Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) v. 2023 (41) : 1–13.
ISSN-0037-8712 in press
doi:10.5269/bspm.46872

On η-Einstein $N(k)$-Contact Metric Manifolds

Sunil Kumar Yadav and Xiaomin Chen*

ABSTRACT: The aim of this paper is to characterize η-Einstein $N(k)$-contact metric manifolds admits η-Ricci soliton. Several consequences of this result are discussed. Beside these, we also study η-Einstein $N(k)$-contact metric manifolds satisfying certain curvature conditions. Among others it is shown that such a manifold is either locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or a Sasakian manifold. Finally, we construct an example to verify some results.

Key Words: $N(k)$-contact metric manifold, torse forming vector field, η-Ricci soliton, η-Einstein manifold.

Contents

1. Introduction

In 1982, Hamilton [9] made the fundamental observation that Ricci flow is an excellent tool for simplifying the structure of a manifold. It is a process which deforms the metric of M by smoothing out the irregularities. It is given by

$$\frac{\partial g}{\partial t} = -2Ric, \tag{1.1}$$

where Ric is the Ricci tensor of M. Ricci soliton is a special solution to the Ricci flow and is a natural generalization of an Einstein metric. It is defined as a triplet (g, V, λ) with g as Riemannian metric, V a vector field and λ a real scalar such that

$$\frac{1}{2}(L_V g)(X,Y) + S(X,Y) + \lambda\, g(X,Y) = 0, \tag{1.2}$$

where S is the Ricci tensor of M and L_V denote the Lie derivative operator along the vector field V.

The Ricci soliton is said to be shrinking, steady and expanding accordingly as λ is negative, zero and positive respectively. In [20], Sharma initiated the study of Ricci solitons in contact Riemannian geometry. Later Tripathi [23], Nagaraja et al. [16] and others extensively studied Ricci solitons in contact metric manifolds. It is well known that, if the potential vector field is zero or Killing then the Ricci soliton is an Einstein metric. In [7], [10], [13] the authors proved that there are no Einstein real hypersurfaces of

* The second author is supported by Science Foundation of China University of Petroleum-Beijing (No.2462020XKJS02, No.2462020YXZZ004).
2010 *Mathematics Subject Classification:* 53C40, 53C25, 53C50.
Submitted March 07, 2019. Published June 09, 2021

non-flat complex space forms. Motivated by this the authors Cho and Kimura [8] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons. Later Blaga [4] studied η-Ricci solitons in para-Kenmotsu manifolds. Recently, this notion have been studied by various authors in different structures of manifolds [11], [12], [17], [19], [24], [25], [26], [27], [28], [29].

A contact manifold is a smooth $(2n+1)$-dimensional manifold M^{2n+1} equipped with a global 1-form η such that $\eta\wedge(d\eta)^n\neq 0$ everywhere. Given a contact form η, there exists a unique vector field ξ, called the characteristic vector field of η, satisfying $\eta(\xi)$=1 and $d\eta(X,\xi)$=0 for any vector field X on M^{2n+1}. A Riemannian metric g is said to be associated metric if there exists a tensor field ϕ of type $(1,1)$ such that

$$\eta(X)=g(X,\xi),\quad d\eta(X,Y)=g(X,\phi Y),\quad \phi^2(X)=-X+\eta(X)\xi, \tag{1.3}$$

for all vector fields X,Y on M^{2n+1}. Then the structure $(\phi,\,\xi,\,\eta,\,g)$ on M^{2n+1} is called a contact metric structure and the manifold M^{2n+1} equipped with such a structure is said to be a contact metric manifold [1]. It can be easily seen that in a contact metric manifold, the following relations hold.

$$\phi\xi=0,\quad \eta\circ\phi=0,\quad g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{1.4}$$

for any vector field X,Y on M^{2n+1}.
Given a contact metric manifold M^{2n+1}, we define a $(1,1)$-tensor field h by $h=\frac{1}{2}L_\xi\phi$. Then h is symmetric and satisfies,

$$h\xi=0,\ \ h\phi=-\phi h,\ \ Tr.h=Tr.\phi h=0. \tag{1.5}$$

If ∇ denotes the Riemannian connection of g, then we have the following relation

$$\nabla_X\xi=-\phi X-\phi hX, \tag{1.6}$$

A contact metric manifold M^{2n+1} for which ξ is a Killing vector field is called a K-contact manifold. A contact metric manifold is Sasakian if and only if

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y, \tag{1.7}$$

where R is the Riemannian curvature tensor of type $(1,3)$.
In 1988, Tanno [22] introduced the notion of k-nullity distribution of a contact metric manifold as a distribution such that the characteristic vector field ξ of the contact metric manifold belongs to the distribution. The contact metric manifold with ξ belonging to the k-nullity distribution is called $N(k)$-contact metric manifold and such a manifold is also studied by various authors. Generalizing this notion in 1995, Blair, Koufogiorgos and Papantoniou [3] introduced the notion of a contact metric manifold with ξ belonging to the (k,μ)-nullity distribution, where k and μ are real constants. In particular, if μ=0, then the notion of (k,μ)-nullity distribution reduces to the notion of k-nullity distribution.

Motivated by these studies, the present paper explores the study of η-Ricci solitons on η-Einstein $N(k)$-contact metric manifold. The paper organized as follows. After introduction. Section 2 is concerned with the fundamental concept of $N(k)$-contact metric manifold. We provided some known results related to η-Einstein $N(k)$-contact metric manifold in section 3. In Section 4 we have investigated η-Ricci soliton on η-Einstein $N(k)$-contact metric manifold and it is observed that such a manifold is Sasakian manifold. In section 5, we consider second order parallel tensor on η-Einstein $N(k)$-contact metric manifold and we obtain several results. Also we have discuss about Ricci semi symmetric η-Einstein $N(k)$-contact metric manifold and prove that either the manifold is locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ or the manifold is an Einstein in section 6. In section 7, we also mention several results for different type of W_2-curvature restrictions on such manifold. Finally, we have constructed an example of $N(k)$-contact metric manifold.

2. $N(k)$-Contact Metric Manifolds

Let us consider a contact metric manifold $M^{2n+1}(\phi,\,\xi,\,\eta,\,g)$. The k-nullity distribution [22] of a Riemainnian manifold (M,g) for a real number k is a distribution

$$N_p(k)=\{Z\in T_pM:R(X,Y)Z=k[g(Y,Z)X-g(X,Z)Y]\} \tag{2.1}$$

for any $X, Y \in T_pM$. Hence if the characteristic vector field ξ of a contact metric manifold belongs to the k-nullity distribution, then we have

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y]. \tag{2.2}$$

Thus a contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ satisfying the relation (2.2) is called a $N(k)$-contact metric manifold. From (1.7) and (2.2) it follows that a $N(k)$-contact metric manifold is a Sasakian manifold if and only if k=1. On the other-hand if k=0, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for n=1 [2]. Also in a $N(k)$-contact metric manifold, k is always constant such that $k \leq 1$ [22].

The (k,μ)-nullity distribution of a contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ is a distribution [3]

$$\begin{aligned} N_p(k,\mu) = \{Z \in T_pM : R(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y] \\ + \mu[g(Y,Z)hX - g(X,Z)hY]\} \end{aligned} \tag{2.3}$$

for any $X, Y \in T_pM$, where k, μ are real constants. Hence if the characteristic vector field ξ belongs to the (k,μ)-nullity distribution, then

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{2.4}$$

A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ satisfying the relation (2.4) is called a $N(k,\mu)$-contact metric manifold or simply a (k,μ)-contact metric manifold. In particular, if μ=0, then the relation (2.4) reduces to (2.2) and hence a $N(k)$-contact metric manifold is a $N(k,0)$-contact metric manifold.

Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a $N(k)$-contact metric manifold. Then the following relations hold [21], [22].

$$Q\phi - \phi Q = 4(n-1)h\phi, \tag{2.5}$$

$$h^2 = (k-1)\phi^2, \; k \leq 1, \tag{2.6}$$

$$Q\xi = 2nk\xi, \tag{2.7}$$

$$R(\xi, X, Y) = k[g(X,Y)\xi - \eta(Y)X], \tag{2.8}$$

where Q is the Ricci operator, i.e., $g(QX,Y)$=$S(X,Y)$. In view of (1.4) and (1.5), it follows from (2.5)-(2.8) that

$$Tr.h^2 = 2n(1-k), \tag{2.9}$$

$$S(X,\phi Y) + S(\phi X, Y) = 2(2n-2)g(\phi X, hY), \tag{2.10}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 2(2n-2)g(hX,Y), \tag{2.11}$$

$$Q\phi + \phi Q = 2\phi Q + 2(2n-2)h\phi, \tag{2.12}$$

$$\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.13}$$

$$S(\phi X, \xi) = 0, \tag{2.14}$$

for any vector field X and Y on M^{2n+1}. Also in a $N(k)$-contact metric manifold the scalar curvature r is given by [3], [21]

$$r = 2n(2n-2+k). \tag{2.15}$$

Given a non-Sasakian (k,μ)-contact manifold, Boeckx [6] introduced an invariant

$$I_M = \frac{1 - \frac{\mu}{2}}{\sqrt{1-k}}$$

and showed that for two non-Sasakian (k,μ)-contact metric manifolds M_1 and M_2, we have $I_{M_1} = I_{M_2}$ if and only if up to a D-homothetic deformation, the two manifolds are locally isometric as contact metric manifolds. Thus, we see that from all non-Sasakian (k,μ)-manifolds of dimension $(2n+1)$ and for every possible value of the invariant I_M, one (k,μ)-manifold M can be obtained. For $I_M > -1$ such examples may be found from the standard contact metric structure on the tangent sphere bundle of a manifold of constant curvature c where $I_M = \frac{1+c}{1-c}$. Boeckx also gives a Lie algebra construction for any odd dimension and value of $I_M < -1$.

Remark 2.1. *Using this invariant, Blair, Kim and Tripathi [5] constructed an example of a $(2n+1)$-dimensional $N(1-\frac{1}{n})$-contact metric manifold $n > 1$. It is given as since the Boeckx invariant for a $(1-\frac{1}{n}, 0)$-manifold is $\sqrt{n} > -1$, we consider the tangent sphere bundle of an $(n+1)$-dimensional manifold of constant curvature c so chosen that the resulting D-homothetic deformation will be a $(1-\frac{1}{n}, 0)$-manifold. That is, for $k = c(2-c)$ and $\mu = -2c$, we solve*

$$1 - \frac{1}{n} = \frac{k + a^2 - 1}{a^2}, \quad 0 = \frac{\mu + 2a - 2}{a},$$

for a and c. We have

$$c = \frac{\sqrt{n} \pm 1}{n-1}, \quad a = 1 + c,$$

and taking c and a to be these values we obtain $N(1-\frac{1}{n})$-contact metric manifold.

Before going to our main work, we recall the following definition and proposition which will be used later on.

Definition 2.2. *[14], [15] A vector field ξ is called torse forming if it satisfies*

$$\nabla_X \xi = \psi X + \gamma(X)\xi, \tag{2.16}$$

for a smooth function $\psi \in C^\infty(M)$ and γ is an 1-form, for all vector field X on M. In particular, if ψ=0 then a torse forming vector field ξ is called recurrent. Also if $\psi = 1$ and $\gamma = 0$ then ξ is called concurrent vector field.

Proposition 2.3. *[2] A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ satisfying the condition $R(X,Y)\xi = 0$ for all X, Y is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive curvature 4, i.e., $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for n=1.*

3. η-Einstein $N(k)$-Contact Metric Manifold

Definition 3.1. *A $N(k)$-contact metric manifold $M^{2n+1}(\phi,\ \xi,\ \eta,\ g)$ is said to be η-Einstein if*

$$S = c_1 g + c_2 \eta \otimes \eta,$$

where c_1, c_2 are smooth functions on M^{2n+1}.

Proposition 3.2. *In an η-Einstein $N(k)$-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g, c_1, c_2)$, $(n > 1)$, the following relations satisfy*

$$S(X,Y) = (2n-2)g(X,Y) + (2n(k-1)+2)\eta(X)\eta(Y), \tag{3.1}$$

$$S(\phi X, \phi Y) = -S(X,Y) - 2nk\eta(X)\eta(Y), \tag{3.2}$$

$$S(\phi X, Y) = -S(X, \phi Y) = (2n-2)g(\phi X, Y), \tag{3.3}$$

$$S(X, \xi) = 2nk\eta(X), \tag{3.4}$$

$$S(\xi, \xi) = 2nk. \tag{3.5}$$

4. η-Ricci solitons on $M^{2n+1}(\phi,\ \xi,\ \eta,\ g)$

The governing equation of η-Ricci soliton is given by [8]

$$\frac{1}{2}(L_V g) + S(X,Y) + \lambda\, g(X,Y) + \mu\eta(X)\eta(Y) = 0, \tag{4.1}$$

where λ, μ are real constants. In view of (1.6), the equation (4.1) becomes

$$S(X,Y) = -\lambda\, g(X,Y) - \mu\eta(X)\eta(Y) + g(\phi h X, Y). \tag{4.2}$$

From (4.2), we have

$$QX = -\lambda X - \mu\eta(X)\xi + \phi hX, \tag{4.3}$$

$$Q\xi = -(\lambda + \mu)\xi, \tag{4.4}$$

$$r = -(2n+1)\lambda - \mu, \tag{4.5}$$

where r is the scalar curvature. There are two natural situations regarding the vector field $V : V \in Span\ \xi$ and $V \perp \xi$. Here we investigate only the case $V{=}\xi$. Consequently, we prove the following result.

Theorem 4.1. *An η-Einstein $N(k)$-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g, c_1, c_2), (n > 1)$ admits η-Ricci soliton whose potential vector field is the Reeb vector field ξ if and only if the manifold is Sasakian.*

Proof. In view of (3.4) and (4.4), we obtain

$$\lambda + \mu = -2nk. \tag{4.6}$$

Using (4.6) in (4.2), we get

$$S(X, Y) = -\lambda\, g(X, Y) + (\lambda + 2nk)\eta(X)\eta(Y) + g(\phi hX, Y). \tag{4.7}$$

Replacing X by ϕX in (4.7) it yield

$$S(\phi X, Y) = -\lambda\, g(\phi X, Y) - g(hX, Y). \tag{4.8}$$

Also from (3.3), we have

$$S(\phi X, Y) = (2n-2)g(\phi X, Y). \tag{4.9}$$

Equating the right hand side of (4.8) and (4.9), we get

$$g(hX, Y) = (\lambda + 2n - 2)g(X, \phi Y). \tag{4.10}$$

Again replacing X by Y in (4.10) it turn up

$$g(hY, X) = (\lambda + 2n - 2)g(Y, \phi X). \tag{4.11}$$

Adding (4.10) and (4.11), we gave $g(hX, Y){=}0$, which gives $h{=}0$ and hence from (2.6) it follows that $k{=}1$. Therefore the manifold is Sasakian. The converse is trivial. This prove the theorem. □

Theorem 4.2. *If $M^{2n+1}(\phi, \xi, \eta, g, \lambda, \mu, c_1, c_2), (n > 1)$ be an η-Ricci soliton on an η-Einstein $N(k)$-contact metric manifold, the we have (i) $\lambda + \mu = -2nk$, (ii) ξ is a geodesic vector field, (iii) $(\nabla_\xi \phi)\xi = 0$, (iv) $\nabla_\xi S = 0$, $\nabla_\xi Q = 0$.*

Proof. Again, we consider $M^{2n+1}(\phi, \xi, \eta, g, \lambda, \mu, c_1, c_2), (n > 1)$ be an η-Ricci soliton on η-Einstein $N(k)$-contact metric manifold. Then from (1.2) and (3.1), we have

$$\begin{aligned} g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + [2\lambda + 2(2n-2)]g(X, Y)& \\ + [2\mu + 2(2n(k-1) + 2)]\eta(X)\eta(X) = 0.& \end{aligned} \tag{4.12}$$

Replacing X and Y by ξ in (4.12) and using (1.6), we get $\lambda + \mu = -2nk$. Since ξ has a constant norm. Thus we get the result (i). Also from (4.12), we have

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + (2\lambda + 2(2n-2))[g(X, Y) - \eta(X)\eta(X)] = 0. \tag{4.13}$$

Taking $Y{=}\xi$ in (4.13), we get $g(\nabla_\xi \xi, X){=}0$ for any vector field X on M. This implies that ξ is a geodesic vector field. So we get the result(ii). As per this consequence we can easily obtain the results (iii) and (iv). Thus the proof is complete □

Theorem 4.3. *If ξ is a torse forming η-Ricci soliton on an η-Einstein $N(k)$-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g, \lambda, \mu, c_1, c_2), (n > 1)$ then we have (i) $\psi = -2(\lambda + 2n - 2), \eta$ is closed, (ii) $k = (2\lambda + 4n - 2)^2$, (iii) $\mu = \lambda + 2n(2n + 4n - 4)^2$.*

Proof. We consider ξ is a torse forming on $M^{2n+1}(\phi,\, \xi,\, \eta,\, g,\, \lambda,\, \mu,\, c_1,\, c_2), (n > 1)$. Then we get from (2.16) $\gamma = -\psi\eta$. So equation (2.16) reduces to

$$\nabla_X \xi = \psi[X - \eta(X)\xi]. \tag{4.14}$$

In view of (4.13) and (4.14), we obtain

$$(\psi + 2(\lambda + 2n - 2))\,\{g(X, Y) - \eta(X)\eta(Y)\} = 0.$$

This implies that $\psi = -2(\lambda + 2n - 2)$. Consequently (4.14) reduces to

$$\nabla_X \xi = 2(\lambda + 2n - 2)[-X + \eta(X)\xi]. \tag{4.15}$$

It is clear that $\nabla_X \xi$ is collinear to $\phi^2 X$ for all X. Thus we get η is closed.

On the other-hand, we have

$$R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi. \tag{4.16}$$

Taking account of equations (4.15) and (4.16), we obtain

$$R(X, Y)\xi = (2\lambda + 2n - 2)^2[\eta(X)Y - \eta(Y)X], \tag{4.17}$$

$$S(X, \xi) = 2n(2\lambda + 2n - 2)^2\eta(X). \tag{4.18}$$

With the help of (3.4), we get from (4.18) that $k = (2\lambda + 4n - 2)^2$ and $\mu = \lambda + 2n(2n + 4n - 4)^2$. Thus we get required result. □

If ξ is recurrent then ψ=0 and hence λ=-$(2n-n)$. Therefore from (4.14), we have the following result.

Corollary 4.4. *If ξ is a recurrent torse forming η-Ricci soliton on an η-Einstein $N(k)$-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g, \lambda, \mu, c_1, c_2), (n > 1)$ then (i) ξ is concurrent vector field, (ii) ξ is Killing vector field.*

Corollary 4.5. *If ξ is a torse forming Ricci soliton on an η-Einstein $N(k)$-contact metric manifold $M^{2n+1}(\phi,\, \xi,\, \eta,\, \lambda,\, c_1,\, c_2, g), (n > 1)$ then the Ricci soliton is always shrinking.*

5. Second order parallel tensors on η-Einstein $N(k)$-Contact Metric manifold

Definition 5.1. *A tensor α of second order is said to be a parallel tensor if $\nabla\alpha = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.*

Let ξ be a torse forming η-Ricci soliton on $M^{2n+1}(\phi,\, \xi,\, \eta,\, g,\, \lambda,\, \mu,\, c_1,\, c_2), (n > 1)$. If α be a symmetric tensor field of type $(0, 2)$ such that $\nabla\alpha = 0$. Then it follows that

$$\alpha(R(X, Y)Z, W) + \alpha(Z, R(X, Y)W) = 0, \tag{5.1}$$

for arbitrary vector fields X, Y, Z and Z on $M^{2n+1}(\phi,\, \xi,\, \eta,\, g,\, \lambda,\, \mu,\, c_1,\, c_2)$. The substitution of X=Z=W =ξ in (5.1) which gives us $\alpha(\xi, R(\xi, Y)\xi) = 0$, since α is symmetric. Using (4.17) in the above equation, we get

$$(2\lambda + 2n - 2)^2\,\{\alpha(Y, \xi) - \eta(Y)\alpha(\xi, \xi)\} = 0. \tag{5.2}$$

From (5.2) it follows that $\lambda + n - 1 \neq 0$. Hence we have

$$\alpha(Y, \xi) - \eta(Y)\alpha(\xi, \xi) = 0. \tag{5.3}$$

Moreover, by differentiating (5.3) covariantly along X and using (4.15), we obtain

$$\alpha(X, Y) = \alpha(\xi, \xi)g(X, Y). \tag{5.4}$$

Differentiating (5.4) covariantly along any vector field on M, it can be easily seen that $\alpha(\xi, \xi)$ is constant. Hence we can state the following theorem

Theorem 5.2. *If the torse forming η-Ricci soliton on an η-Einstein $N(k)$-contact metric manifold $M^{2n+1}(\phi,$
$, \xi, \eta, g, \lambda, \mu, c_1, c_2), (n > 1)$ is regular, then any parallel symmetric $(0,2)$ tensor field is a constant multiple of the metric.*

Corollary 5.3. *The η-Ricci soliton on an η-Einstein $N(k)$-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g,$
$, \lambda, \mu, c_1, c_2)$ is regular if $\lambda + n - 1 \neq 0$.*

Next, we prove the following result.

Theorem 5.4. *If the Ricci tensor S of an η-Einstein $N(k)$-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g, c_1, c_2)$, $(n > 1)$ is one of the followings: (i) cyclic parallel, then $k{=}1 - \frac{1}{n}$, that is, it is locally isometric to Example 2.1 or Sasakian. (ii) cyclic parallel η-recurrent, then the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$.*

Proof. It is well known that

$$(\nabla_X S)(Y, Z) = XS(Y, Z) + S(\nabla_X Y, Z) + S(Y, \nabla_X Z). \tag{5.5}$$

In view of (1.6) and (3.1), the equation (5.5) reduces to

$$\begin{aligned}(\nabla_X S)(Y, Z) = -(2n(k-1)+2)\,&\{g(Y, \phi X) + g(Y, \phi hX)\}\,\eta(Z)\\ &+ \{g(Z, \phi X) + g(Z, \phi hX)\}\,\eta(Z)\end{aligned} \tag{5.6}$$

If possible, we suppose that the Ricci tensor S of M is cyclic parallel, that is,

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0. \tag{5.7}$$

The cyclic sum of (5.6) together with the last argument at $Y{=}\xi$ and (5.7) give

$$(2n(k-1)+2)\,\{g(\phi Z, \phi hX) - g(\phi Z, \phi X)\} = 0. \tag{5.8}$$

It follows from (5.8) that either $k{=}1 - \frac{1}{n}$ or,

$$g(\phi Z, \phi hX) - g(\phi Z, \phi X) = 0. \tag{5.9}$$

If $k = 1 - \frac{1}{n}$. Then we required result (i), that is, it is locally isometric to Remark 2.1.
Next, putting Z by ϕZ in (5.9) and using (1.3), we get

$$g(Z, \phi X) - g(Z, \phi hX) = 0. \tag{5.10}$$

Again, replacing X by ϕX in (5.10) and using (1.5) we obtain

$$-g(Z, X) + \eta(Z)\eta(X) - g(Z, hX) = 0. \tag{5.11}$$

In view of (5.10) and (5.11), it yield

$$g(Z, hX) = 0,$$

for all X. Therefore, we must have $h{=}0$ and hence from (2.6) it follows that $k{=}1$. So the manifold is Sasakian.
To prove the result (ii), we suppose that manifold is η-recurrent, that is, $(\nabla_X S)(Y, Z) = \eta(X)S(Y, Z)$, $\forall X, Y, Z \in \chi(M)$. If the Ricci tensor S of the η-recurrent η-Einstein $N(k)$-contact metric manifold is cyclic parallel, then

$$\begin{aligned}&\eta(X)S(Y, Z) + \eta(Y)S(Z, X) + \eta(Z)S(X, Y)\\ =&(2n-2)[g(Y, Z)\eta(X) + g(Z, X)\eta(Y) + g(X, Y)\eta(Z)]\\ &+ 6n(k-1) + 2[\eta(X)\eta(Y)\eta(Z)],\end{aligned} \tag{5.12}$$

for any $X, Y, Z \in \chi(M)$.
Taking $Y{=}Z{=}\xi$ in (5.12), we obtain $6nk\,\eta(X){=}0$, for any $X \in \chi(M)$. It follows that $k{=}0$, then from (2.2) we have $R(X, Y)\xi{=}0$, for all X, Y. So by Proposition 2.1, it follows that the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$. Thus theorem is proved. □

In particular, if consider a cyclic parallel 3-dimensional η-Einstein $N(k)$-contact metric manifold. Then n=1 and in that case, we have k=0. Hence, in view of Proposition 2.1, we have the result.

Corollary 5.5. *A 3-dimensional η-Einstein $N(k)$-contact metric manifold is cyclic parallel if and only if the manifold is flat.*

.

6. Ricci semisymmetric η-Einstein $N(k)$-Contact Metric manifold

In this section we discuss about the Ricci semisymmetric η-Einstein $N(k)$-contact metric manifolds. Then

$$R(X,Y)\cdot S=0.$$

This is equivalent to

$$(R(X,Y)\cdot S)(U,V)=0, \tag{6.1}$$

for any $X,Y,U,V\in\chi(M)$. From (6.1), we have

$$S(R(X,Y)U,V)+S(U,R(X,Y)V)=0. \tag{6.2}$$

Substituting X=U=ξ in (6.2), we get

$$S(R(\xi,Y)\xi,V)+S(\xi,R(\xi,Y)V)=0. \tag{6.3}$$

Using (2.8) and (3.4) we have from (6.3) that

$$k\left\{2nkg(Y,V)-S(Y,V)\right\}=0.$$

This implies either k=0 or

$$S(Y,V)=2nkg(Y,V). \tag{6.4}$$

If k=0, then from (2.2) we have

$$R(X,Y)\xi=0,$$

for all X,Y.

Therefore by virtue of Proposition 2.3, it follows that the manifold is locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$. Also (6.4) implies that the manifold is an Einstein. Hence we can state the following result.

Theorem 6.1. *If an η-Einstein $N(k)$-contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g),(n>1)$, is Ricci semisymmetric then either the manifold is locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ or the manifold is an Einstein.*

Again, Ricci symmetry ($\nabla S=0$) implies Ricci semisymmetric ($R\cdot S=0$), thus we have the following.

Corollary 6.2. *If an η-Einstein $N(k)$-contact metric manifold $M^{2n+1}(\phi,\xi,\eta,g),(n>1)$, is Ricci symmetric then either the manifold is locally isometric to the Riemannian product $E^{n+1}(0)\times S^n(4)$ or the manifold is an Einstein.*

7. η-Einstein $N(k)$-contact metric manifold satisfying certain curvature conditions

In 1970, Pokhariyal et al. [18] defined and studied the properties of W_2-curvature tenor, and is given by

$$W_2(X,Y)Z=R(X,Y)Z+\frac{1}{n-1}\left\{g(X,Z)QY-g(Y,Z)QX\right\} \tag{7.1}$$

for any $X,Y,Z\in\chi(M)$.

We discuss certain curvature conditions, that is, $R(\xi,X)\cdot S=0$, $S\cdot R(\xi,X)=0$, $W_2(\xi,X)\cdot S=0$ and $S\cdot W_2(\xi,X)=0$ on an η-Einstein $N(k)$-contact metric manifolds and deduce some results.

Theorem 7.1. *If an η-Einstein $N(k)$-contact metric manifold $M^{2n+1}(\phi,\, \xi,\, \eta,\, g,\, c_1,\, c_2), (n > 1)$ satisfies $R(\xi, X) \cdot S = 0$ then either the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or the manifold is an Einstein.*

Proof. Suppose $M^{2n+1}(\phi,\, \xi,\, \eta,\, g, c_1,\, c_2)$ satisfies the condition $R(\xi, X) \cdot S = 0$. Then we have

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0, \tag{7.2}$$

for any $X, Y, Z \in \chi(M)$. Using (2.8) and (3.4) in (7.2), we get

$$k\, \{2nkg(X, Y)\eta(Z) - S(X, Z)\eta(X) + 2nkg(X, Z)\eta(Y) - S(X, Y)\eta(Z)\} = 0. \tag{7.3}$$

For $Z{=}\xi$, equation (7.3), have

$$k[2nkg(X, Y) - S(X, Y)] = 0. \tag{7.4}$$

This implies either $k = 0$ or,

$$S(X, Y) = 2nkg(X, Y).$$

If $k = 0$, then from (2.2) we obtain

$$R(X, Y)\xi = 0,$$

for all X, Y.

Consequently, it follows that the manifold is either locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or an Einstein. This complete the proof. □

Theorem 7.2. *If an η-Einstein $N(k)$-contact metric manifold $M^{2n+1}(\phi,\, \xi,\, \eta,\, g,\, c_1,\, c_2), (n > 1)$ satisfies $S(\xi, X) \cdot R = 0$ then either the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or an η-Einstein.*

Proof. Let the condition $S(\xi, X) \cdot R = 0$ holds on $M^{2n+1}(\phi,\, \xi,\, \eta,\, g,\, c_1,\, c_2)$. Then this implies that

$$S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0, \tag{7.5}$$

for any $X, Y, Z \in \chi(M)$.

Equation (7.5) can be written as

$$\begin{aligned} &S(X, R(Y, Z)W)\xi - S(\xi, R(Y, Z)W)X + S(X, Y)R(\xi, Z)W \\ &- S(\xi, Y)R(X, Z)W + S(X, Z)R(Y, \xi)W - S(\xi, Z)R(Y, X)W \\ &+ S(X, W)R(Y, Z)\xi - S(\xi, W)R(Y, Z)X = 0. \end{aligned} \tag{7.6}$$

Taking the inner product with ξ, the relation (7.6) turn up

$$\begin{aligned} &S(X, R(Y, Z)W) - S(\xi, R(Y, Z)W)\eta(X) + S(X, Y)\eta(R(\xi, Z)W) \\ &- S(\xi, Y)\eta(R(X, Z)W) + S(X, Z)\eta(R(Y, \xi)W) \\ &-S(\xi, Z)\eta(R(Y, X)W) + S(X, W)\eta(R(Y, Z)\xi) - S(\xi, W)\eta(R(Y, Z)X) = 0. \end{aligned} \tag{7.7}$$

With the help of (2.2), (2.8), (2.13) and (3.4), Equation (7.7) reduces to for $Z{=}W{=}\xi$.

$$k[S(X, Y) + 2nkg(X, Y) - 4nk\eta(X)\eta(Y)] = 0.$$

This implies either $k{=}0$ or,

$$S(X, Y) = -2nkg(X, Y) + 4nk\eta(X)\eta(Y).$$

If $k{=}0$, then from (2.2) we have

$$R(X, Y)\xi = 0,$$

for all X, Y. Thus by Proposition 2.3, it follows that the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or an η-Einstein. This leads to the proof. □

Theorem 7.3. *If an η-Einstein $N(k)$-contact metric manifolds $M^{2n+1}(\phi, \xi, \eta, g, c_1, c_2), (n > 1)$ satisfies $W_2(\xi, X) \cdot S = 0$ then either the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or an η-Einstein.*

Proof. The condition satisfied by S is $W_2(\xi, X) \cdot S = 0$. Therefore

$$S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z) = 0, \tag{7.8}$$

for any $X, Y, Z \in \chi(M)$.

Using (2.8),(3.4),(3.5) and (7.1) in (7.8) it yields

$$\begin{aligned} &k\{2nkg(X,Y)\eta(Z) - S(X,Z)\eta(Y) + 2nkg(X,Z)\eta(Y) - S(X,Y)\eta(Z)\} \\ &+ \frac{1}{2n}\{2nkS(X,Z)\eta(Y) - (2nk)^2 g(X,Y)\eta(Z) \\ &+ 2nkS(X,Y)\eta(Z) - (2nk)^2 g(X,Z)\eta(Y)\} = 0 \end{aligned} \tag{7.9}$$

Replacing Z by ξ in (7.9), we obtain

$$k[S(X,Y) + 2nkg(X,Y) - 4nk\eta(X)\eta(Y)] = 0.$$

This implies either k=0 or

$$S(X,Y) = -2nkg(X,Y) + 4nk\eta(X)\eta(Y).$$

If k=0, then from (2.2) we have

$$R(X,Y)\xi = 0,$$

for all X, Y.

As per guideline of Proposition 2.3, it follows that the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or an η-Einstein. This complete the proof. □

Theorem 7.4. *If an η-Einstein $N(k)$-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g, c_1, c_2), (n > 1)$ satisfies $S(\xi, X) \cdot W_2 = 0$ then the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or an η-Einstein.*

Proof. The condition $S(\xi, X) \cdot W_2 = 0$ on $M^{2n+1}(\phi, \xi, \eta, g, c_1, c_2)$ indicate that

$$\begin{aligned} &S(X, W_2(Y,Z)V)\xi - S(\xi, W_2(Y,Z)V)X + S(X,Y)W_2(\xi,Z)V \\ &- S(\xi,Y)W_2(X,Z)V + S(X,Z)W_2(Y,\xi)V - S(\xi,Z)W_2(Y,X)V \\ &+ S(X,V)W_2(Y,Z)\xi - S(\xi,V)W_2(Y,Z)X = 0 \end{aligned} \tag{7.10}$$

for any $X, Y, Z, V \in \chi(M)$.

Taking the inner product with ξ, the relation (7.10) reduces to

$$\begin{aligned} &S(X, W_2(Y,Z)V) - S(\xi, W_2(Y,Z)V)\eta(X) + S(X,Y)\eta(W_2(\xi,Z)V) \\ &- S(\xi,Y)\eta(W_2(X,Z)V) + S(X,Z)\eta(W_2(Y,\xi)V) - S(\xi,Z)\eta(W_2(Y,X)V) \\ &+ S(X,V)\eta(W_2(Y,Z)\xi) - S(\xi,V)\eta(W_2(Y,Z)X) = 0 \end{aligned} \tag{7.11}$$

For Z=V=ξ, Equation (7.11) takes the form

$$\begin{aligned} &S(X, W_2(Y,\xi)\xi) - S(\xi, W_2(Y,\xi)\xi)\eta(X) + S(X,Y)\eta(W_2(\xi,\xi)\xi) \\ &- S(\xi,Y)\eta(W_2(X,\xi)\xi) + S(X,\xi)\eta(W_2(Y,\xi)\xi) - S(\xi,\xi)\eta(W_2(Y,X)\xi) \\ &+ S(X,\xi)\eta(W_2(Y,\xi)\xi) - S(\xi,\xi)\eta(W_2(Y,\xi)X) = 0. \end{aligned} \tag{7.12}$$

In view of (2.2), (2.8), (3.4), (3.5) and (7.12), we get

$$k[S(X,Y) + 2nkg(X,Y) - 4nk\eta(X)\eta(Y)] = 0.$$

This implies either k=0 or

$$S(X,Y) = -2nkg(X,Y) + 4nk\eta(X)\eta(Y).$$

If k=0, then from (2.2) we have

$$R(X,Y)\xi = 0,$$

for all X, Y.

According to Proposition 2.3, it follows that the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or an η-Einstein. Thus we get desired result. □

8. Example of 5-dimensional $N(k)$-Contact Metric manifold

We consider a 5-dimensional differentiable manifold

$$M^5 = \{(x,y,z,u,v) \in \mathbb{R}^5 \mid (x,y,z,u,v) \neq (0,0,0)\},$$

where (x,y,z,u,v) denote the standard coordinate in $\mathbb{R}^5$. Let $(e_1, e_2, e_3, e_4, e_5)$ are five vector fields in $\mathbb{R}^5$ which satisfies

$$[e_1, e_2] = -\lambda e_2, \ \ [e_1, e_3] = -\lambda e_3, \ \ [e_1, e_4] = 0, \ \ [e_1, e_5] = 0,$$
$$[e_i, e_j] = 0, \ \ where \ i, j = 2, 3, 4, 5.$$

We also define the Riemannian metric g by

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1.$$

$$g(e_1, e_i) = g(e_i, e_j) = 0, \ \ for \ \ i \neq j; i, j = 2, 3, 4, 5.$$

Let the 1−form η be $\eta(Z) = g(Z, e_1)$ for any $Z \in \chi(M^5)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1) = 0, \ \phi(e_2) = e_4, \ \ \phi(e_3) = e_5, \ \ \phi(e_4) = -e_2, \ \ \phi(e_5) = -e_3.$$

By the linearity properties of ϕ and g, we have

$$\phi^2 X = -X + \eta(X)e_1, \eta(e_1) = 1, g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for arbitrary vector fields $X, Y \in \chi(M^5)$. Moreover,

$$he_1 = 0, \ he_2 = \frac{\lambda}{2}e_4, \ he_3 = \frac{\lambda}{2}e_5, \ he_4 = \frac{\lambda}{2}e_2, \ \ he_5 = \frac{\lambda}{2}e_3.$$

We recall the Koszul's formula as

$$\begin{aligned} 2g(\nabla_X Y, Z) \ &= \ Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]) \end{aligned}$$

for arbitrary vector fields $X, Y, Z \in \chi(M^5)$. It is obvious from Koszul's formula that

$$\begin{aligned}
&\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \quad \nabla_{e_1} e_4 = 0, \quad \nabla_{e_1} e_5 = e_1, \\
&\nabla_{e_2} e_1 = -e_4 + \frac{\lambda}{2}e_2, \quad \nabla_{e_2} e_2 = -\lambda e_1, \quad \nabla_{e_2} e_3 = 0, \quad \nabla_{e_2} e_4 = 0, \quad \nabla_{e_2} e_5 = 0, \\
&\nabla_{e_3} e_1 = -e_5 + \frac{\lambda}{2}e_3, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = -\lambda e_1, \quad \nabla_{e_3} e_4 = 0, \quad \nabla_{e_3} e_5 = 0, \\
&\nabla_{e_4} e_1 = e_2 - \frac{\lambda}{2}e_4, \quad \nabla_{e_4} e_2 = 0, \quad \nabla_{e_4} e_3 = 0, \quad \nabla_{e_4} e_4 = 0, \quad \nabla_{e_4} e_5 = 0, \\
&\nabla_{e_5} e_1 = e_3 - \frac{\lambda}{2}e_5, \quad \nabla_{e_5} e_2 = 0, \quad \nabla_{e_5} e_3 = 0, \quad \nabla_{e_5} e_4 = 0, \quad \nabla_{e_5} e_5 = 0.
\end{aligned}$$

With the help of above relation, it is notice that $\nabla_X \xi = -\phi X - \phi hX$ for $\xi = e_1$. Therefore, the manifold is a contact metric manifold with the contact structure (ϕ, η, ξ, g).

Now, we find the curvature tensors as follows

$$\begin{aligned}
&R(e_1,e_2)e_1=\lambda^2 e_2,\quad R(e_1,e_2)e_2=-\lambda^2 e_1,\quad R(e_1,e_3)e_1=\lambda^2 e_3,\quad R(e_1,e_3)e_3=-\lambda^2 e_1,\\
&R(e_1,e_4)e_1=0,\quad R(e_1,e_4)e_4=0,\quad R(e_1,e_5)e_1=0,\quad R(e_1,e_5)e_5=0,\\
&R(e_2,e_3)e_2=-\lambda^2 e_3,\quad R(e_2,e_3)e_3=-\lambda^2 e_2,\quad R(e_2,e_4)e_2=0, R(e_2,e_4)e_4=0,\\
&R(e_2,e_5)e_2=0,\quad R(e_2,e_5)e_5=0,\quad R(e_3,e_4)e_3=0,\quad R(e_3,e_4)e_4=0,\\
&R(e_3,e_5)e_3=0,\quad R(e_3,e_5)e_5=0,\quad R(e_4,e_5)e_4=0,\quad R(e_4,e_5)e_5=0.
\end{aligned}$$

In view of the expressions of the curvature tensors we conclude that the manifold is a $N(-\lambda^2)$-contact metric manifold. Using the expressions of the curvature tensor we find the values of the Ricci tensors S as follows

$$S(e_1,e_1)=S(e_2,e_2)=S(e_3,e_3)=-2\lambda^2,\ S(e_4,e_4)=S(e_5,e_5)=0.$$

This shows that the manifold is Ricci semisymmetric. Let X and Y are any two vector fields given by

$$\begin{aligned}
X=&\quad a_1e_1+a_2e_2+a_3e_3+a_4e_4+a_5e_5,\\
Y=&\quad b_1e_1+b_2e_2+b_3e_3+b_4e_4+b_5e_5,
\end{aligned}$$

where $a_1,a_2,a_3,a_4,a_5,b_1,b_2,b_3,b_4,b_5\in\mathbb{R}\setminus\{0\},\ a_4b_4+a_5b_5\neq 0.$

Then we have

$$g(X,Y)=a_1b_1+a_2b_2+a_3b_3\quad S(X,Y)=-2\lambda^2[a_1b_1+a_2b_2+a_3b_3]$$

Therefore, we notice that $S(X,Y)$=-2$\lambda^2 g(X,Y)$,that is, the manifold M is an Einstein manifold. Thus Theorem 6.1 is verified.

Acknowledgments

The authors would like to express their sincere thanks to the referee and Editor-in-chief for their valuable suggestions towards to the improvement of the paper. The first author (Sunil Kumar Yadav) would like to express his gratitude to Poornima Collage of Engineering for providing administrative and technical support. The second author (Xiaomin Chen) would like to express his gratitude to the Science Foundation of China University of Petroleum-Beijing, China.

References

1. Blair, D. E., *Contact manifolds in Riemannian geometry*, Lecture Notes in Math., 509, Springer-Verlag, (1976).
2. Blair, D. E., *Two remarks on contact metric structure*, Tohoku Math. J., 29, 319-324, (1977).
3. Blair, D. E., Koufogiorgos, T., Papantoniou, B. J., *Contact metric manifolds satisfying a nullity condition*, Israel J. of Math., 19, 189-214, (1995).
4. Blaga, A. M., *η-Ricci solitons on para-Kenmotsu, manifolds*, arXiv:1402.0223v1, [math DG], (2014).
5. Blair, D. E., Kim, J. S., Tripathi, M. M., *On the concircular curvature tensor of a contact metric manifold*, J. Korean Math. Soc. 42, 883-892, (2005).
6. Boeckx, E. *A full classification of contact metric (κ,μ)-spaces*, Illinois J. Math., 44(1), 212-219, (2000).
7. Cecil, T. E., Ryan, P. J., *Focal sets and real hypersurfaces in complex projective space*, Trans. Amer. Math. Soc., 269, 481-499, (1982).
8. Cho, J. T., Kimura, M., *Ricci solitons and Real hypersurfaces in a complex space form*, Tohoku math. J., 61, 205-212, (2009).
9. Hamilton, R. S., *The Ricci flow on surfaces, Mathematical and general relativity*(Santa Cruz,CA,1986), American Math. Soc., Contemp. Math., 71, 237-262, (1988).
10. Ki, U-H., *Real hypersurfaces with parallel Ricci tensor of a complex space form*, Tsukaba J. Math., 13, 73-81, (1989).
11. Hui, S. K., Yadav, S. K., Chaubey, S. K. *η-Ricci soliton on 3-dimensional f-Kenmotsu manifolds*, Appl. Appl. Math., 13(2), 933-951, (2018).
12. Hui, S. K., Yadav, S. K., Patra, A. *Almost conformal Ricci soliton on f-Kenmotsu manifolds*, Khayam J. Math., 5(1), 89-104, (2019).

13. Montiel, S., *Real hypersurfaces of complex hyperbolic space*, J.Math. Soc. Japan, 35, 515-535, (1985).

14. Mikes, J., Rachånek, L., *Torse forming vector fields in T-semisymmetric Riemannian spaces.* In: Steps in Diff. Geom., Proc. of the Colloquium on Diff. Geom., Univ. Debrecen, Debrecen, Hungary, 219-229, (2000).

15. Mikes, J. et al., *Differential Geometry of Special Mappings*, Palacky Univ. Press, Olomouc, (2015).

16. Nagaraja, H. G., Prematha, C. R., *Ricci solitons in Kenmotsu manifolds*, Journal of Mathematical analysis, 3 (2), 18-24, (2012).

17. Prakasha, D. G., Hadimani, B. S., *η-Ricci solitons on para-Sasakian manifolds*, J. Geom.,DOI 10.1007/s00022-016-0345-z.

18. Pokhariyal, G. P., Mishra, R. S., *The curvature tensor and their relativistic significance*, Yokohama Math. J., 18, 105-108, (1970).

19. Pokhariyal, G. P., Yadav, S. K., Chaubey, S. K. *Ricci solitons on trans-Sasakian manifolds*, Diff. Geom. and Dyn. Sys. 20, 138-158, (2018).

20. Sharma, R., *Certain results on K-contact and (κ, μ)-contact manifolds*, J.Geom., 89, 138-147, (2008).

21. Shaikh, A. A., Baishya, K. K., *On (k, μ)-contact metric manifolds*, J. Diff. Geom. and Dyn. Sys., 11, 253-261, (1906).

22. Tanno, S., *Ricci curvatures of contact Riemannian manifolds*, Tohoku Math. J., 40, 441-448, (1988).

23. Tripathi, M. M., *Ricci solitons in contact metric manifolds*, arXiv:0801, 4222v1, [math DG], (2008).

24. Yadav, S. K.,Chaubey, S. K., Suthar, D. L. *Some results on η-Ricci solitons on $(LCS)_n$-manifolds*, Surveys in Mathematics and its Applications, 13, 237-250, (2018).

25. Yadav, S. K., Chaubey, S. K., Suthar, D. L. *Some geometric properties of η-Ricci solitons and gradient Ricci solitons on $(LCS)_n$-manifolds*, CUBO A Mathematical Journal, 2(19), 33-48 (2017).

26. Yadav, S. K.,*Ricci solitons on Para-Kaehler manifolds*, Extracta Mathematikae, 34(2), 269-284, (2019).

27. Yadav, S. K., Ozturk, H., *On (ϵ)-Almost paracontact metric manifolds with conformal η-Ricci soliton*, Differential Geometry-Dynamical Systems, 19, 1-10,(2019).

28. Yadav, S. K., Kushwaha, A., Narain, D., *Certain results for η-Ricci soliton and Yamabe soliton on quasi-Sasakian 3-manifolds*, Cubo A mathematical Journal, 21(2), 77-98, (2019).

29. Yadav, S.K., Chaubey, S. K., Suthar, D.L.,*Certain geometric properties of η-Ricci soliton on η-Einstein Para-Kenmotsu manifolds*, Palestine Journal of Mathematics, 9(1), 237-244, (2020).

Sunil Kumar Yadav,
Department of Mathematics,
Poornima Collage of Engineering,Jaipur-Rajasthan
India.
E-mail address: prof_sky16@yahoo.com

and

Xiaomin Chen,
College of Science,
China University of petroleum-Beijing, Beijing, 102249
China.
E-mail address: xmchen@cup.edu.cn