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Generalized Lacunary Statistical Convergence of Order β of Difference Sequences of Fractional Order

Nazlım Deniz Aral

ABSTRACT: In this paper, using a modulus function we generalize the concepts of Δ^m-lacunary statistical convergence and Δ^m-lacunary strongly convergence $(m \in \mathbb{N})$ to $\Delta^\alpha-$lacunary statistical convergence of order β with the fractional order of α and $\Delta^\alpha-$lacunary strongly convergence of order β with the fractional order of α (where $0 < \beta \leq 1$ and α be a fractional order).

Key Words: Difference sequence, statistical convergence, lacunary sequence.

Contents

1. Introduction

The idea of statistical convergence was given by Zygmund [40] in the first edition of his monograph published in Warsaw in 1935. The concept of statistical convergence was introduced by Steinhaus [34] and Fast [19] and later reintroduced by Schoenberg [32]. Over the years and under different names statistical convergence was discussed in the theory of Fourier analysis, Ergodic theory, Number theory, Measure theory, Trigonometric series, Turnpike theory and Banach spaces. Later on it was further investigated from the sequence space point of view and linked with summability theory by Caserta et al. [5], Connor [6], Çakallı et al. ([7], [8], [9]), Çınar et al. [10], Et et al. ([15], [16]), Fridy [21], Fridy and Orhan [22], Işık et al. ([23], [24]), Mursaleen [28], Salat [30], Şengül [35] and many others.

The idea of statistical convergence depends upon the density of subsets of the set $\mathbb{N}$ of natural numbers. The density of a subset $\mathbb{E}$ of $\mathbb{N}$ is defined by

$$\delta(\mathbb{E}) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{\mathbb{E}}(k), \quad \text{provided that the limit exists.}$$

A sequence $x = (x_k)$ is said to be statistically convergent to L if for every $\varepsilon > 0$,

$$\delta\left(\{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}\right) = 0.$$

Recently, Çolak [11] generalized the statistical convergence by ordering the interval $(0, 1]$ and defined the statistical convergence of order β and strong $p-$Cesàro summability of order β, where $0 < \beta \leq 1$ and p is a positive real number. Şengül and Et ([17], [36]) generalized the concepts such as lacunary statistical convergence of order β and lacunary strong $p-$Cesàro summability of order β for sequences of real numbers.

Difference sequence spaces was defined by Kızmaz [27] and the concept was generalized by Et et al. ([12], [13]) as follows:

$$\Delta^m (X) = \{x = (x_k) : (\Delta^m x_k) \in X\},$$

where X is any sequence space, $m \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^m x = (\Delta^m x_k) = \left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)$ and so $\Delta^m x_k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$.

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If $x \in \Delta^m(X)$ then there exists one and only one sequence $y = (y_k) \in X$ such that $y_k = \Delta^m x_k$ and

$$x_k = \sum_{v=1}^{k-m} (-1)^m \left(\begin{array}{c} k-v-1 \\ m-1 \end{array} \right) y_v = \sum_{v=1}^{k} (-1)^m \left(\begin{array}{c} k+m-v-1 \\ m-1 \end{array} \right) y_{v-m}, \tag{1.1}$$

$$y_{1-m} = y_{2-m} = \cdots = y_0 = 0$$

for sufficiently large k, for instance $k > 2m$. After then some properties of difference sequence spaces have been studied in ([1], [2], [14], [26], [31]).

By $\Gamma(r)$, we denote the Gamma function of a real number r and $r \notin \{0, -1, -2, -3, ...\}$. By the definition, it can be expressed as an improper integral as:

$$\Gamma(r) = \int_0^\infty e^{-t} t^{r-1} dt.$$

From the definition, it is observed that:
(i) For any natural number n, $\Gamma(n+1) = n!$,
(ii) For any real number n and $n \notin \{0, -1, -2, -3, ...\}, \Gamma(n+1) = n\Gamma(n)$,
(iii) For particular cases, we have $\Gamma(1) = \Gamma(2) = 1, \Gamma(3) = 2!, \Gamma(4) = 3!, ...$

For a proper fraction α, we define a fractional difference operator $\Delta^\alpha : w \to w$ defined by

$$\Delta^\alpha(x_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}. \tag{1.2}$$

In particular, we have $\Delta^{\frac{1}{2}} x_k = x_k - \frac{1}{2}x_{k+1} - \frac{1}{8}x_{k+2} - \frac{1}{16}x_{k+3} - \frac{5}{128}x_{k+4} - \frac{7}{256}x_{k+5} - \frac{21}{1024}x_{k+6} \cdots$
$\Delta^{-\frac{1}{2}} x_k = x_k + \frac{1}{2}x_{k+1} + \frac{3}{8}x_{k+2} + \frac{5}{16}x_{k+3} + \frac{35}{128}x_{k+4} + \frac{63}{256}x_{k+5} + \frac{231}{1024}x_{k+6} \cdots$
$\Delta^{\frac{1}{3}} x_k = x_k - \frac{1}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{5}{81}x_{k+3} - \frac{10}{243}x_{k+4} - \frac{22}{729}x_{k+5} - \frac{154}{6561}x_{k+6} \cdots$
$\Delta^{\frac{2}{3}} x_k = x_k - \frac{2}{3}x_{k+1} - \frac{1}{9}x_{k+2} - \frac{4}{81}x_{k+3} - \frac{7}{243}x_{k+4} - \frac{14}{729}x_{k+5} - \frac{91}{6561}x_{k+6} \cdots$

Without loss of generality, we assume throughout that the series defined in (1.2) is convergent. Moreover, if α is a positive integer, then the infinite sum defined in (1.2) reduces to a finite sum i.e., $\sum_{i=0}^{\alpha} (-1)^i \frac{\Gamma(\alpha+1)}{i!\Gamma(\alpha-i+1)} x_{k+i}$. In fact, this operator is generalized the difference operator introduced by Et and Çolak [12].

Recently, using fractional operator Δ^α (fractional order of α) Baliarsingh et al. ([3], [4], [29]) defined the sequence space $\Delta^\alpha(X)$ such as:

$$\Delta^\alpha(X) = \{x = (x_k) : (\Delta^\alpha x_k) \in X\},$$

where X is any sequence space.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ of non-negative integers such that $k_0 = 0$ and $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r, and $q_1 = k_1$ for convenience. In recent years, lacunary sequences have been studied in ([7], [8], [9], [18], [20], [22], [25], [37], [38], [33]).

2. Main Results

Definition 2.1. [2] *Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1]$ and α be a proper fraction. The sequence $x = (x_k)$ is said to be $\Delta^\alpha-$lacunary statistically convergent of order β of fractional order of α (or $\Delta^\alpha(S_\theta^\beta)-$convergent to L) to the number L, if there is a real number L such that*

$$\lim_{r\to\infty} \frac{1}{h_r^\beta} |\{k \in I_r : |\Delta^\alpha x_k - L| \geq \varepsilon\}| = 0$$

for all $\varepsilon > 0$. In this case, we write $x_k \to L(\Delta^\alpha(S_\theta^\beta))$.

The set of all $\Delta^{\alpha}(S_{\theta}^{\beta})-$convergent sequences will be denoted by $\Delta^{\alpha}(S_{\theta}^{\beta})$. If $\theta = (2^r)$, then we write $\Delta^{\alpha}(S^{\beta})$ instead of $\Delta^{\alpha}(S_{\theta}^{\beta})$. In the special cases $\theta = (2^r)$ and $\beta = 1$, we write $\Delta^{\alpha}(S)$ instead of $\Delta^{\alpha}(S_{\theta}^{\beta})$.

In particular, $\Delta^{\alpha}(S_{\theta}^{\beta})-$convergence includes many special cases; for example, in case of $\alpha = m \in \mathbb{N}, \beta = 1, \Delta^{\alpha}-$lacunary statistical convergence of order β reduces to the $\Delta^{m}-$lacunary statistical convergence which was defined and studied by Tripathy and Et [39].

Definition 2.2. [2] *Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1]$ and α be a proper fraction, then the sequence (x_n) is said to be $\Delta^{\alpha}-$Cesàro summable of order β to L if*

$$\lim_{n\to\infty} \frac{1}{n^{\beta}} \sum_{k=1}^{n} (\Delta^{\alpha} x_k - L) = 0.$$

The class of all $\Delta^{\alpha}-$Cesàro summable sequences of order β is denoted by $\Delta^{\alpha}(\sigma_1^{\beta})$. In case of $\beta = 1$, we will write $\Delta^{\alpha}(\sigma_1)$ instead of $\Delta^{\alpha}(\sigma_1^{\beta})$ and say that $x = (x_n)$ is $\Delta^{\alpha}-$Cesàro summable to L.

Theorem 2.3. [2] *If a $\Delta^{\alpha}-$bounded sequence (that is $x \in \Delta^{\alpha}(\ell_{\infty})$) is $\Delta^{\alpha}-$statistically convergent to L, then it is $\Delta^{\alpha}-$Cesàro summable to L.*

Converse of Theorem 2.1 does not holds. For this choose $\alpha = 1$, then the sequence

$$x = (0, -1, -1, -2, -2, -3-, 3, -4, -4, ...)$$

belongs to $\Delta(\sigma_1)$ and does not belong to $\Delta(S)$.

Definition 2.4. [2] *Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1]$, α be a proper fraction and p be a fixed positive real number. A sequence $x = (x_k)$ is said to be $\Delta_p^{\alpha}-$lacunary strongly summable of order β to L if*

$$\lim_{r\to\infty} \frac{1}{h_r^{\beta}} \sum_{k\in I_r} |\Delta^{\alpha} x_k - L|^p = 0.$$

In this case we write $x_k \to L(\Delta^{\alpha}(N_{\theta}^{\beta}, p))$. We denote the class of all $\Delta_p^{\alpha}-$lacunary strongly summable sequences of order β by $\Delta^{\alpha}(N_{\theta}^{\beta}, p)$. If $\beta = 1$, then we write $\Delta^{\alpha}(N_{\theta}, p)$ instead of $\Delta^{\alpha}(N_{\theta}^{\beta}, p)$ and in the special case $\theta = (2^r)$ we write $\Delta^{\alpha}(|\sigma_1|^{\beta}, p)$ instead of $\Delta^{\alpha}(N_{\theta}^{\beta}, p)$. Also in the special case $p = 1$ we shall write $\Delta^{\alpha}(|\sigma_1|^{\beta})$ instead of $\Delta^{\alpha}(|\sigma_1|^{\beta}, p)$.

The proof the following theorems are straightforward, so we choose to state these results without proof.

Theorem 2.5. *Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1]$ and α be a proper fraction. If*

$$\liminf_{r} q_r > 1,$$

then $\Delta^{\alpha}(|\sigma_1|^{\beta}, p) \subset \Delta^{\alpha}(N_{\theta}^{\beta}, p)$.

Theorem 2.6. *Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1]$and α be a proper fraction. If*

$$\limsup_{r} \frac{k_r}{k_{r-1}^{\beta}} < \infty,$$

then $\Delta^{\alpha}(N_{\theta}, p) \subset \Delta^{\alpha}(|\sigma_1|^{\beta}, p)$.

Theorem 2.7. *Let $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1]$ and α be a proper fraction. If $\Delta^\alpha(|\sigma_1|^\beta) \cap \Delta^\alpha(N_\theta^\beta)$ and $\sup_r \frac{k_r}{k_{r-1}^\beta} < \infty$, then $x_k \to L(\Delta^\alpha(|\sigma_1|^\beta))$ and $x_k \to L(\Delta^\alpha(N_\theta^\beta))$ that is these limits equal.*

Theorem 2.8. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subset J_r$ for all $r \in \mathbb{N}$, $\beta, \gamma \in (0,1]$ be real numbers such that $\beta \leqslant \gamma$ and α be a proper fraction.*

i) If

$$\lim_{r\to\infty} \inf \frac{h_r^\beta}{\ell_r^\gamma} > 0, \tag{2.1}$$

then $\Delta^\alpha(S_{\theta'}^\gamma) \subseteq \Delta^\alpha(S_\theta^\beta)$,

ii) If

$$\lim_{r\to\infty} \inf \frac{\ell_r}{h_r^\gamma} = 1, \tag{2.2}$$

then $\Delta^\alpha(S_\theta^\beta) \subseteq \Delta^\alpha(S_{\theta'}^\gamma)$.

Proof. i) Omitted.

ii) Let $x = (x_k) \in \Delta^\alpha(S_\theta^\beta)$ and be (2.2) satisfied. Since $I_r \subset J_r$, for $\varepsilon > 0$ we may write

$$\begin{aligned}
\frac{1}{\ell_r^\gamma}|\{k \in J_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| &= \frac{1}{l_r^\gamma}|\{s_{r-1} < k \leqslant k_{r-1} : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| \\
&+ \frac{1}{\ell_r^\gamma}|\{k_r < k \leqslant s_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| + \frac{1}{\ell_r^\gamma}|\{k_{r-1} < k \leqslant k_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| \\
&\leqslant \frac{k_{r-1} - s_{r-1}}{l_r^\gamma} + \frac{s_r - k_r}{\ell_r^\gamma} + \frac{1}{\ell_r^\gamma}|\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| \\
&= \frac{\ell_r - h_r}{l_r^\gamma} + \frac{1}{\ell_r^\gamma}|\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| \\
&\leqslant \frac{\ell_r - h_r^\gamma}{h_r^\gamma} + \frac{1}{h_r^\gamma}|\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| \\
&\leqslant \left(\frac{\ell_r}{h_r^\gamma} - 1\right) + \frac{1}{h_r^\beta}|\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}|
\end{aligned}$$

for all $r \in \mathbb{N}$, where $I_r = (k_{r-1}, k_r]$, $J_r = (s_{r-1}, s_r]$, $h_r = k_r - k_{r-1}$ and $\ell_r = s_r - s_{r-1}$. This implies that $\Delta^\alpha(S_\theta^\beta) \subseteq \Delta^\alpha(S_{\theta'}^\gamma)$.

Theorem 2.9. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, β and γ be fixed real numbers such that $0 < \beta \leqslant \gamma \leqslant 1$ and $0 < p < \infty$. Then we have,*

i) If (2.1) holds then $\Delta^\alpha(N_{\theta'}^\gamma, p) \subset \Delta^\alpha(N_\theta^\beta, p)$,

ii) If (2.2) holds and $x \in \Delta^\alpha(\ell_\infty)$ then $\Delta^\alpha(N_\theta^\beta, p) \subset \Delta^\alpha(N_{\theta'}^\gamma, p)$.

Proof. Omitted.

Theorem 2.10. *Let $\theta = (k_r)$ and $\theta' = (s_r)$ be two lacunary sequences such that $I_r \subseteq J_r$ for all $r \in \mathbb{N}$, β and γ be fixed real numbers such that $0 < \beta \leqslant \gamma \leqslant 1$ and $0 < p < \infty$. Then,*

i) Let (2.1) holds, if a sequence is strongly $\Delta^\alpha(N_{\theta'}^\gamma, p)$-summable to L, then it is $\Delta^\alpha(S_\theta^\beta)$-statistically convergent to L.

ii) Let (2.2) holds, if a $\Delta^\alpha-$bounded sequence is $\Delta^\alpha(S_\theta^\beta)$-statistically convergent to L, then it is strongly $\Delta^\alpha(N_{\theta'}^\gamma, p)-$summable to L.

Proof. i) For any sequence $x=(x_k)$ and $\varepsilon>0$, we have

$$\begin{aligned}\sum_{k\in J_r}|\Delta^\alpha x_k-L|^p &= \sum_{\substack{k\in J_r\\ |\Delta^\alpha x_k-L|\geqslant\varepsilon}}|\Delta^\alpha x_k-L|^p+\sum_{\substack{k\in J_r\\ |\Delta^\alpha x_k-L|<\varepsilon}}|\Delta^\alpha x_k-L|^p\\ &\geqslant \sum_{\substack{k\in I_r\\ |\Delta^\alpha x_k-L|\geqslant\varepsilon}}|\Delta^\alpha x_k-L|^p\\ &\geqslant |\{k\in I_r:|\Delta^\alpha x_k-L|\geqslant\varepsilon\}|\varepsilon^p\end{aligned}$$

and so that

$$\begin{aligned}\frac{1}{\ell_r^\gamma}\sum_{k\in J_r}|\Delta^\alpha x_k-L|^p &\geqslant \frac{1}{\ell_r^\gamma}|\{k\in I_r:|\Delta^\alpha x_k-L|\geqslant\varepsilon\}|\varepsilon^p\\ &\geqslant \frac{h_r^\beta}{\ell_r^\gamma}\frac{1}{h_r^\beta}|\{k\in I_r:|\Delta^\alpha x_k-L|\geqslant\varepsilon\}|\varepsilon^p.\end{aligned}$$

Hence $x=(x_k)$ is $\Delta^\alpha(S_\theta^\beta)-$statistically convergent to L.

ii) Suppose that $\Delta^\alpha(S_\theta^\beta)$-statistically convergent to L and $x=(x_k)\in\Delta^\alpha(\ell_\infty)$. Then there exists some $M>0$ such that $|\Delta^\alpha x_k-L|\leqslant M$ for all k. Then for every $\varepsilon>0$ we may write

$$\begin{aligned}\frac{1}{\ell_r^\gamma}\sum_{k\in J_r}|\Delta^\alpha x_k-L|^p &= \frac{1}{\ell_r^\gamma}\sum_{k\in J_r-I_r}|\Delta^\alpha x_k-L|^p+\frac{1}{\ell_r^\gamma}\sum_{k\in I_r}|\Delta^\alpha x_k-L|^p\\ &\leqslant \left(\frac{\ell_r-h_r}{l_r^\gamma}\right)M^p+\frac{1}{\ell_r^\gamma}\sum_{k\in I_r}|\Delta^\alpha x_k-L|^p\\ &\leqslant \left(\frac{\ell_r-h_r^\gamma}{\ell_r^\gamma}\right)M^p+\frac{1}{\ell_r^\gamma}\sum_{k\in I_r}|\Delta^\alpha x_k-L|^p\\ &\leqslant \left(\frac{\ell_r}{h_r^\gamma}-1\right)M^p+\frac{1}{h_r^\gamma}\sum_{\substack{k\in I_r\\ |\Delta^\alpha x_k-L|\geqslant\varepsilon}}|\Delta^\alpha x_k-L|^p+\frac{1}{h_r^\gamma}\sum_{\substack{k\in I_r\\ |\Delta^\alpha x_k-L|<\varepsilon}}|\Delta^\alpha x_k-L|^p\\ &\leqslant \left(\frac{\ell_r}{h_r^\gamma}-1\right)M^p+\frac{M^p}{h_r^\gamma}|\{k\in I_r:|\Delta^\alpha x_k-L|\geqslant\varepsilon\}|+\frac{h_r}{h_r^\gamma}\varepsilon^p\\ &\leqslant \left(\frac{\ell_r}{h_r^\gamma}-1\right)M^p+\frac{M^p}{h_r^\beta}|\{k\in I_r:|\Delta^\alpha x_k-L|\geqslant\varepsilon\}|+\frac{\ell_r}{h_r^\gamma}\varepsilon^p\end{aligned}$$

for all $r\in\mathbb{N}$. Using (2.2) we obtain that $\Delta^\alpha(N_{\theta'}^\gamma,p)$-statistically convergent to L, whenever $\Delta^\alpha(S_\theta^\beta)$-summable to L.

3. Results Related to Modulus Function

In this section , we give the inclusion relations between the sets of $\Delta^\alpha-$ lacunary statistically convergent sequences of order β and strongly $\Delta^\alpha(N_\theta^\beta(f,(p))-$summable sequences with respect to the modulus function f.

A modulus function is a function $f:[0,\infty)\to[0,\infty)$, such that

i) $f(x)=0$ if and only if $x=0$,
ii)$f(x+y)\leqslant f(x)+f(y)$ for $x,y\geqslant 0$,
iii) f is increasing,
iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0,\infty)$.

Definition 3.1. *Let f be a modulus function, $\theta = (k_r)$ be a lacunary sequence, $\beta \in (0,1]$, α be a proper fraction and $p = (p_k)$ be a sequence of strictly positive real numbers. Now, we define the sequence space $\Delta^\alpha(N_\theta^\beta, f, (p))$ as follows:*

$$\Delta^\alpha(N_\theta^\beta, f, (p)) := \left\{ x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r^\beta} \sum_{k\in I_r} \left[f\left(|\Delta^\alpha x_k - L| \right) \right]^{p_k} = 0, \quad \text{for some} \quad L \right\}.$$

In the special case $p_k = p$ (for all $k \in \mathbb{N}$) and $f(x) = x$, we shall write $\Delta^\alpha(N_\theta^\beta, p)$ instead of $\Delta^\alpha(N_\theta^\beta, f, (p))$. If $x \in \Delta^\alpha(N_\theta^\beta, f, (p))$, then we say that x is strongly $\Delta^\alpha(N_\theta^\beta, f, (p))-$summable with respect to the modulus function f and write $x_k \to L(\Delta^\alpha(N_\theta^\beta, f, (p)))$.

In the following theorems, we shall assume that the sequence $p = (p_k)$ is bounded and $0 < h = \inf_k p_k \leqslant p_k \leqslant \sup_k p_k = H < \infty$.

Theorem 3.2. *Let $\beta, \gamma \in (0,1]$ be real numbers such that $\beta \leqslant \gamma$, f be a modulus function and $\theta = (k_r)$ be a lacunary sequence, then $\Delta^\alpha(N_\theta^\beta, f, (p)) \subset \Delta^\alpha(S_\theta^\gamma)$.*

Proof. Let $x \in \Delta^\alpha(N_\theta^\beta, f, (p))$ and $\varepsilon > 0$ be given and $\sum_1$ and $\sum_2$ denote the sums over $k \in I_r$, $|\Delta^\alpha x_k - L| \geqslant \varepsilon$ and $|\Delta^\alpha x_k - L| < \varepsilon$ respectively. Since $h_r^\beta \leqslant h_r^\gamma$ for each r, we may write

$$\begin{aligned}
\frac{1}{h_r^\beta} \sum_{k\in I_r} \left[f\left(|\Delta^\alpha x_k - L| \right) \right]^{p_k} &= \frac{1}{h_r^\beta} \left[\sum_1 \left[f\left(|\Delta^\alpha x_k - L| \right) \right]^{p_k} + \sum_2 \left[f\left(|\Delta^\alpha x_k - L| \right) \right]^{p_k} \right] \\
&\geqslant \frac{1}{h_r^\gamma} \left[\sum_1 \left[f\left(|\Delta^\alpha x_k - L| \right) \right]^{p_k} + \sum_2 \left[f\left(|\Delta^\alpha x_k - L| \right) \right]^{p_k} \right] \\
&\geqslant \frac{1}{h_r^\gamma} \left[\sum_1 f(\varepsilon) \right]^{p_k} \\
&\geqslant \frac{1}{h_r^\gamma} \sum_1 \min\left([f(\varepsilon)]^h, [f(\varepsilon)]^H \right) \\
&\geqslant \frac{1}{h_r^\gamma} |\{k \in I_r : |\Delta^\alpha x_k - L| \geqslant \varepsilon\}| \min\left([f(\varepsilon)]^h, [f(\varepsilon)]^H \right).
\end{aligned}$$

Since $x \in \Delta^\alpha(N_\theta^\beta(f, (p))$, the left hand side of the above inequality tends to zero as $r \to \infty$. Therefore, the right hand side of the above inequality tends to zero as $r \to \infty$, hence $x \in \Delta^\alpha(S_\theta^\gamma)$.

Theorem 3.3. *If the modulus function f is bounded and $\lim_{r\to\infty} \frac{h_r}{h_r^\beta} = 1$, then $\Delta^\alpha(S_\theta^\beta) \subset \Delta^\alpha(N_\theta^\beta, f, (p))$.*

Proof. Let $x \in \Delta^\alpha(S_\theta^\beta)$ and suppose that f is bounded and $\varepsilon > 0$ be given. Since f is bounded there exists an integer K such that $f(x) \leqslant K$, for all $x \geqslant 0$. Then for each $r \in \mathbb{N}$ we may write

$$\begin{aligned}
\frac{1}{h_r^\beta} \sum_{k\in I_r} \left[f\left(|\Delta^\alpha x_k - L| \right) \right]^{p_k} &= \frac{1}{h_r^\beta} \sum_1 \left[f\left(|\Delta^\alpha x_k - L| \right) \right]^{p_k} + \frac{1}{h_r^\beta} \sum_2 \left[f\left(|\Delta^\alpha x_k - L| \right) \right]^{p_k} \\
&\leqslant \frac{1}{h_r^\beta} \sum_1 \max\left(K^h, K^H \right) + \frac{1}{h_r^\beta} \sum_2 \left[f(\varepsilon) \right]^{p_k} \\
&\leqslant \max\left(K^h, K^H \right) \frac{1}{h_r^\beta} |\{k \in I_r : f\left(|\Delta^\alpha x_k - L| \right) \geqslant \varepsilon\}| \\
&\quad + \frac{h_r}{h_r^\beta} \max\left(f(\varepsilon)^h, f(\varepsilon)^H \right).
\end{aligned}$$

Hence $x \in \Delta^\alpha(N_\theta^\beta, f, (p))$.

Theorem 3.4. *If* $\lim p_k > 0$ *and* $x = (x_k)$ *is strongly* $\Delta^{\alpha}(N_{\theta}^{\beta}, f, (p))$*-summable to* L *with respect to the modulus function* f*, then that limit* L *is unique.*

Proof. Omitted.

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Nazlım Deniz Aral,
Department of Mathematics,
Faculty of Arts and Sciences, Bitlis Eren University, Bitlis, Turkey.
E-mail address: `ndaral@beu.edu.tr`