, in obtaining strong convergence of their algorithm, Petrot et al. [33] imposed the condition of compactness on the mapping in the first algorithm. They incorporated the hybrid step into the proposed parallel algorithm and proved strong convergence theorems based on this algorithms.
In this paper, inspired and motivated by the research going on in this direction, most especially what has preceded above in the literature, we introduce a parallel iterative algorithm for finding a solution of split feasibility problem for a generalized mixed equilibrium problem, a point in zeroes of a finite sum of an α-inverse strongly operator and maximal monotone operator and a common fixed point of a finite family of η-demimetric mappings. Our algorithm does not require the compactness assumption on the underlining space or any of the demimetric mappings. Also, our proposed method does not require a projection onto a convex set for obtaining its strong convergence.

2. Preliminaries

In this section, we give some definitions and important results which will be useful in establishing our main results. We denote the weak and the strong convergence of a sequence $\{x_n\}$ to a point x by $x_n \rightharpoonup x$ and $x_n \to x$ respectively.
Throughout this paper, we suppose C is a nonempty, closed and convex subset of a real Banach space E with norm $||\cdot||$. The normalized duality mapping $J : E \to 2^{E^*}$ is defined by

$$J(x) = \{x^* \in E^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2, \ \forall x \in E\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot\rangle$ the duality pairing between the elements of E and E^*. By considering the Lyapunov functional $\phi : E \times E \to \mathbb{R}^+$ which is defined by

$$\phi(x, y) = ||x||^2 - 2\langle x, Jy\rangle + ||y||^2, \ \forall x, y \in E,$$

Alber [7], introduced a generalized projection operator $\Pi_C : E \to C$ given by

$$\Pi_C(x) = \inf_{y\in C}\{\phi(y, x), \ \forall x \in E\}.$$

Note that, for the Hilbert spaces, we observe that $\phi(x, y) = ||x - y||^2$ and $\Pi_C(x) \equiv P_C(x)$, where $P_C : H \to C$ is the usual metric projection. It is obvious from the definition of the functional ϕ that

$$(||x|| - ||y||)^2 \leq \phi(x, y) \leq (||x|| + ||y||)^2.$$

Moreover, the functional ϕ also satisfy the following important properties:

(N_1) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy\rangle$;

(N_2) $\phi(x, y) + \phi(y, x) = 2\langle x - y, Jx - Jy\rangle$;

(N_3) $\phi(x, y) = ||x||||Jx - Jy|| + ||y||||x - y||$.

In this work, we are also concerned with the functional $V : E \times E^* \to \mathbb{R}$ which is defined by

$$V(x, x^*) = ||x||^2 - 2\langle x, x^*\rangle + ||x^*||^2 \tag{2.1}$$

for all $x \in E$ and $x^* \in E^*$. Observe that, if E is a reflexive, strictly convex and smooth Banach space, we have $V(x, x^*) = \phi(x, J^{-1}x^*)$, and

$$V(x, x^*) \le V(x, x^* + y^*) - 2\langle J^{-1}x^* - x, y^*\rangle \tag{2.2}$$

for all $x \in E$ and all $x^*, y^* \in E^*$, see [42].
A point $x \in C$ is called a fixed point of a mapping $T : C \to C$, if $x = Tx$. We denote the set of fixed points of T by $F(T)$. A point $p \in C$ is called an asymptotic fixed point of T, if C contains a sequence $\{x^k\}$ such that $x^k \rightharpoonup p$ and $||x^k - Tx^k|| \to 0$ as $k \to \infty$ (see [34]). We denote by $\widehat{F(T)}$ the set of asymptotic fixed points of T.
A mapping $T : C \to C$ is said to be:

(a) relatively nonexpansive if $\widehat{F(T)} = F(T)$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$ (see [11,12]);

(b) ϕ-nonexpansive if $\phi(Tx, Ty) \le \phi(x, y)$ for all $x, y \in C$ and quasi-ϕ-nonexpansive if $F(T) \neq \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$;

(c) firmly nonexpansive type mapping if, for all $x, y \in C$,

$$\phi(Tx, Ty) + \phi(Ty, Tx) \le \phi(Tx, y) + \phi(Ty, x) - \phi(Tx, x) - \phi(Ty, y)$$

or equivalently

$$\langle Tx - Ty, Jx - JTx - (Jy - JTy)\rangle \ge 0, \tag{2.3}$$

where ϕ is the Lyapunov functional.

It is known that the class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mapping which requires the strict condition $F(T) = \widehat{F(T)}$, see ([11,12]).
Let C be a nonempty, closed and convex subset of a Banach space E and let η be a real number with $\eta \in (-\infty, 1)$. Then a mapping $T : C \to E$ with $F(T) \neq \emptyset$ is called η-demimetric [39], if

$$\langle x - p, j(x - Tx)\rangle \ge \frac{(1-\eta)}{2}||x - Tx||^2, \tag{2.4}$$

for all $x \in C$ and $p \in F(T)$. It has been shown in [39] that $F(T)$ is closed and convex. It is known that the class of quasi-ϕ-nonexpansive is 0-demimetric. For more on demimetric mappings see [20,39] and the references therein.
Recall that an operator $A : C \to E^*$ is α-inverse strongly monotone, where $\alpha > 0$, if

$$\langle x - y, Ax - Ay\rangle \ge \alpha||Ax - Ay||^2, \tag{2.5}$$

for all $x, y \in C$. A mapping $T : E \to E$ is said to be L-Lipschitz continuous if $||Tx - Ty|| \le L||x - y||$, for all $x, y \in E$. It is known that the class of α-inverse strongly monotone operator is $\frac{1}{\alpha}$-Lipschitz continuous.
A multivalued mapping $A : E \to 2^{E^*}$ is said to be monotone if $\langle u - v, x - y\rangle \ge 0$, for all $x, y \in E$, $u \in Ax$ and $v \in Ay$. A monotone operator is said to be maximal if the graph of A is not properly contained in the graph of any other monotone operator in the same space. We denote by $A^{-1}(0)$ the set of zeros of A, that is $A^{-1}(0) = \{x \in E : 0 \in Ax\}$. It is well known that, if A is maximal monotone then the solution set $A^{-1}(0)$ is closed and convex. Let $A : E \to 2^{E^*}$ be a maximal monotone operator, then for each $r > 0$ and $x \in E$, there exists a unique element $x_r \in D(T)$ satisfying

$$J(x) \in J(x_r) + rA(x_r);$$

(see [40]). For each $r > 0$, define the resolvent operator of A by $Q_r^A x$. In other words, $Q_r^A = (J + rA)^{-1} J$ for all $r > 0$. It is easy show that $A^{-1}(0) = F(Q_r^A)$ for all $r > 0$, where $F(Q_r^A)$ denotes the set of all fixed points of Q_r^A.

Remark: If we put $G_r^{A,B} = Q_r^A(J^{-1}(J - rB)) = (J + rA)^{-1} J[J^{-1}(J - rB)]$, where r is a real number. Then,

$$\begin{aligned} x &= G_r^{A,B} x \\ &\iff x \in (J + rA)^{-1} J[J^{-1}(Jx - rBx)] = (J + rA)^{-1}(Jx - rBx) \\ &\iff (Jx - rBx) \in (Jx + rAx) \\ &\iff 0 \in Ax + Bx. \end{aligned} \tag{2.6}$$

This means that, $F(G_r) = (A + B)^{-1}(0)$.

Now, we recall some geometric properties on the Banach space, see [37,38].
For a real Banach space E. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{1}{2}||x + y|| : ||x|| = ||y|| = 1, ||x - y|| \geq \epsilon\}. \tag{2.7}$$

Recall that E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$. E is said to be strictly convex if $\frac{||x + y||}{2} < 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$. Also, E is p-uniformly convex if there exists a constant $c_p > 0$ such that $\delta_E(\epsilon) > c_p \epsilon^p$ for any $\epsilon \in (0, 2]$.

The modulus of smoothness of E is the function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(||x + ty|| - ||x - ty||) - 1 : ||x|| = ||y|| = 1\}. \tag{2.8}$$

E is said to be uniformly smooth if $\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0$. Let $1 < q \leq 2$, then E is q-uniformly smooth if there exists $c_q > 0$ such that $\rho_E(t) \leq c_q t^q$ for $t > 0$. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth, where $p^{-1} + q^{-1} = 1$. It is also known that every q-uniformly smooth Banach space is uniformly smooth.
It is widely known that if E is uniformly smooth, then the duality mapping J is norm-to-norm continuous on each bounded subset of E. The following are some important and useful properties of duality mapping J, for further details, see [1]:

- For every $x \in E$, Jx is nonempty, closed, convex and bounded subset of E^*.
- If E is smooth or E^* is strictly convex, then J is single valued. Also, If E is reflexive, then J is onto.
- If E is strictly convex, then J is strictly monotone, that is

$$\langle x - y, Jx - Jy \rangle > 0, \quad \forall x, y \in E.$$

- If E is smooth, strictly convex and reflexive and $J^* : E^* \to 2^E$ is the normalized duality mapping on E^*, then $J^{-1} = J^*$, $JJ^* = I_{E^*}$ and $J^* J = I_E$, where I_E and I_{E^*} are the identity mappings on E and E^* respectively.
- If E is uniformly convex and uniformly smooth, then J is uniformly norm-to-norm continuous on bounded subsets of E and $J^* = J^{-1}$ is also uniformly norm-to-norm continuous on bounded subsets of E^*.

We now give the following useful and important lemmas that are needed in establishing our main results:

Lemma 2.1. *[44] Given a number $s > 0$. A real Banach space X is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$||\lambda x + (1-\lambda)y||^2 \leq \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)g(||x-y||),$$

for all $x, y \in X$, $\lambda \in [0,1]$, with $||x|| < s$ and $||y|| < s$.

Lemma 2.2. *[19] Let E be a smooth and uniformly convex real Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $\phi(x_n, y_n) \to 0$ as $n \to \infty$, then $||x_n - y_n|| \to 0$ as $n \to \infty$.*

Remark 2.3. *Using N_3, it is easy to see that the converse of Lemma 2.2 is also true whenever $\{x_n\}$ and $\{y_n\}$ are both bounded.*

Lemma 2.4. *[7] Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth Banach space E. If $x \in E$ and $q \in C$, then*

$$q = \Pi_C x \iff \langle y - q, Jx - Jq\rangle \leq 0, \ \forall y \in C \tag{2.9}$$

and

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \ \forall y \in C, \ x \in E. \tag{2.10}$$

Lemma 2.5. *[44] Let E be a 2-uniformly smooth Banach space with the best smoothness constant $k > 0$. Then, the following inequality holds:*

$$||x + y||^2 \leq ||x||^2 + 2\langle y, Jx\rangle + 2||ky||^2, \ \forall x, y \in E.$$

Lemma 2.6. *[8] Suppose that E is 2-uniformly convex Banach space. Then, there exists a constant $c \geq 1$ such that*

$$\phi(x, y) \geq \frac{1}{c}||x - y||^2, \ \forall x, y \in E.$$

Lemma 2.7. *[43] Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\delta_n, \ n \geq 0,$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfy the conditions $\sum\limits_{n=0} \alpha_n = \infty$ and $\limsup\limits_{n\to\infty} \delta_n \leq 0$. Then, $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.8. *[21] Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing subsequence $\{m_n\} \subset \mathbb{N}$ such that $m_n \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $n \in \mathbb{N}$: $a_{m_n} < a_{m_n+1}$ and $a_n < a_{m_n+1}$. In fact, $m_n = \max\{i \leq k : a_i < a_{i+1}\}$.*

We end this section by recalling some knowledge on the concept of equilibrium problem.
Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction, $\varphi : C \to \mathbb{R}$ be a real-valued function and $\Psi : C \to E^*$ be a nonlinear mapping, where C is a nonempty, closed and convex subset of a real Banach space E with E^* its dual. We consider the following Generalized Mixed Equilibrium Problem (GMEP):
Find $x \in C$ such that

$$\Theta(x, y) + \langle y - x, \Psi x\rangle + \varphi(y) \geq \varphi(x), \ \forall \ y \in C. \tag{2.11}$$

The set of solutions of (2.11) is denoted by $GMEP(\Theta, \Psi, \varphi)$. In the case when $\Psi = 0$, problem (2.11) reduces to the Mixed Equilibrium Problem (MEP) with solution set $MEP(\Theta, \varphi)$. In the case $\varphi = 0$, then (2.11) reduces to Generalized Equilibrium Problem (GEP) with solution set $GEP(\Theta, \Psi)$. Note that, if $\Psi = \varphi = 0$, then problem (2.11) becomes the classical equilibrium introduced by Blum and Oetlli [10]. This means that the problem (2.11) is very general in the sense that it include as special cases, the optimization problem, variational inequalities, min-max problems, the Nash equilibrium in non co-operative games and so on (see [2,4,14,16,18]).
For solving the GMEP, we will assume the bifunction Θ satisfies the following:
Assumption I: The bifunction $\Theta : C \times C \to \mathbb{R}$ satisfies the following conditions (see [5,**?**]:

(I_1) $\Theta(x,x) = 0$ for all $x \in C$;

(I_2) Θ is monotone, i.e, $g(x,y) + g(y,x) \leq 0$ for all $x, y \in C$;

(I_3) $\limsup_{t\downarrow 0} g(x + t(z-x), y) \leq g(x,y),\ \forall x, y, z \in C$;

(I_4) the function $y \mapsto \Theta(x,y)$ is convex and lower semi-continuous.

The following Lemma is useful in our work.

Lemma 2.9. *[41] Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty, closed and convex subset of E. Let $\Psi : C \to E^*$ be a continuous and monotone mapping, $\varphi : C \to \mathbb{R}$ be a lower semi-continuous and convex function and $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying the Assumption I. Let $s > 0$ be any given number and $x \in E$ be any given point. Then, the following hold:*

(i) there exists $z \in C$, such that

$$\Theta(z,y) + \langle y - z, \Psi z\rangle + \varphi(y) + \frac{1}{s}\langle y - z, Jz - Jx\rangle \geq \varphi(z),\ \forall\ y \in Q;$$

(ii) the mapping $K_s^\Theta : E \to C$ defined by

$$K_s^\Theta(x) = \{z \in C : \Theta(z,y) + \langle y - z, \Psi z\rangle + \varphi(y) + \frac{1}{s}\langle z - y, Jz - Jx\rangle \geq \varphi(z),\ \forall y \in C\}, \forall x \in E,$$

has the following properties:

(a) for all $x \in E$, $K_s^\Theta(x) \neq \emptyset$.

(b) K_s^Θ is single valued

(c) K_s^Θ is firmly nonexpansive-type, i.e

$$\langle K_s^\Theta z - K_s^\Theta y, JK_s^\Theta z - JK_s^\Theta y\rangle \leq \langle K_s^\Theta z - K_s^\Theta y, Jz - Jy\rangle,\ \forall\ z, y \in E;$$

(d) $F(K_s^\Theta) = GMEP(\Theta, \Psi, \varphi)$;

(e) $F(K_s^\Theta)$ is closed and convex;

(f) $\phi(p, K_s^\Theta z) + \phi(K_s^\Theta z, z) \leq \phi(p, z),\ \forall p \in F(K_s^\Theta),\ z \in E$.

3. Main Results

In this section we state and prove our main results. Firstly, we explicitly state the problem considered in this paper, we introduce a parallel iterative method for obtaining the solution of the problem and finally discuss its convergence analysis.

Let C be a nonempty, closed and convex subset of a 2-uniformly convex, and uniformly smooth Banach space E_1, Q be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E_2 and $S : E_1 \to E_2$ be a bounded linear operator with $S^* : E_2^* \to E_1^*$ its adjoint. For each $j = 1, \cdots M$, let $T_j : E_2 \to E_2$ be a finite family of η_j-demimetric mappings. For each $i = 1, \cdots, N$, let $A_i : E_1 \to 2^{E_1^*}$ be a finite family of maximal monotone operators and $B_i : E_1 \to E_1^*$ be a finite family of α_i-inverse strongly monotone operators. Let $\Theta : C \times C \to \mathbb{R}$ be a monotone bifunction satisfying Assumption I, let $\Psi : C \to E_1^*$ be a continuous and monotone mapping and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous and convex function. From now $\frac{1}{c}$, where $c \in (0, 1]$, and k represent the uniformly convex constant and smoothness constant of E_1 respectively.

We consider the problem of finding a point $p \in C$ such that

$$p \in GMEP(\Theta, \Psi, \varphi) \cap (\cap_{i=1}^N (A_i + B_i)^{-1}(0)) \cap S^{-1}(\cap_{j=1}^M F(T_j)). \tag{3.1}$$

We denote the set of solution of (3.1) by Γ and assume $\Gamma \neq \emptyset$. To solve (3.1), we introduce the following parallel iterative algorithm: Choose $u, x_0 \in C$, define $\{x_n\}$ by the following process:

$$\begin{cases} y_{j,n} = J_1^{-1}(J_1 x_n - \mu_n S^* J_2(I - T_j)Sx_n), \\ \text{choose } j_n : \ ||y_{j_n,n} - x_n|| = \max\{||y_{j,n} - x_n|| : j = 1, \cdots, M\}, \\ \text{set } y_{j_n,n} = y_n, \\ z_{i,n} = Q_r^{A_i} \circ J_1^{-1}(J_1 y_n - rB_i y_n), \\ \text{choose } i_n : ||z_{i_n,n} - x_n|| = \max\{||z_{i,n} - x_n|| : i = 1, \cdots, N\}, \\ \text{set } z_{i_n,n} = z_n, \\ v_{i,n} = z_n - r(B_i z_n - B_i y_n), \\ \text{choose } i_n : ||v_{i_n,n} - x_n|| = \max\{||v_{i,n} - x_n|| : i = 1, \cdots, N\}, \\ \text{set } v_{i_n,n} = v_n, \\ \Theta(w_n, y) + \langle y - w_n, \Psi w_n \rangle + \varphi(y) - \varphi(w_n) + \dfrac{1}{s}\langle y - w_n, J_1 w_n - J_1 v_n \rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = J_1^{-1}(\beta_n J_1 u + (1 - \beta_n) J_1 w_n), \ n \geq 0, \end{cases} \tag{3.2}$$

where $s > 0$, is a positive parameter, $\{\mu_n\}$ is a sequence of positive real numbers and $\{\beta_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(i) $\lim\limits_{n\to\infty} \beta_n = 0$, $\sum\limits_{n=0}^{\infty} \beta_n = \infty$;

(ii) $0 \leq \mu_n \leq \dfrac{1-\eta}{2k^2||S||^2}$, where $\eta = \max\limits_{1\leq j\leq M}\{\eta_j\}$;

(iii) $0 < a \leq r \leq b < \dfrac{\alpha}{k\sqrt{2c}}$, where $\alpha = \min\limits_{1\leq i\leq N}\{\alpha_i\}$.

Firstly, we prove the following lemma which is important for establishing the zero of the sum $(A_i + B_i)$.

Lemma 3.1. *Define the sequences $\{y_n\}$ and $\{z_{i,n}\}$ as in (3.2). Choose subsequences $\{y_{n_k}\}$ of $\{y_n\}$ and $\{z_{i,n_k}\}$ of $\{z_{i,n}\}$, such that $y_{n_k} \rightharpoonup p$ and $||y_{n_k} - z_{i,n_k}|| \to 0$ as $n \to \infty$. Then, $p \in (A_i + B_i)^{-1}(0)$.*

Proof. Suppose that $(v, u) \in Grp(A_i + B_i)$. Then we have $J_1 u - B_i v \in A_i v$. Furthermore, we obtain from $z_{i,n_k} = (J_1 + rA_i)^{-1} J_1 \circ J_1^{-1}(J_1 y_{n_k} - rB_i y_{n_k})$ that $(J_1 y_{n_k} - rB_i y_{n_k}) \in (J_1 + rA_i)z_{i,n_k}$, and thus

$$\frac{1}{r}(J_1 y_{n_k} - J_1 z_{i,n_k} - rB_i y_{n_k}) \in A_i z_{i,n_k}.$$

Since A_i is maximal monotone and $(v, u) \in Grp(A_i + B_i)$ for each $i = 1, 2 \cdots, N$, we obtain

$$\langle v - z_{i,n_k}, J_1 u - B_i v - \frac{1}{r}(J_1 y_{n_k} - J_1 z_{i,n_k} - rB_i y_{n_k})\rangle \geq 0.$$

Therefore,

$$\begin{aligned} \langle v - z_{i,n_k}, J_1 u \rangle &\geq \langle v - z_{i,n_k}, B_i v + \frac{1}{r}(J_1 y_{n_k} - J_1 z_{i,n_k} - rB_i y_{n_k})\rangle \\ &= \langle v - z_{i,n_k}, B_i v - B_i y_{n_k}\rangle + \langle v - z_{i,n_k}, \frac{1}{r}(J_1 y_{n_k} - J_1 z_{i,n_k})\rangle \\ &= \langle v - z_{i,n_k}, B_i v - B_i z_{i,n_k}\rangle + \langle v - z_{i,n_k}, B_i z_{i,n_k} - B_i y_{n_k}\rangle + \langle v - z_{i,n_k}, \frac{1}{r}(J_1 y_{n_k} - J_1 z_{i,n_k})\rangle \\ &\geq \langle v - z_{i,n_k}, B_i z_{i,n_k} - B_i y_{n_k}\rangle + \langle v - z_{i,n_k}, \frac{1}{r}(J_1 y_{n_k} - J_1 z_{i,n_k})\rangle. \end{aligned}$$

Since $||z_{i,n_k} - y_{n_k}|| \to 0$ as $n \to \infty$ and B_i is Lipschitz continuous for each i, we obtain $||B_i z_{i,n_k} - B_i y_{n_k}|| \to 0$ as $n \to \infty$. Consequently, $\langle v - p, J_1 u\rangle \geq 0$. By the maximal monotonicity of $(A_i + B_i)$ for each $i = 1, 2, \cdots N$, we have $0 \in (A_i + B_i)p$, for each i. Hence $p \in \cap_{i=1}^{N}(A_i + B_i)^{-1}(0)$. □

Next, we prove a lemma which guarantees the existence of weak cluster points of the sequence. That is, we show that $\{x_n\}$ is bounded.

Lemma 3.2. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E_1 and Q is a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E_2. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption I, $\Psi : C \to E_1^*$ be a continuous and monotone mapping and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. For each $i = 1, \cdots N$, let $A_i : E_1 \to 2^{E_1^*}$ be a finite family of maximal monotone operators and $B_i : E_1 \to E_1^*$ be a finite family of α_i-inverse strongly monotone operators, and let $T_j : E_2 \to E_2$ be a finite family of η-demimetric and demiclosed mappings. If $\Gamma \neq \emptyset$ then, the sequence $\{x_n\}$ generated by (3.2) is bounded.*

Proof. Fix $p \in \Gamma$, we have from (3.2), that

$$\begin{aligned}
\phi(p, y_n) &= \phi(p, J_1^{-1}(J_1 x_n - \mu_n S^* J_2(I - T_{j_n})Sx_n)) \\
&= ||p||^2 - 2\langle p, J_1 x_n - \mu_n S^* J_2(I - T_{j_n})Sx_n\rangle + ||J_1 x_n - \mu_n S^* J_2(I - T_{j_n})Sx_n||^2 \\
&= ||p||^2 - 2\langle p, J_1 x_n\rangle + 2\mu_n\langle p, S^* J_2(I - T_{j_n})Sx_n\rangle + ||x_n||^2 - 2\mu_n\langle x_n, S^* J_2(I - T_{j_n})Sx_n\rangle \\
&\quad +2\mu_n^2 k^2 ||S||^2 ||(I - T_{j_n})Sx_n||^2 \\
&= \phi(p, x_n) - 2\mu_n\langle Sx_n - Sp, J_2(I - T_{j_n})Sx_n\rangle + 2\mu_n^2 k^2 ||S||^2 ||(I - T_{j_n})Sx_n||^2 \\
&\leq \phi(p, x_n) - \mu_n(1 - \eta)||(I - T_{j_n})Sx_n||^2 + 2\mu_n^2 k^2 ||S||^2 ||(I - T_{j_n})Sx_n||^2 \\
&= \phi(p, x_n) - \mu_n[(1 - \eta) - 2\mu_n ||S||^2 k^2]||(I - T_{j_n})Sx_n||^2. \qquad (3.3)
\end{aligned}$$

Since $0 < \mu_n \leq \dfrac{1 - \eta}{2k^2||S||^2}$, we have

$$\phi(p, y_n) \leq \phi(p, x_n). \qquad (3.4)$$

Again from (3.2), we have

$$\begin{aligned}
\phi(p, v_n) &= \phi(p, J_1^{-1}(J_1 z_n - r(B_{i_n} z_n - B_{i_n} y_n))) \\
&= ||p||^2 - 2\langle p, J_1 z_n - r(B_{i_n} z_n - B_{i_n} y_n)\rangle + ||J_1^{-1}(J_1 z_n - r(B_{i_n} z_n - B_{i_n} y_n))||^2 \\
&= ||p||^2 - 2\langle p, J_1 z_n - r(B_{i_n} z_n - B_{i_n} y_n)\rangle + ||J_1 z_n - r(B_{i_n} z_n - B_{i_n} y_n)||^2 \\
&= ||p||^2 - 2\langle p, J_1 z_n\rangle + 2r\langle p, B_{i_n} z_n - B_{i_n} y_n\rangle + ||J_1 z_n - r(B_{i_n} z_n - B_{i_n} y_n)||^2. \qquad (3.5)
\end{aligned}$$

By using Lemma 2.5, we have

$$||J_1 z_n - r(B_{i_n} z_n - B_{i_n} y_n)||^2 \quad \leq \quad ||J_1 z_n||^2 - 2r\langle z_n, B_{i_n} z_n - B_{i_n} y_n\rangle + 2k^2||r(B_{i_n} z_n - B_{i_n} y_n)||^2.$$

Substituting this into (3.5), we obtain

$$\begin{aligned}
\phi(p, v_n) &= ||p||^2 - 2\langle p, J_1 z_n\rangle + ||z_n||^2 - 2r\langle z_n, B_{i_n} z_n - B_{i_n} y_n\rangle \\
&\quad +2r\langle p, B_{i_n} z_n - B_{i_n} y_n\rangle + 2k^2||r(B_{i_n} z_n - B_{i_n} y_n)||^2 \\
&= \phi(p, y_n) + \phi(y_n, z_n) + 2\langle y_n - p, J_1 z_n - J_1 y_n\rangle - 2r\langle z_n - p, B_{i_n} z_n - B_{i_n} y_n\rangle \\
&\quad +2k^2||r(B_{i_n} z_n - B_{i_n} y_n)||^2. \qquad (3.6)
\end{aligned}$$

By N_2, we have

$$\phi(y_n, z_n) = -\phi(z_n, y_n) + 2\langle z_n - y_n, J_1 z_n - J_1 y_n\rangle$$

Substituting this into (3.6), we get

$$\begin{aligned}
\phi(p, v_n) &= \phi(p, y_n) - \phi(z_n, y_n) + 2\langle z_n - y_n, J_1 z_n - J_1 y_n\rangle + 2\langle y_n - p, J_1 z_n - J_1 y_n\rangle \\
&\quad -2r\langle z_n - p, B_{i_n} z_n - B_{i_n} y_n\rangle + 2k^2||r(B_{i_n} z_n - B_{i_n} y_n)||^2 \\
&= \phi(p, y_n) - \phi(z_n, y_n) + 2\langle z_n - p, J_1 z_n - J_1 y_n\rangle - 2r\langle z_n - p, B_{i_n} z_n - B_{i_n} y_n\rangle \\
&\quad +2k^2||r(B_{i_n} z_n - B_{i_n} y_n)||^2 \\
&= \phi(p, y_n) - \phi(z_n, y_n) - 2\langle z_n - p, J_1 y_n - J_1 z_n - r(B_{i_n} y_n - B_{i_n} z_n)\rangle \\
&\quad +2k^2||r(B_{i_n} z_n - B_{i_n} y_n)||. \qquad (3.7)
\end{aligned}$$

Since $z_n = (J_1 + rA_{i_n})^{-1}J_1 \circ J_1^{-1}(J_1y_n - rB_{i_n}y_n)$, we have $J_1y_n - rB_{i_n}y_n \in (J_1 + rA_{i_n})z_n$. Using the fact A_{i_n} is maximal monotone for each i, there exists $u_n \in A_{i_n}z_n$ such that $J_1y_n - rB_{i_n}y_n = J_1z_n + ru_n$. Therefore,

$$u_n = \frac{1}{r}(J_1y_n - J_1z_n - rB_{i_n}y_n) \tag{3.8}$$

On the other hand, we have $0 \in (A_{i_n} + B_{i_n})p$ and $u_n + B_{i_n}z_n \in (A_{i_n} + B_{i_n})z_n$. By the maximal monotonicity of $(A_{i_n} + B_{i_n})$, we obtain

$$\langle u_n + B_{i_n}z_n, z_n - p\rangle \geq 0.$$

Using (3.8) in the last inequality, we get

$$\langle \frac{1}{r}(J_1y_n - J_1z_n - rB_{i_n}y_n) + B_{i_n}z_n, z_n - p\rangle \geq 0,$$

this implies that

$$\langle J_1y_n - J_1z_n - r(B_{i_n}y_n - B_{i_n}z_n), z_n - p\rangle \geq 0. \tag{3.9}$$

Now using (3.7), (3.9) and Lemma 2.6, we have

$$\begin{aligned}
\phi(p, v_n) &\leq \phi(p, y_n) - \phi(z_n, y_n) + \frac{2k^2r^2c}{\alpha^2}\phi(z_n, y_n) \\
&= \phi(p, y_n) - (1 - \frac{2k^2r^2c}{\alpha^2})\phi(z_n, y_n),
\end{aligned} \tag{3.10}$$

by condition (iii), we obtain

$$\phi(p, v_n) \leq \phi(p, y_n).$$

Using this in (3.4), we have

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, J_1^{-1}(\beta_nJ_1u + (1-\beta_n)J_1w_n)) \\
&= ||p||^2 - 2\langle p, \beta_nJ_1u + (1-\beta_n)J_1w_n\rangle + ||\beta_nJ_1u + (1-\beta_n)J_1w_n||^2 \\
&\leq ||p||^2 - 2\beta_n\langle p, J_1u\rangle - 2(1-\beta_n)\langle p, J_1w_n\rangle + \beta_n||u||^2 + (1-\beta_n)||w_n||^2 \\
&\quad -\beta_n(1-\beta_n)g(||J_1u - J_1w_n||) \\
&= \beta_n\phi(p, u) + (1-\beta_n)\phi(p, w_n) - \beta_n(1-\beta_n)g(||J_1u - J_1w_n||) \\
&= \beta_n\phi(p, u) + (1-\beta_n)\phi(p, K_s^{\Theta}v_n) - \beta_n(1-\beta_n)g(||J_1u - J_1w_n||) \\
&\leq \beta_n\phi(p, u) + (1-\beta_n)\phi(p, v_n) - \beta_n(1-\beta_n)g(||J_1u - J_1w_n||) \\
&\leq \beta_n\phi(p, u) + (1-\beta_n)\phi(p, y_n) - \beta_n(1-\beta_n)g(||J_1u - J_1w_n||) \\
&\leq \beta_n\phi(p, u) + (1-\beta_n)\phi(p, x_n) - \beta_n(1-\beta_n)g(||J_1u - J_1w_n||) \\
&\leq \beta_n\phi(p, u) + (1-\beta_n)\phi(p, x_n) \\
&\leq \max\{\phi(p, u), \phi(p, x_n)\} \\
&\ \ \vdots \\
&\leq \max\{\phi(p, u), \phi(p, x_0)\}.
\end{aligned} \tag{3.11}$$

Therefore, $\{\phi(p, x_n)\}$ is bounded. Consequently, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded. □

We now present our main theorem.

Theorem 3.3. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E_1 and Q is a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E_2. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions I, $\Psi : C \to E_1^*$ be a continuous and monotone mapping and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. For each $i = 1, \cdots N$, let $A_i : E_1 \to 2^{E_1^*}$ be a finite family of maximal monotone operators and $B_i : E_1 \to E_1^*$ be a finite family α_i-inverse strongly monotone operators, such that $\cap_{i=1}^N (A_i + B_i)^{-1}(0) \neq \emptyset$. Let $T_j : E_2 \to E_2$, be a finite family of η-demimetric and demiclosed mapping such that $\cap_{j=1}^M F(T_j) \neq \emptyset$. Assume $\Gamma = \{p \in GMEP(\Theta, \Psi, \varphi) \cap (\cap_{i=1}^N (A_i + B_i)^{-1}(0) \cap S^{-1}(\cap_{j=1}^M F(T_j))\} \neq \emptyset$. Then, the sequence $\{x_n\}$ generated by $x_0 \in C$ and (3.2) converges strongly to $p \in \Gamma$.*

Proof. Fix $p \in \Gamma$, then by (2.2) and (3.2), we have

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, J_1^{-1}(\beta_n J_1 u + (1-\beta_n) J_1 w_n)) \\
&= V(p, \beta_n J_1 u + (1-\beta_n) J_1 w_n) \\
&\leq V(p, \beta_n J_1 u + (1-\beta_n) J_1 w_n - \beta_n (J_1 u - J_1 p)) + 2\langle J_1^{-1}(\beta_n J_1 u + (1-\beta_n) J_1 w_n) \\
&\quad -p, \beta_n (J_1 u - J_1 p)\rangle \\
&= \beta_n V(p, J_1 p) + (1-\beta_n) V(p, J_1 w_n) + 2\beta_n \langle x_{n+1} - p, J_1 u - J_1 p\rangle \\
&= \beta_n \phi(p, p) + (1-\beta_n)\phi(p, w_n) + 2\beta_n \langle x_{n+1} - p, J_1 u - J_1 p\rangle \\
&\leq (1-\beta_n)\phi(p, z_n) + 2\beta_n \langle x_{n+1} - p, J_1 u - J_1 p\rangle \\
&\leq (1-\beta_n)\phi(p, y_n) + 2\beta_n \langle x_{n+1} - p, J_1 u - J_1 p\rangle \\
&\leq (1-\beta_n)\phi(p, x_n) + 2\beta_n \langle x_{n+1} - p, J_1 u - J_1 p\rangle. \qquad (3.12)
\end{aligned}$$

Now, we consider the following two possible cases:
Case 1: Suppose there exists $n_0 \in \mathbb{N}$ such that $\{\phi(p, x_n)\}$ is motonically nonincreasing for $n \geq n_0$. Then, $\{x_n\}$ is a convergent sequence. We have from (3.3), that

$$\mu_n[(1-\eta) - 2\mu_n ||S||^2 k^2] ||(I - T_{j_n}) S x_n||^2 \leq \phi(p, x_n) - \phi(p, x_{n+1}) \to 0, \quad n \to \infty. \qquad (3.13)$$

Since $\mu_n[(1-\eta) - 2\mu_n ||S||^2 k^2] > 0$, we obtain

$$\lim_{n\to\infty} ||S x_n - T_{j_n} S x_n|| = 0. \qquad (3.14)$$

Observe that

$$\begin{aligned}
\phi(x_n, y_n) &= \phi(x_n, J_1^{-1}(J_1 x_n - \mu_n S^* J_2 (I - T_{j_n}) S x_n)) \\
&= ||x_n||^2 - 2\langle x_n, J_1 x_n - \mu_n S^* J_2 (I - T_{j_n})\rangle + ||J_1 x_n - \mu_n S^* J_2 (I - T_{j_n}) S x_n||^2 \\
&\leq ||x||^2 - 2\langle x_n, J_1 x_n\rangle + 2\mu_n \langle x_n, S^* J_2 (I - T_{j_n}) S x_n\rangle + ||x_n||^2 - 2\mu_n \langle x_n, S^* J_2 (I - T_{j_n}) S x_n\rangle \\
&\quad + 2\mu_n^2 k^2 ||S||^2 ||(I - T_{j_n}) S x_n||^2 \\
&= \phi(x_n, x_n) + 2\mu_n^2 k^2 ||S||^2 ||(I - T_{j_n}) S x_n||^2.
\end{aligned}$$

Thus, by (3.14), we have $\phi(x_n, y_n) \to 0$ as $n \to \infty$. By Lemma 2.2 and the boundedness of $\{x_n\}$, we obtain

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \qquad (3.15)$$

So it follows from the definition of y_n, that

$$\lim_{n\to\infty} ||x_n - y_{j,n}|| = 0. \qquad (3.16)$$

Since J_1 is uniformly norm-to-norm continuous on bounded subsets of E, we have

$$\lim_{n\to\infty} ||J_1 x_n - J_1 y_{j,n}|| = 0.$$

Thus, we have from (3.2), that

$$\lim_{n\to\infty} ||Sx_n - T_jSx_n|| = \lim_{n\to\infty} \frac{1}{\mu_n||S||^2}||J_1x_n - J_1y_{j,n}|| = 0. \tag{3.17}$$

Observe also from (3.10) and (3.11), that

$$\begin{aligned}
\phi(p, x_{n+1}) &\leq \beta\phi(p, u) + (1-\beta_n)\phi(p, w_n) \\
&\leq \beta\phi(p, u) + (1-\beta_n)\phi(p, v_n) \\
&\leq \beta_n\phi(p, u) + (1-\beta_n)\phi(p, y_n) - (1-\beta_n)(1-\frac{2k^2r^2c}{\alpha^2})\phi(z_n, y_n) \\
&\leq \beta_n\phi(p, u) + (1-\beta_n)\phi(p, x_n) - (1-\beta_n)(1-\frac{2k^2r^2c}{\alpha^2})\phi(z_n, y_n),
\end{aligned}$$

which implies

$$(1-\beta_n)(1-\frac{2k^2r^2c}{\alpha^2})\phi(z_n, y_n) \quad \leq \quad \beta_n\phi(p, u) + (1-\beta_n)\phi(p, x_n) - \phi(p, x_{n+1}) \to 0, \text{ as } n \to \infty.$$

Now since $(1-\beta_n)(1-\frac{2k^2r^2c}{\alpha^2}) > 0$, we obtain that $\phi(z_n, y_n) \to 0$ as $n \to \infty$. By Lemma 2.2 and the boundedness of $\{z_n\}$, we get

$$\lim_{n\to\infty} ||z_n - y_n|| = 0. \tag{3.18}$$

From (3.2), we have

$$\begin{aligned}
\phi(z_n, v_n) &= \phi(z_n, J_1^{-1}(J_1z_n - r(B_{i_n}z_n - B_{i_n}y_n))) \\
&= ||z_n||^2 - 2\langle z_n, J_1z_n - r(B_{i_n}z_n - B_{i_n}y_n)\rangle + ||J_1z_n - r(B_{i_n}z_n - B_{i_n}y_n)||^2 \\
&\leq ||z_n||^2 - 2\langle z_n, J_1z_n\rangle + 2r\langle z_n, B_{i_n}z_n - B_{i_n}y_n\rangle + ||z_n||^2 - 2r\langle z_n, (B_{i_n}z_n - B_{i_n}y_n)\rangle \\
&\quad +2k^2||r(B_{i_n}z_n - B_{i_n}y_n)||^2 \\
&= \phi(z_n, z_n) + 2k^2r^2||B_{i_n}z_n - B_{i_n}y_n||^2 \\
&\leq \phi(z_n, z_n) + \frac{2k^2r^2}{\alpha^2}||z_n - y_n|| \to 0, \text{ as } n \to \infty.
\end{aligned}$$

By Lemma 2.2 and the boundedness of $\{y_n\}$, we get

$$\lim_{n\to\infty} ||z_n - v_n|| = 0. \tag{3.19}$$

Now, using Lemma 2.9 (f) and (3.11), we have

$$\begin{aligned}
\phi(v_n, K_s^{\Theta}v_n) &\leq \phi(p, v_n) - \phi(p, K_s^{\Theta}v_n) \\
&= \phi(p, v_n) - \phi(p, w_n) \\
&\leq \phi(p, y_n) + \beta_n\phi(p, u) - \beta_n\phi(p, w_n) - \phi(p, x_{n+1}) \\
&\leq \phi(p, x_n) + \beta_n\phi(p, u) - \beta_n\phi(p, w_n) - \phi(p, x_{n+1}) \to 0, \text{ as } n \to \infty. \qquad (3.20)
\end{aligned}$$

It follows immediately by Lemma 2.2 and the boundedness of $\{v_n\}$ that $||v_n - K_s^{\Theta}v_n|| \to 0$ as $n \to \infty$. Next, we show that $\phi(v_n, x_{n+1}) \to 0$, as $n \to \infty$. From (3.2), we have

$$\begin{aligned}
\phi(v_n, x_{n+1}) &= \phi(v_n, J_1^{-1}(\beta_nJ_1u + (1-\beta_n)J_1w_n)) \\
&= ||v_n||^2 - 2\langle v_n, \beta_nJ_1u + (1-\beta_n)J_1w_n\rangle + ||\beta_nJ_1 + (1-\beta_n)J_1w_n||^2 \\
&\leq ||v_n||^2 - 2\beta_n\langle v_n, J_1u\rangle - 2(1-\beta_n)\langle v_n, J_1w_n\rangle + \beta_n||u||^2 + (1-\beta_n)||w_n||^2 \\
&\quad - \beta_n(1-\beta_n)g(||J_1u - J_1w_n||) \\
&= \beta_n\phi(v_n, u) + (1-\beta_n)\phi(v_n, w_n) - \beta_n(1-\beta_n)g(||J_1u - J_1w_n||) \\
&= \beta_n\phi(v_n, u) + (1-\beta_n)\phi(v_n, K_s^{\Theta}v_n) - \beta_n(1-\beta_n)g(||J_1u - J_1w_n||), \qquad (3.21)
\end{aligned}$$

hence, by using condition (i) and (3.20), we obtain

$$\phi(v_n, x_{n+1}) \to 0, \text{ as } n \to \infty.$$

By Lemma 2.2, we have

$$\lim_{n\to\infty} ||v_n - x_{n+1}|| = 0. \tag{3.22}$$

Now, from (3.2), (3.15) and (3.18), we obtain

$$||x_n - z_{i,n}|| \le ||x_n - z_n|| \le ||x_n - y_n|| + ||y_n - z_n|| \to 0, \text{ as } n \to \infty. \tag{3.23}$$

Also

$$\lim_{n\to\infty} ||x_n - v_n|| = \lim_{n\to\infty} (||x_n - y_n|| + ||y_n - z_n|| + ||z_n - v_n||) = 0. \tag{3.24}$$

Using (3.22) and (3.24), we get

$$||x_{n+1} - x_n|| = ||x_{n+1} - v_n|| + ||v_n - x_n|| \to 0 \text{ as } n \to \infty. \tag{3.25}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q \in C$. We have from (3.15) and (3.24), that $\{y_{n_k}\}$ of $\{y_n\}$ and $\{z_{n_k}\}$ of $\{z_n\}$ both converge weakly to q. Hence, by (3.15), (3.23) and Lemma 3.1, we have $q \in \cap_{i=1}^N (A_i + B_i)^{-1}(0)$. Also, by using (3.20) , we obtain $q \in F(K_s^\Theta) = GMEP(\Theta, \Psi, \varphi)$. Thus, we have $q \in GMEP(\Theta, \Psi, \varphi) \cap (\cap_{i=1}^N (A_i + B_i)^{-1}(0))$. Now, using the linearity of S, we obtain $Sx_{n_k} \rightharpoonup Sq$, by (3.17) and the demiclosedeness of T_j for each j, we have $q \in S^{-1}F(T_j)$. Hence $q \in \Gamma$.

Finally, we show that $\{x_n\}$ converges strongly to $p \in \Gamma$. To do this, we only need to show that $\limsup_{n\to\infty}\langle x_{n+1} - p, J_1 u - J_1 p\rangle \le 0$ in (3.12). Indeed, choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup q$ and

$$\lim_{n\to\infty} \langle x_{n+1} - p, J_1 u - J_1 p\rangle = \limsup_{k\to\infty} \langle x_{n_k+1} - p, J_1 u - J_1 p\rangle.$$

Since $||x_{n+1} - x_n|| \to 0$ as $n \to \infty$, we obtain $x_{n_k+1} \rightharpoonup q$. From (2.9) in Lemma 2.4, we obtain

$$\begin{aligned} \lim_{n\to\infty} \langle x_{n+1} - p, J_1 u - J_1 p\rangle &= \limsup_{k\to\infty} \langle x_{n_k+1} - p, J_1 u - J_1 p\rangle. \\ &= \langle q - p, J_1 u - J_1 p\rangle \\ &\le 0. \end{aligned} \tag{3.26}$$

By using Lemma 2.4, Lemma 2.7 and (3.26), we obtain that $\{x_n\}$ converges strongly to p.

Case 2: Assume $\{\Phi_n = ||x_n - p||\}$ is monotonically nondecreasing. For some n_0 large enough, define a mapping

$$\tau(n) := \max\{j \in \mathbb{N} : j \le n, \Phi_j \le \Phi_{j+1}\}.$$

Clearly, τ so defined is a nondecreasing sequence, $\tau(n) \to 0$ as $n \to \infty$ and

$$0 \le \Phi_{\tau(n)} \le \Phi_{\tau(n)+1}, \ \forall n \ge n_0. \tag{3.27}$$

By the same argument as in **Case 1**, we have $||(I - T_{j_{\tau(n)}})Sx_{\tau(n)}|| \to 0$, $||v_{\tau(n)} - K_s^\Theta v_{\tau(n)}|| \to 0$ and $||x_{\tau(n)+1} - x_{\tau(n)}|| \to 0$ as $n \to \infty$ and $\limsup_{n\to\infty}\langle x_{n+1} - p, J_{E_1} u - J_{E_1} p\rangle \le 0$. Since $\{x_{\tau(n)}\}$ is bounded, there exists a subsequence $\{x_{\tau(n)}\}$ denoted again by $\{x_{\tau(n)}\}$ which converges weakly to $\bar{q} \in C$. Also by the linearity of S, we have $Sx_{\tau(n)} \rightharpoonup S\bar{q} \in Q$. Following the same arguments as in the first case, we conclude that $\bar{q} \in \Gamma$. Recall from (3.12), that

$$\Phi_{\tau(n)+1} \le (1 - \beta_{\tau(n)})\Phi_{\tau(n)} + \beta_{\tau(n)}\delta_{\tau(n)}. \tag{3.28}$$

Where $\delta_{\tau(n)} = 2\langle x_{\tau(n)+1} - p, J_{E_1}u - J_{E_1}p\rangle$. Note that $\beta_{\tau(n)} \to 0$ as $n \to \infty$ and $\limsup_{n\to\infty} \delta_{\tau(n)} \le 0$. Since $\Phi_{\tau(n)} \le \Phi_{\tau(n+1)}$ and $\beta_{\tau(n)} > 0$, we have

$$||x_{\tau(n)} - p|| \le \delta_{\tau(n)}.$$

This implies

$$\limsup_{n\to\infty} ||x_{\tau(n)} - p||^2 \le 0,$$

hence

$$\lim_{n\to\infty} ||x_{\tau(n)} - p|| = 0. \tag{3.29}$$

By using $\lim_{n\to\infty} ||x_{\tau(n)+1} - x_{\tau(n)}|| = 0$ and (3.29), we have that

$$\lim_{n\to\infty} ||x_{\tau(n)+1} - p|| \le \lim_{n\to\infty} (||x_{\tau(n)+1} - x_{\tau(n)}|| + ||x_{\tau(n)} - p||) = 0. \tag{3.30}$$

Furthermore, for $n \ge n_0$, it is easy to see that $\Phi_{\tau(n)} \le \Phi_{\tau(n)+1}$ if $n \ne \tau(n)$ (that is $\tau(n) < n$), because $\Phi_j \ge \Phi_{j+1}$ for $\tau(n) + 1 \le j \le n$. As a consequence, we obtain for all $n \ge n_0$

$$0 \le \Phi_n \le \max\{\Phi_{\tau(n)}, \Phi_{\tau(n)+1}\} = \Phi_{\tau(n)+1}. \tag{3.31}$$

By using (3.30), we can conclude that $\lim_{n\to\infty} \Phi_n = 0$, that is $\{x_n\}$ converges strongly to p. Thus completing the proof. □

The following are deduced results from our main Theorem 3.3:
If we take $M = N = 1$ (3.2), we have the following corollary:

Corollary 3.4. *Let C be a nonempty, closed and convex subset of a 2-uniformly convex and uniformly smooth Banach space E_1 and Q is a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E_2. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions I, $\Psi : C \to E_1^*$ be a continuous and monotone mapping and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ is a nonlinear mapping. Let $A : E_1 \to 2^{E_1^*}$ be a maximal monotone operator and $B : E_1 \to E_1^*$ be an α-inverse strongly monotone operator, such that $(A + B)^{-1}(0) \ne \emptyset$. Let $T : E_2 \to E_2$ be an η-demimetric and demiclosed mapping such that $F(T) \ne \emptyset$. Assume $\Gamma = \{p \in GMEP(\Theta, \Psi, \varphi) \cap (A + B)^{-1}(0) \cap S^{-1}(F(T))\} \ne \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_0 \in C$ and*

$$\begin{cases} y_n = J_1^{-1}(J_1 x_n - \mu_n S^* J_2(I - T)Sx_n), \\ z_n = J_1^{-1}((J_1 + rA)^{-1} J_1 J_1^{-1}(J_1 - rB)y_n), \\ v_n = J_1^{-1}(J_1 z_n - r(Bz_n - By_n)), \\ \Theta(w_n, y) + \langle y - w_n, \Psi w_n\rangle + \varphi(y) - \varphi(w_n) + \dfrac{1}{s}\langle y - w_n, J_1 w_n - J_1 v_n\rangle \ge 0, \ \forall y \in C, \\ x_{n+1} = J_1^{-1}(\beta_n J_1 u + (1 - \beta_n) J_1 w_n), \ n \ge 0, \end{cases}$$

where $s > 0$, is a positive parameter, $\{\mu_n\}$ is a sequence of positive real numbers and $\{\beta_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0, \ \sum_{n=0}^{\infty} \beta_n = 0;$

(ii) $0 \le \mu_n \le \dfrac{1 - \eta}{k^2||S||^2};$

(iii) $0 < a \le r \le b < \dfrac{\alpha}{k\sqrt{2c}}.$

Then, $\{x_n\}$ converges strongly to $p \in \Gamma$.

If the underlying spaces are Hilbert spaces, that is $E_1 = H_1$ and $E_2 = H_2$ in Theorem 3.3, we obtain the following result:

Corollary 3.5. *Let C and Q be nonempty, closed and convex subset of real Hilbert spaces H_1 and H_2 respectively. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions I, $\Psi : C \to H_1$ be a continuous and monotone mapping and $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ is a nonlinear mapping. For each $i = 1, \cdots N$, let $A_i : H_1 \to 2^{H_1}$ be a finite family of maximal monotone operators and $B_i : H_1 \to H_1$ be a finite family of α_i-inverse strongly monotone operators with $\alpha = \min\{\alpha_1, \cdots, \alpha_N\}$, such that $\cap_{i=1}^N (A_i + B_i)^{-1}(0) \neq \emptyset$. Let $T_j : Q \to Q$ be a finite family of η_j-demimetric and demiclosed mapping with $\eta = \max\{\eta_1, \eta_2, \ldots, \eta_M\}$ such that $\cap_{j=1}^M F(T_j) \neq \emptyset$. Assume $\Gamma = \{p \in GMEP(\Theta, \Psi, \varphi) \cap (\cap_{i=1}^N (A_i + B_i)^{-1}(0) \cap S^{-1}(\cap_{j=1}^M F(T_j))\} \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by $x_0 \in C$ and*

$$\begin{cases} y_{j,n} = x_n - \mu_n S^*(I - T_j)Sx_n \\ \textit{choose } j_n : \ ||y_{j_n,n} - x_n|| = \max\{||y_{j,n} - x_n|| : j = 1, \cdots, M\}, \\ y_{j_n,n} = y_n, \\ z_{i,n} = (I + rA_i)^{-1}(I - rB_i)y_n \\ \textit{choose } i_n : ||z_{i_n,n} - x_n|| = \max\{||z_{i,n} - x_n|| : i = 1, \cdots, N\}, \\ z_{i_n,n} = z_n, \\ v_n = z_n - r(B_i z_n - B_i y_n), \\ \textit{choose } i_n : ||v_{i_n,n} - x_n|| = \max\{||v_{i,n} - x_n|| : i = 1, \cdots, N\}, \\ v_{i_n,n} = v_n, \\ \Theta(w_n, y) + \langle y - w_n, \Psi w_n \rangle + \varphi(y) - \varphi(w_n) + \dfrac{1}{s}\langle y - w_n, w_n - v_n \rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \beta_n u + (1 - \beta_n)w_n, \ n \geq 0, \end{cases} \tag{3.32}$$

where $s > 0$, is a positive parameter, $\{\mu_n\}$ is a sequence of positive real numbers and $\{\beta_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \beta_n = 0, \ \sum_{n=0}^{\infty} \beta_n = 0;$

(ii) $0 \leq \mu_n \leq \dfrac{1 - \eta}{||S||^2};$

(iii) $0 < a \leq r \leq b < \alpha.$

Then, $\{x_n\}$ converges strongly to $p \in \Gamma$.

4. Numerical Experiment

In this section, we present a numerical example to illustrate the performance of our algorithm.

Example 4.1. *Let $E_1 = E_2 = \ell_2(\mathbb{R})$ be the linear spaces whose elements are all 2-summable sequences $\{x_k\}_{k=1}^{\infty}$ of scalars in $\mathbb{R}$, that is*

$$\ell_2(\mathbb{R}) := \left\{ x = (x_1, x_2 \cdots, x_k \cdots), \ x_k \in \mathbb{R} \textit{ and } \sum_{k=1}^{\infty} |x_k|^2 < \infty \right\},$$

with the inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x, y \rangle := \sum_{k=1}^{\infty} x_k y_k$ and the norm $||\cdot|| : \ell_2 \to \mathbb{R}$ by $||x|| := \sqrt{\sum_{k=1}^{\infty} |x_k|^2}$, where $x = \{x_k\}_{k=1}^{\infty}$, $y = \{y_k\}_{k=1}^{\infty}$. Let $S : \ell_2 \to \ell_2$ be given by $Sx = \left(\frac{x_1}{5}, \frac{x_2}{5}, \cdots, \frac{x_k}{5}, \cdots, \right)$

Table 1: Computation result for Example 4.1, Case (i); Time: 0.0316sec.

Iteration	x^{k+1}	$\|\|x^{k+1}-x^k\|\|_{\ell_2}$
1	$(3.6048, -1.9835, 1.9555, 0, \dots, 0, \dots)^T$	3.3814
2	$(3.5826, -1.9520, 1.9682, 0, \dots, 0, \dots)^T$	0.0406
3	$(3.5829, -1.9524, 1.9680, 0, \dots, 0, \dots)^T$	$4.87e^{-4}$
4	$(3.5829, -1.9524, 1.9680, 0, \dots, 0, \dots)^T$	$5.84e^{-6}$

Table 2: Computation result for Example 4.1, Case (ii); Time: 0.1568sec.

Iteration	x^{k+1}	$\|\|x^{k+1}-x^k\|\|_{\ell_2}$
1	$(3.6048, -1.9441, 1.9327, 0, \dots, 0, \dots)^T$	3.5745
2	$(3.5826, -1.9525, 1.9684, 0, \dots, 0, \dots)^T$	0.0429
3	$(3.5829, -1.9524, 1.9680, 0, \dots, 0, \dots)^T$	$5.15e^{-6}$
4	$(3.5829, -1.9524, 1.9680, 0, \dots, 0, \dots)^T$	$6.17e^{-9}$

for all $x = \{x_k\}_{k=1}^{\infty} \in \ell_2$*. Then* $S^*y = \left(\frac{y_1}{5}, \frac{y_2}{5}, \cdots, \frac{y_k}{5}, \cdots,\right)$ *for each* $y = \{y_k\}_{k=1}^{\infty} \in \ell_2$*. Define the sets* $C := \{x \in \ell_2 : ||x|| \leq 1\}$ *and* $Q := \{y \in \ell_2 : ||y|| \leq 1\}$*. Let the bifunction* $\Theta : C \times C \to \mathbb{R}$ *be defined by* $g(x, y) = xy + 5y - 5x - x^2$*. Let* $\varphi(x) := x$*, for all* $x = \{x_k\}_{k=1}^{\infty} \in \ell_2$ *and* $y = \{y_k\}_{k=1}^{\infty} \in \ell_2$*. It is easy to check that*

$$K_s^{\Theta}(x) = \frac{x - 30s}{5(s+1)}.$$

For $j = 1, 2, \dots, M$*, define* $T_j : C \to C$ *by* $T_j(x) = \left(\frac{x_1}{2j}, \frac{x_2}{2j}, \cdots, \frac{x_k}{2j}, \cdots,\right)$ *for all* $x = \{x_k\}_{k=1}^{\infty} \in \ell_2$ *and all* j*. It is easy to show that* $F(T) = \{0\}$*.*
Also, for $i = 1, 2, \dots, N$*, let* $A_i : \ell_2 \to \ell_2$ *be defined by* $A_i(x) = (2ix_1, 2ix_2, \cdots, 2ix_k, \cdots)$*, for each* $x = \{x_k\}_{k=1}^{\infty} \in \ell_2$*, it is clear that* A_i *is maximal monotone for each* $x = \{x_k\}_{k=1}^{\infty} \in \ell_2$*, and each* i*. Let* $B_i : \ell_2 \to \ell_2$ *be defined by* $B_i(x) = (ix_1, ix_2, \cdots ix_k, \cdots)$*, for all* $x = \{x_k\}_{k=1}^{\infty} \in \ell_2$ *and each* i*. Then,* B_i *is* α*-inverse strongly monotone with* $\alpha = \min\{\alpha_i\}_{i=1}^{\infty} = 1$*. It is easy to see that*

$$G_r(x) = \left(\frac{i - r}{r + 2i}\right) x.$$

We choose $N = M = 30, r = \frac{1}{40}, s = 0.25, \mu_n = 0.01$, $\beta_n = \frac{1}{n+3}$, $\alpha_n = \frac{1}{5} + \frac{2n-1}{3n+7}$, $u = (5.2345, -2.2344, 3.0555, 0, \dots, 0, \dots)^T$*. Using* $\frac{||x^{k+1}-x^k||_{\ell_2}}{||x^2-x^1||_{\ell_2}} < 10^{-5}$ *as the stopping criterion, we test our Algorithm (3.2) for different values of* x_0 *as follows:*

Case (i) $x_0 = (1.7601, 0.6457, 3.0129, 0, \dots, 0, \dots)^T$,

Case (ii) $x_0 = (1.7601, -2.6457, 4.9129, 0, \dots, 0, \dots)^T$.

We then plot the graphs of error '$||x^{k+1} - x^k||_{\ell_2}$*' against the number of iteration in each case. The computational results can be found in Table 1, 2 and Figure 1. These show that the change in the initial value does not have a significant effect on the performance of the algorithm.*

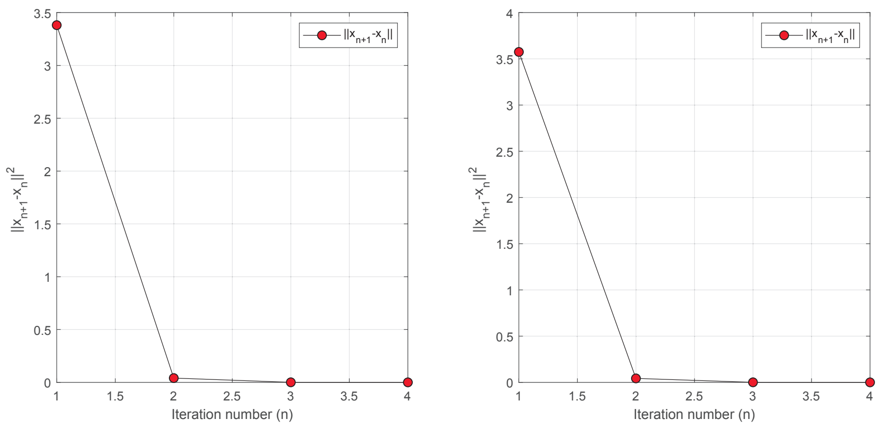

Figure 1: Example 4.1, Top Left: Case (i); Top Right: Case (ii).

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