Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–10.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.52758

Differential Equations for Certain Hybrid Special Matrix Polynomials

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ABSTRACT: The main aim of this article is to find the matrix recurrence relation and shift operators for the Gould-Hopper-Laguerre-Appell matrix polynomials. The matrix differential, matrix integro-differential and matrix partial differential equations are derived for these polynomials via factorization method. Certain examples are constructed in order to illustrate the applications of the results.

Key Words: Gould-Hopper-Laguerre-Appell matrix polynomials, recurrence relation, differential equations.

Contents

1. Introduction and preliminaries

It is known that the special polynomials of two variables provided new means of analysis for the solutions of large classes of partial differential equations often encountered in physical problems. The two variable forms of special polynomials are very important and perform a crucial role in solving problems of mathematical physics and engineering. These special polynomials are useful and possess potential for applications in numerous problems of number theory, combinatorics, theoretical physics, approximation theory and other fields of pure and applied mathematics.

Recently, Çekim and Aktaş [1] introduced the matrix generalization of the Gould-Hopper polynomials (GHMaP) $\mathsf{g}_n^m(x,y;A,B)$ by means of the following generating function:

$$\exp(xt\sqrt{2A})\exp(Byt^m)=\sum_{n=0}^{\infty}\mathsf{g}_n^m(x,y;A,B)\frac{t^n}{n!}, \tag{1.1}$$

where A,B are matrices in $\mathbb{C}^{N\times N}$, such that A is positive stable and m is a positive integer. The GHMaP $\mathsf{g}_n^m(x,y;A,B)$ are specified by the following series:

$$\mathsf{g}_n^m(x,y;A,B)=\sum_{k=0}^{[\frac{n}{m}]}\frac{n!\;(\sqrt{2A})^{n-mk}B^k}{(n-mk)!\;k!}x^{n-mk}y^k. \tag{1.2}$$

To introduce the hybrid forms of multi-variable special polynomials and to characterize their properties via generating function are beneficial and interesting approach. The introduction of multi-variable special functions serves as an analytical foundation for the majority of problems in mathematical physics that have been solved exactly and finds broad practical applications. Inspired by the usefulness and applications of multi-variable hybrid special polynomials, a hybrid family of special matrix polynomials, namely the Gould-Hopper-Laguerre-Appell matrix polynomials ${}_{\mathsf{g}L}\mathcal{A}_n^m(x,y,z;A,B)$ is introduced by Nahid and Khan [7] and are specified by the following generating function:

$$\mathcal{A}(t)\exp\left((x\sqrt{2A}-\hat{D}_z^{-1})t+Byt^m\right)=\sum_{n=0}^{\infty}{}_{\mathsf{g}L}\mathcal{A}_n^m(x,y,z;A,B)\frac{t^n}{n!}. \tag{1.3}$$

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2010 *Mathematics Subject Classification:* 65D20, 65F60, 35R09, 65Lxx.
Submitted March 24, 2020. Published June 10, 2021

Based on appropriate selection for the function $\mathcal{A}(t)$, different members belonging to the family of GHLAMaP ${}_{\mathtt{g}L}\mathcal{A}_n^m(x,y,z;A,B)$ are obtained. These members along with their names, generating functions and series definitions are given in Table 1.

Table 1. Certain members belonging to the GHLAMaP ${}_{\mathtt{g}L}\mathcal{A}_n^m(x,y,z;A,B)$ family

S. No.	Name and notation of the hybrid polynomials	$\mathcal{A}(t)$	Generating function	Series definition
I.	Gould-Hopper-Laguerre-Bernoulli matrix polynomials (GHLBMaP)	$\left(\frac{t}{e^t-1}\right)$	$\left(\frac{t}{e^t-1}\right)e^{(x\sqrt{2A}-\hat{D}_z^{-1})t+Byt^m} = \sum_{n=0}^{\infty}{}_{\mathtt{g}L}B_n^m(x,y,z;A,B)\frac{t^n}{n!}$	${}_{\mathtt{g}L}B_n^m(x,y,z;A,B) = n!\sum_{k=0}^{n}\sum_{l=0}^{[\frac{n}{m}]}\frac{B_{n-k-ml}(x\sqrt{2A})\ (By)^l(-z)^k}{l!\ (k!)^2(n-k-ml)!}$
II.	Gould-Hopper-Laguerre-Euler matrix polynomials (GHLEMaP)	$\left(\frac{2}{e^t+1}\right)$	$\left(\frac{2}{e^t+1}\right)e^{(x\sqrt{2A}-\hat{D}_z^{-1})t+Byt^m} = \sum_{n=0}^{\infty}{}_{\mathtt{g}L}E_n^m(x,y,z;A,B)\frac{t^n}{n!}$	${}_{\mathtt{g}L}E_n^m(x,y,z;A,B) = n!\sum_{k=0}^{n}\sum_{l=0}^{[\frac{n}{m}]}\frac{E_{n-k-ml}(x\sqrt{2A})\ (By)^l(-z)^k}{l!\ (k!)^2(n-k-ml)!}$
III.	Gould-Hopper-Laguerre-Genocchi matrix polynomials (GHLGMaP)	$\left(\frac{2t}{e^t+1}\right)$	$\left(\frac{2t}{e^t+1}\right)e^{(x\sqrt{2A}-\hat{D}_z^{-1})t+Byt^m} = \sum_{n=0}^{\infty}{}_{\mathtt{g}L}G_n^m(x,y,z;A,B)\frac{t^n}{n!}$	${}_{\mathtt{g}L}G_n^m(x,y,z;A,B) = n!\sum_{k=0}^{n}\sum_{l=0}^{[\frac{n}{m}]}\frac{G_{n-k-ml}(x\sqrt{2A})\ (By)^l(-z)^k}{l!\ (k!)^2(n-k-ml)!}$

Differential equations besides playing indispensable role in pure mathematics constitute significant part of mathematical description of physical processes. Many researchers obtained the differential equations for the mixed type special polynomials and 2D special polynomials associated with the Appell family, see for example [8,10,12,6]. Further, the q-difference equations for the q-special polynomials are also established by many authors [5,9]. In 2002, He and Ricci [3] obtained the differential equation for the Appell polynomials using the factorization method introduced by Infeld and Hull [4]. We recall some preliminaries related to the factorization method:

Let $\{p_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials such that $\deg(p_n(x)) = n,\ (n \in \mathbb{N}_0\ := \{0,1,2,\cdots\})$. The differential operators Θ_n^- and Θ_n^+ satisfying the properties:

$$\Theta_n^-\{p_n(x)\} = p_{n-1}(x) \tag{1.4a}$$

and

$$\Theta_n^+\{p_n(x)\} = p_{n+1}(x) \tag{1.4b}$$

are called lowering and raising operators, respectively. Obtaining the lowering and raising operators of a given family of polynomials gives rise to the following useful property:

$$(\Theta_{n+1}^-\Theta_n^+)\{p_n(x)\} = p_n(x). \tag{1.5}$$

The technique used in obtaining differential equations via equation (1.5) is known as the factorization method. The main idea of the factorization method is to find lowering operator Θ_n^- and raising operator Θ_n^+ such that equation (1.5) holds.

In this article, the matrix recurrence relations and shift operators for the Gould-Hopper-Laguerre-Appell matrix polynomials are obtained. In addition, the matrix differential, matrix partial differential and matrix integro-differential equations for these polynomials are derived. Certain examples are considered to derive the results for the Gould-Hopper-Laguerre-Bernoulli, Gould-Hopper-Laguerre-Euler and Gould-Hopper-Laguerre-Genocchi matrix polynomials.

2. Main results

First, we derive the matrix recurrence relation for the GHLAMaP ${}_{\mathtt{g}L}\mathcal{A}_n^m(x,y,z;A,B)$ by proving the following result:

Theorem 2.1. *For the Gould-Hopper-Laguerre-Appell matrix polynomials* ${}_{gL}\mathcal{A}_n^m(x,y,z;A,B)$*, the following matrix recurrence relation holds true:*

$$\begin{aligned} {}_{gL}\mathcal{A}_{n+1}^m(x,y,z;A,B) &= (x\sqrt{2A}+\alpha_0-\hat{D}_z^{-1})\,{}_{gL}\mathcal{A}_n^m(x,y,z;A,B) + m!\binom{n}{m-1}By \\ &\times {}_{gL}\mathcal{A}_{n-m+1}^m(x,y,z;A,B) + \sum_{k=1}^{n}\binom{n}{k}\alpha_k\ {}_{gL}\mathcal{A}_{n-k}^m(x,y,z;A,B), \end{aligned} \tag{2.1}$$

where the coefficients $\{\alpha_k\}_{k\in\mathbb{N}_0}$ *are given by*

$$\frac{\mathcal{A}'(t)}{\mathcal{A}(t)} = \sum_{k=0}^{n}\alpha_k\ \frac{t^k}{k!}. \tag{2.2}$$

Proof. Differentiating both sides of generating function (1.3) with respect to t, we find

$$\left(x\sqrt{2A}-\hat{D}_z^{-1}+mByt^{m-1}+\frac{\mathcal{A}'(t)}{\mathcal{A}(t)}\right)\mathcal{A}(t)\ e^{((x\sqrt{2A}-\hat{D}_z^{-1})t+Byt^m)} = \sum_{n=0}^{\infty}{}_{gL}\mathcal{A}_{n+1}^m(x,y,z;A,B)\frac{t^n}{n!}. \tag{2.3}$$

Using equations (1.3) and (2.2) in above equation and then on equating the coefficients of identical powers of t in both sides of the resultant equation, assertion (2.1) is proved. □

Next, the shift operators for the GHLAMaP ${}_{gL}\mathcal{A}_n^m(x,y,z;A,B)$ are obtained by proving the following result:

Theorem 2.2. *The Gould-Hopper-Laguerre-Appell matrix polynomials* ${}_{gL}\mathcal{A}_n^m(x,y,z;A,B)$ *possess the following shift operators:*

$${}_x\mathcal{L}_n^- := \frac{1}{n\sqrt{2A}}\hat{D}_x, \tag{2.4}$$

$${}_z\mathcal{L}_n^- := \frac{-1}{n}\hat{D}_z, \tag{2.5}$$

$${}_y\mathcal{L}_n^- := \frac{(\sqrt{2A})^{m-1}}{nB}\hat{D}_x^{1-m}\hat{D}_y, \tag{2.6}$$

$${}_x\mathcal{L}_n^+ := x\sqrt{2A}+\alpha_0-\hat{D}_z^{-1}+mBy(\sqrt{2A})^{1-m}\hat{D}_x^{m-1}+\sum_{k=1}^{n}\frac{\alpha_k}{k!}(\sqrt{2A})^{-k}\hat{D}_x^k, \tag{2.7}$$

$${}_z\mathcal{L}_n^+ := x\sqrt{2A}+\alpha_0-\hat{D}_z^{-1}+mBy(-1)^{m-1}\hat{D}_z^{m-1}+\sum_{k=1}^{n}\frac{\alpha_k}{k!}(-1)^k\hat{D}_z^k, \tag{2.8}$$

$$\begin{aligned} {}_y\mathcal{L}_n^+ &:= x\sqrt{2A}+\alpha_0-\hat{D}_z^{-1}+mB^{2-m}y(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{-(m-1)^2}\hat{D}_y^{m-1} \\ &+\sum_{k=1}^{n}\frac{\alpha_k}{k!}(\sqrt{2A})^{k(m-1)}\hat{D}_x^{k(1-m)}\hat{D}_y^k. \end{aligned} \tag{2.9}$$

Proof. Differentiating both sides of generating relation (1.3) with respect to x and then equating the coefficients of identical powers of t in both sides of the resultant equation, it follows that

$$\hat{D}_x\{{}_{gL}\mathcal{A}_n^m(x,y,z;A,B)\} = n\sqrt{2A}\ {}_{gL}\mathcal{A}_{n-1}^m(x,y,z;A,B),$$

so that

$${}_x\mathcal{L}_n^-\{{}_{gL}\mathcal{A}_n^m(x,y,z;A,B)\} = \frac{1}{n\sqrt{2A}}\hat{D}_x\{{}_{gL}\mathcal{A}_n^m(x,y,z;A,B)\} = {}_{gL}\mathcal{A}_{n-1}^m(x,y,z;A,B), \tag{2.10}$$

which in view of equation (1.4a), yields assertion (2.4).

Again, by taking the derivative of generating function (1.3) with respect to z and then equating the coefficients of identical powers of t in both sides of the resultant equation, it follows that

$$\hat{D}_z\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\} = -n\ {}_{\mathrm{g}L}\mathcal{A}_{n-1}^m(x,y,z;A,B).$$

Consequently, we have

$${}_z\mathcal{L}_n^-\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\} = \frac{-1}{n}\hat{D}_z\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\} = {}_{\mathrm{g}L}\mathcal{A}_{n-1}^m(x,y,z;A,B), \tag{2.11}$$

which yields assertion (2.5).

Further, differentiating both sides of generating relation (1.3) with respect to y and then equating the coefficients of identical powers of t in both sides of the resultant equation, it follows that

$$\begin{aligned}\hat{D}_y\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\} &= Bn(n-1)\cdots(n-m+1)\ {}_{\mathrm{g}L}\mathcal{A}_{n-m}^m(x,y,z;A,B)\\ &= nB(\sqrt{2A})^{1-m}\hat{D}_x^{m-1}\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\},\end{aligned}$$

which gives

$${}_y\mathcal{L}_n^-\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\} = \frac{(\sqrt{2A})^{m-1}}{nB}\hat{D}_x^{1-m}\hat{D}_y\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\} = {}_{\mathrm{g}L}\mathcal{A}_{n-1}^m(x,y,z;A,B). \tag{2.12}$$

Thus assertion (2.6) follows.

Next, in order to derive the expression for raising shift operator (2.7), the following relation is used:

$${}_{\mathrm{g}L}\mathcal{A}_k^m(x,y,z;A,B) = \big({}_x\mathcal{L}_{k+1}^-\ {}_x\mathcal{L}_{k+2}^-\ \cdots {}_x\mathcal{L}_n^-\big)\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\}, \tag{2.13}$$

which in view of equation (2.10) can be simplified as:

$${}_{\mathrm{g}L}\mathcal{A}_k^m(x,y,z;A,B) = \frac{k!}{n!}(\sqrt{2A})^{k-n}\ \hat{D}_x^{n-k}\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\}. \tag{2.14}$$

Putting $k = n-m+1$ and $k = n-k$ in equation (2.14), the following relations are obtained:

$${}_{\mathrm{g}L}\mathcal{A}_{n-m+1}^m(x,y,z;A,B) = \frac{(n-m+1)!}{n!}(\sqrt{2A})^{1-m}\ \hat{D}_x^{m-1}\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\}, \tag{2.15}$$

$${}_{\mathrm{g}L}\mathcal{A}_{n-k}^m(x,y,z;A,B) = \frac{(n-k)!}{n!}(\sqrt{2A})^{-k}\ \hat{D}_x^{k}\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\}. \tag{2.16}$$

Making use of equations (2.15) and (2.16) in recurrence relation (2.1) and in view of the relation:

$${}_x\mathcal{L}_n^+\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\} = {}_{\mathrm{g}L}\mathcal{A}_{n+1}^m(x,y,z;A,B),$$

it follows that

$$\begin{aligned}{}_x\mathcal{L}_n^+ := &\left(x\sqrt{2A} + \alpha_0 - \hat{D}_z^{-1} + mBy(\sqrt{2A})^{1-m}\hat{D}_x^{m-1} + \sum_{k=1}^{n}\frac{\alpha_k}{k!}(\sqrt{2A})^{-k}\hat{D}_x^k\right)\\ &\times\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\} = {}_{\mathrm{g}L}\mathcal{A}_{n+1}^m(x,y,z;A,B),\end{aligned} \tag{2.17}$$

which proves assertion (2.7).

In order to derive the expression for raising shift operator (2.8), the following relation is used:

$${}_{\mathrm{g}L}\mathcal{A}_k^m(x,y,z;A,B) = \big({}_z\mathcal{L}_{k+1}^-\ {}_z\mathcal{L}_{k+2}^-\ \cdots {}_z\mathcal{L}_n^-\big)\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\}, \tag{2.18}$$

which in view of equation (2.11) can be simplified as:

$${}_{\mathrm{g}L}\mathcal{A}_k^m(x,y,z;A,B) = \frac{k!}{n!}(-1)^{n-k}\ \hat{D}_z^{n-k}\big\{{}_{\mathrm{g}L}\mathcal{A}_n^m(x,y,z;A,B)\big\}. \tag{2.19}$$

From equation (2.19), the following relations are obtained:

$$ {}_{gL}\mathcal{A}^m_{n-m+1}(x,y,z;A,B) = \frac{(n-m+1)!}{n!}(-1)^{m-1}\ \hat{D}_z^{m-1}\{{}_{gL}\mathcal{A}^m_n(x,y,z;A,B)\}, \tag{2.20}$$

$$ {}_{gL}\mathcal{A}^m_{n-k}(x,y,z;A,B) = \frac{(n-k)!}{n!}(-1)^k\ \hat{D}_z^k\{{}_{gL}\mathcal{A}^m_n(x,y,z;A,B)\}. \tag{2.21}$$

Making use of equations (2.20) and (2.21) in recurrence relation (2.1) and in view of the relation:

$$ {}_z\mathcal{L}_n^+\{{}_{gL}\mathcal{A}^m_n(x,y,z;A,B)\} = {}_{gL}\mathcal{A}^m_{n+1}(x,y,z;A,B),$$

it follows that

$$\begin{aligned} {}_z\mathcal{L}_n^+ := &\left(x\sqrt{2A}+\alpha_0-\hat{D}_z^{-1}+mBy(-1)^{m-1}\hat{D}_z^{m-1}+\sum_{k=1}^{n}\frac{\alpha_k}{k!}(-1)^k\hat{D}_z^k\right)\\ &\times\{{}_{gL}\mathcal{A}^m_n(x,y,z;A,B)\} = {}_{gL}\mathcal{A}^m_{n+1}(x,y,z;A,B),\end{aligned} \tag{2.22}$$

which proves assertion (2.8).

Finally, to derive the expression for raising shift operator (2.9), the following relation is used:

$$ {}_{gL}\mathcal{A}^m_k(x,y,z;A,B) = \left({}_y\mathcal{L}^-_{k+1}\ {}_y\mathcal{L}^-_{k+2}\ \cdots {}_y\mathcal{L}^-_n\right)\{{}_{gL}\mathcal{A}^m_n(x,y,z;A,B)\}, \tag{2.23}$$

which in view of equation (2.12) can be simplified as:

$$ {}_{gL}\mathcal{A}^m_k(x,y,z;A,B) = \frac{k!}{n!}B^{k-n}(\sqrt{2A})^{(m-1)(n-k)}\ \hat{D}_x^{(1-m)(n-k)}\hat{D}_y^{n-k}\{{}_{gL}\mathcal{A}^m_n(x,y,z;A,B)\}. \tag{2.24}$$

From equation (2.24), the following relations are obtained:

$$ {}_{gL}\mathcal{A}^m_{n-m+1}(x,y,z;A,B) = \frac{(n-m+1)!}{n!}B^{1-m}(\sqrt{2A})^{(m-1)^2}\ \hat{D}_x^{-(m-1)^2}\hat{D}_y^{m-1}\{{}_{gL}\mathcal{A}^m_n(x,y,z;A,B)\}, \tag{2.25}$$

$$ {}_{gL}\mathcal{A}^m_{n-k}(x,y,z;A,B) = \frac{(n-k)!}{n!}B^{-k}\ (\sqrt{2A})^{k(m-1)}\ \hat{D}_x^{k(1-m)}\hat{D}_y^k\{{}_{gL}\mathcal{A}^m_n(x,y,z;A,B)\}. \tag{2.26}$$

Making use of equations (2.25) and (2.26) in recurrence relation (2.1) and in view of the relation:

$$ {}_y\mathcal{L}_n^+\{{}_{gL}\mathcal{A}^m_n(x,y,z;A,B)\} = {}_{gL}\mathcal{A}^m_{n+1}(x,y,z;A,B),$$

it follows that

$$\begin{aligned} {}_y\mathcal{L}_n^+ := &\Big(x\sqrt{2A}+\alpha_0-\hat{D}_z^{-1}+mB^{2-m}y(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{-(m-1)^2}\hat{D}_y^{m-1}\\ &+\sum_{k=1}^{n}\frac{\alpha_k}{k!}(\sqrt{2A})^{k(m-1)}\hat{D}_x^{k(1-m)}\hat{D}_y^k\Big)\{{}_{gL}\mathcal{A}^m_n(x,y,z;A,B)\} = {}_{gL}\mathcal{A}^m_{n+1}(x,y,z;A,B),\end{aligned} \tag{2.27}$$

which proves assertion (2.9). □

In order to derive the matrix differential, matrix integro-differential and matrix partial differential equations for the GHLAMaP ${}_{gL}\mathcal{A}^m_n(x,y,z;A,B)$, the following results are proved:

Theorem 2.3. *For the Gould-Hopper-Lagurre-Appell matrix polynomials ${}_{gL}\mathcal{A}^m_n(x,y,z;A,B)$, the following matrix differential equation holds true:*

$$\left((x\sqrt{2A}+\alpha_0)\hat{D}_z+mBy(-1)^{m-1}\hat{D}_z^m+\sum_{k=1}^{n}\frac{\alpha_k}{k!}(-1)^k\hat{D}_z^{k+1}+(n-1)\right){}_{gL}\mathcal{A}^m_n(x,y,z;A,B) = 0. \tag{2.28}$$

Proof. Consider the following factorization relation:

$$ {}_{z}\mathcal{L}_{n+1}^{-}\ {}_{z}\mathcal{L}_{n}^{+}\{{}_{gL}\mathcal{A}_{n}^{m}(x,y,z;A,B)\} = {}_{gL}\mathcal{A}_{n}^{m}(x,y,z;A,B). \tag{2.29} $$

Use of expressions (2.5) and (2.8) of the shift operators in the l.h.s of above equation, yields assertion (2.28). □

Theorem 2.4. *For the Gould-Hopper-Lagurre-Appell matrix polynomials* ${}_{gL}\mathcal{A}_{n}^{m}(x,y,z;A,B)$*, the following matrix integro-differential equations hold true:*

$$ \begin{aligned} &\Big((x\sqrt{2A}+\alpha_0-\hat{D}_z^{-1})\hat{D}_x + mBy(\sqrt{2A})^{1-m}\hat{D}_x^{m} + \sum_{k=1}^{n}\frac{\alpha_k}{k!}(\sqrt{2A})^{-k}\hat{D}_x^{k+1} \\ &- n\sqrt{2A}\Big){}_{gL}\mathcal{A}_{n}^{m}(x,y,z;A,B) = 0 \end{aligned} \tag{2.30} $$

and

$$ \begin{aligned} &\Big((x\sqrt{2A}+\alpha_0-\hat{D}_z^{-1})\hat{D}_y + mB^{2-m}(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{-(m-1)^2}\hat{D}_y^{m-1} \\ &+ myB^{2-m}(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{-(m-1)^2}\hat{D}_y^{m} + \sum_{k=1}^{n}\frac{\alpha_k}{k!}(\sqrt{2A})^{k(m-1)}B^{-k}\hat{D}_x^{-k(m-1)}\hat{D}_y^{k+1} \\ &- (n+1)B(\sqrt{2A})^{1-m}\hat{D}_x^{m-1}\Big){}_{gL}\mathcal{A}_{n}^{m}(x,y,z;A,B) = 0. \end{aligned} \tag{2.31} $$

Proof. Consider the following factorization relation:

$$ {}_{x}\mathcal{L}_{n+1}^{-}\ {}_{x}\mathcal{L}_{n}^{+}\{{}_{gL}\mathcal{A}_{n}^{m}(x,y,z;A,B)\} = {}_{gL}\mathcal{A}_{n}^{m}(x,y,z;A,B), \tag{2.32} $$

$$ {}_{y}\mathcal{L}_{n+1}^{-}\ {}_{y}\mathcal{L}_{n}^{+}\{{}_{gL}\mathcal{A}_{n}^{m}(x,y,z;A,B)\} = {}_{gL}\mathcal{A}_{n}^{m}(x,y,z;A,B). \tag{2.33} $$

Making use of expressions (2.4) and (2.7) of the shift operators in the l.h.s of equation (2.32) and using expressions (2.6) and (2.9) of the shift operators in the l.h.s of equation (2.33), we get assertions (2.30) and (2.31), respectively. □

Theorem 2.5. *For the Gould-Hopper-Lagurre-Appell matrix polynomials* ${}_{gL}\mathcal{A}_{n}^{m}(x,y,z;A,B)$*, the following matrix partial differential equations hold true:*

$$ \begin{aligned} &\Big((x\sqrt{2A}+\alpha_0)\hat{D}_x\hat{D}_z^{n} - \hat{D}_z^{n-1}\hat{D}_x + mBy(\sqrt{2A})^{1-m}\hat{D}_x^{m}\hat{D}_z^{n} + \sum_{k=1}^{n}\frac{\alpha_k}{k!}(\sqrt{2A})^{-k}\hat{D}_x^{k+1}\hat{D}_z^{n} \\ &- n\sqrt{2A}\hat{D}_z^{n}\Big){}_{gL}\mathcal{A}_{n}^{m}(x,y,z;A,B) = 0 \end{aligned} \tag{2.34} $$

and

$$ \begin{aligned} &\Big((x\sqrt{2A}+\alpha_0-\hat{D}_z^{-1})\hat{D}_y\hat{D}_x^{(m-1)(n-1)} + (m-1)(n-1)\hat{D}_x^{(m-1)(n-1)-1}\sqrt{2A}\hat{D}_y + m(\sqrt{2A})^{(m-1)^2} \\ &\times B^{2-m}\hat{D}_x^{(n-m)(m-1)}\hat{D}_y^{m-1} + myB^{2-m}(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{(n-m)(m-1)}\hat{D}_y^{m} + \sum_{k=1}^{n}\frac{\alpha_k}{k!}(\sqrt{2A})^{k(m-1)}B^{-k} \\ &\times\hat{D}_x^{(m-1)(n-1-k)}\hat{D}_y^{k+1} - (n+1)B(\sqrt{2A})^{1-m}\hat{D}_x^{n(m-1)}\Big){}_{gL}\mathcal{A}_{n}^{m}(x,y,z;A,B) = 0. \end{aligned} \tag{2.35} $$

Proof. Differentiation of equation (2.30) n times with respect to z, yields assertion (2.34). Similarly differentiating equation (2.31) $(m-1)(n-1)$ times with respect to x, we get assertion (2.35). □

In the next section, certain examples are considered in order to give the applications of the results derived above.

3. Applications

The matrix recurrence relation, shift operators, matrix differential, matrix integro-differential and matrix partial differential equations for certain members belonging to the family of GHLAMaP ${}_{gL}A_n^m(x,y,z;A,B)$ are derived by considering the following examples:

Example 3.1. *In view of relation (2.2) and using the generating function Table 1 (I), it follows that*

$$\frac{\mathcal{A}'(t)}{\mathcal{A}(t)}=\sum_{n=0}^{\infty}\alpha_n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{-B_{n+1}(1)}{(n+1)}\frac{t^n}{n!},$$

which gives

$$\alpha_n=\frac{-B_{n+1}(1)}{(n+1)},\ n\geq 1;\ \alpha_0=-\frac{1}{2}. \tag{3.1}$$

Substituting the values of the coefficients from equation (3.1) in equations (2.1), (2.4)–(2.9), (2.28), (2.30), (2.31), (2.34) and (2.35), the corresponding results for the GHLBMaP ${}_{gL}B_n^m(x,y,z;A,B)$ are obtained. These results are given in Table 2.

Table 2. Results for the GHLBMaP ${}_{gL}B_n^m(x,y,z;A,B)$

S. No.	Results	Expressions
I.	*Matrix recurrence relation*	${}_{gL}B_{n+1}^m(x,y,z;A,B)=\left(x\sqrt{2A}-\frac{1}{2}-\hat{D}_z^{-1}\right){}_{gL}B_n^m(x,y,z;A,B)+m!\binom{n}{m-1}By$ $\times{}_{gL}B_{n-m+1}^m(x,y,z;A,B)-\sum_{k=1}^{n}\binom{n}{k}\frac{B_{k+1}(1)}{k+1}\,{}_{gL}B_{n-k}^m(x,y,z;A,B).$
II.	*Shift operators*	${}_x\mathcal{L}_n^-:=\frac{1}{n\sqrt{2A}}\hat{D}_x.$ ${}_z\mathcal{L}_n^-:=\frac{-1}{n}\hat{D}_z.$ ${}_y\mathcal{L}_n^-:=\frac{(\sqrt{2A})^{m-1}}{nB}\hat{D}_x^{1-m}\hat{D}_y.$ ${}_x\mathcal{L}_n^+:=x\sqrt{2A}-\frac{1}{2}-\hat{D}_z^{-1}+mBy(\sqrt{2A})^{1-m}\hat{D}_x^{m-1}-\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}(\sqrt{2A})^{-k}\hat{D}_x^k.$ ${}_z\mathcal{L}_n^+:=x\sqrt{2A}-\frac{1}{2}-\hat{D}_z^{-1}+mBy(-1)^{m-1}\hat{D}_z^{m-1}-\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}(-1)^k\hat{D}_z^k.$ ${}_y\mathcal{L}_n^+:=x\sqrt{2A}-\frac{1}{2}-\hat{D}_z^{-1}+mB^{2-m}y(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{-(m-1)^2}\hat{D}_y^{m-1}$ $-\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}(\sqrt{2A})^{k(m-1)}\hat{D}_x^{k(1-m)}\hat{D}_y^k.$
III.	*Matrix differential equation*	$\left(\left(x\sqrt{2A}-\frac{1}{2}\right)\hat{D}_z+mBy(-1)^{m-1}\hat{D}_z^m-\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}(-1)^k\hat{D}_z^{k+1}+(n-1)\right){}_{gL}B_n^m(x,y,z;A,B)=0.$
IV.	*Matrix integro differential equations*	$\left(\left(x\sqrt{2A}-\frac{1}{2}-\hat{D}_z^{-1}\right)\hat{D}_x+mBy(\sqrt{2A})^{1-m}\hat{D}_x^m-\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}(\sqrt{2A})^{-k}\hat{D}_x^{k+1}\right.$ $\left.-n\sqrt{2A}\right){}_{gL}B_n^m(x,y,z;A,B)=0.$ $\left(\left(x\sqrt{2A}-\frac{1}{2}-\hat{D}_z^{-1}\right)\hat{D}_y+mB^{2-m}(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{-(m-1)^2}\hat{D}_y^{m-1}\right.$ $+myB^{2-m}(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{-(m-1)^2}\hat{D}_y^m-\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}(\sqrt{2A})^{k(m-1)}B^{-k}\hat{D}_x^{-k(m-1)}\hat{D}_y^{k+1}$ $\left.-(n+1)B(\sqrt{2A})^{1-m}\hat{D}_x^{m-1}\right){}_{gL}B_n^m(x,y,z;A,B)=0.$
V.	*Matrix partial differential equations*	$\left(\left(x\sqrt{2A}-\frac{1}{2}\right)\hat{D}_x\hat{D}_z^n-\hat{D}_z^{n-1}\hat{D}_x+mBy(\sqrt{2A})^{1-m}\hat{D}_x^m\hat{D}_z^n-\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}(\sqrt{2A})^{-k}\hat{D}_x^{k+1}\hat{D}_z^n\right.$ $\left.-n\sqrt{2A}\hat{D}_z^n\right){}_{gL}B_n^m(x,y,z;A,B)=0.$ $\left(\left(x\sqrt{2A}-\frac{1}{2}-\hat{D}_z^{-1}\right)\hat{D}_y\hat{D}_x^{(m-1)(n-1)}+(m-1)(n-1)\hat{D}_x^{(m-1)(n-1)-1}\sqrt{2A}\hat{D}_y+m(\sqrt{2A})^{(m-1)^2}\right.$ $\times B^{2-m}\hat{D}_x^{(n-m)(m-1)}\hat{D}_y^{m-1}+myB^{2-m}(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{(n-m)(m-1)}\hat{D}_y^m-\sum_{k=1}^{n}\frac{B_{k+1}(1)}{(k+1)!}B^{-k}$ $\left.\times(\sqrt{2A})^{k(m-1)}\hat{D}_x^{(m-1)(n-1-k)}\hat{D}_y^{k+1}-(n+1)B(\sqrt{2A})^{1-m}\hat{D}_x^{n(m-1)}\right){}_{gL}B_n^m(x,y,z;A,B)=0.$

Example 3.2. *In view of relation* (2.2) *and using the generating function Table 1 (II), it follows that*

$$\frac{\mathcal{A}'(t)}{\mathcal{A}(t)}=\sum_{n=0}^{\infty}\alpha_n\frac{t^n}{n!}=\sum_{n=0}^{\infty}\frac{\mathcal{E}_n}{2}\frac{t^n}{n!},$$

which gives

$$\alpha_n=\frac{\mathcal{E}_n}{2},\ n\geq 1;\ \alpha_0=-\frac{1}{2}, \tag{3.2}$$

where the numerical coefficients $\mathcal{E}_k$ $(k=1,2,\cdots,n-2,n-1)$ *are linked to the Euler number by the relation:*

$$\mathcal{E}_n=\frac{-1}{2^n}\sum_{k=0}^{n}\binom{n}{k}E_{n-k}.$$

Substituting the values of the coefficients from equation (3.2) *in equations* (2.1), (2.4)–(2.9), (2.28), (2.30), (2.31), (2.34) *and* (2.35), *the corresponding results for the Gould-Hopper-Lagurre-Euler matrix polynomials (GHLEMaP)* ${}_{gL}E_n^m(x,y,z;A,B)$ *are obtained. These results are given in Table 3.*

Table 3. Results for the GHLEMaP ${}_{gL}E_n^m(x,y,z;A,B)$

S. No.	Results	Expressions
I.	*Matrix recurrence relation*	${}_{gL}E_{n+1}^m(x,y,z;A,B)=\left(x\sqrt{2A}-\frac{1}{2}-\hat{D}_z^{-1}\right){}_{gL}E_n^m(x,y,z;A,B)+m!\binom{n}{m-1}By$ $\times{}_{gL}E_{n-m+1}^m(x,y,z;A,B)+\frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}\mathcal{E}_k\,{}_{gL}E_{n-k}^m(x,y,z;A,B).$
II.	*Shift operators*	${}_x\mathcal{L}_n^-:=\frac{1}{n\sqrt{2A}}\hat{D}_x.$ ${}_z\mathcal{L}_n^-:=\frac{-1}{n}\hat{D}_z.$ ${}_y\mathcal{L}_n^-:=\frac{(\sqrt{2A})^{m-1}}{nB}\hat{D}_x^{1-m}\hat{D}_y.$ ${}_x\mathcal{L}_n^+:=x\sqrt{2A}-\frac{1}{2}-\hat{D}_z^{-1}+mBy(\sqrt{2A})^{1-m}\hat{D}_x^{m-1}+\frac{1}{2}\sum_{k=1}^{n}\frac{\mathcal{E}_k}{k!}(\sqrt{2A})^{-k}\hat{D}_x^k.$ ${}_z\mathcal{L}_n^+:=x\sqrt{2A}-\frac{1}{2}-\hat{D}_z^{-1}+mBy(-1)^{m-1}\hat{D}_z^{m-1}+\frac{1}{2}\sum_{k=1}^{n}\frac{\mathcal{E}_k}{k!}(-1)^k\hat{D}_z^k.$ ${}_y\mathcal{L}_n^+:=x\sqrt{2A}-\frac{1}{2}-\hat{D}_z^{-1}+mB^{2-m}y(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{-(m-1)^2}\hat{D}_y^{m-1}$ $+\frac{1}{2}\sum_{k=1}^{n}\frac{\mathcal{E}_k}{k!}(\sqrt{2A})^{k(m-1)}\hat{D}_x^{k(1-m)}\hat{D}_y^k.$
III.	*Matrix differential equation*	$\left(\left(x\sqrt{2A}-\frac{1}{2}\right)\hat{D}_z+mBy(-1)^{m-1}\hat{D}_z^m+\frac{1}{2}\sum_{k=1}^{n}\frac{\mathcal{E}_k}{k!}(-1)^k\hat{D}_z^{k+1}+(n-1)\right){}_{gL}E_n^m(x,y,z;A,B)=0.$
IV.	*Matrix integro differential equations*	$\Bigg(\left(x\sqrt{2A}-\frac{1}{2}-\hat{D}_z^{-1}\right)\hat{D}_x+mBy(\sqrt{2A})^{1-m}\hat{D}_x^m+\frac{1}{2}\sum_{k=1}^{n}\frac{\mathcal{E}_k}{k!}(\sqrt{2A})^{-k}\hat{D}_x^{k+1}$ $-n\sqrt{2A}\Bigg){}_{gL}E_n^m(x,y,z;A,B)=0.$ $\Bigg(\left(x\sqrt{2A}-\frac{1}{2}-\hat{D}_z^{-1}\right)\hat{D}_y+mB^{2-m}(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{-(m-1)^2}\hat{D}_y^{m-1}$ $+myB^{2-m}(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{-(m-1)^2}\hat{D}_y^m+\frac{1}{2}\sum_{k=1}^{n}\frac{\mathcal{E}_k}{k!}(\sqrt{2A})^{k(m-1)}B^{-k}\hat{D}_x^{-k(m-1)}\hat{D}_y^{k+1}$ $-(n+1)B(\sqrt{2A})^{1-m}\hat{D}_x^{m-1}\Bigg){}_{gL}E_n^m(x,y,z;A,B)=0.$
V.	*Matrix partial differential equations*	$\Bigg(\left(x\sqrt{2A}-\frac{1}{2}\right)\hat{D}_x\hat{D}_z^n-\hat{D}_z^{n-1}\hat{D}_x+mBy(\sqrt{2A})^{1-m}\hat{D}_x^m\hat{D}_z^n+\frac{1}{2}\sum_{k=1}^{n}\frac{\mathcal{E}_k}{k!}(\sqrt{2A})^{-k}\hat{D}_x^{k+1}\hat{D}_z^n$ $-n\sqrt{2A}\hat{D}_z^n\Bigg){}_{gL}E_n^m(x,y,z;A,B)=0.$ $\Bigg(\left(x\sqrt{2A}-\frac{1}{2}-\hat{D}_z^{-1}\right)\hat{D}_y\hat{D}_x^{(m-1)(n-1)}+(m-1)(n-1)\hat{D}_x^{(m-1)(n-1)-1}\sqrt{2A}\hat{D}_y+m(\sqrt{2A})^{(m-1)^2}$ $\times B^{2-m}\hat{D}_x^{(n-m)(m-1)}\hat{D}_y^{m-1}+myB^{2-m}(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{(n-m)(m-1)}\hat{D}_y^m+\frac{1}{2}\sum_{k=1}^{n}\frac{\mathcal{E}_k}{k!}B^{-k}$ $\times(\sqrt{2A})^{k(m-1)}\hat{D}_x^{(m-1)(n-1-k)}\hat{D}_y^{k+1}-(n+1)B(\sqrt{2A})^{1-m}\hat{D}_x^{n(m-1)}\Bigg){}_{gL}E_n^m(x,y,z;A,B)=0.$

Example 3.3. *In view of relation (2.2) and using the generating function Table 1 (III), it follows that*

$$\frac{\mathcal{A}'(t)}{\mathcal{A}(t)} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{G_n}{2} \frac{t^n}{n!},$$

which gives

$$\alpha_n = \frac{G_n}{2},\ n \geq 1;\ \alpha_0 = 1. \tag{3.3}$$

Substituting the values of the coefficients from equation (3.3) in equations (2.1), (2.4)–(2.9), (2.28), (2.30), (2.31), (2.34) and (2.35), the corresponding results for the Gould-Hopper-Lagurre-Genocchi matrix polynomials (GHLGMaP) ${}_{gL}G_n^m(x,y,z;A,B)$ *are obtained. These results are given in Table 4.*

Table 4. Results for the GHLGMaP ${}_{gL}G_n^m(x,y,z;A,B)$

S. No.	Results	Expressions
I.	*Matrix recurrence relation*	${}_{gL}G_{n+1}^m(x,y,z;A,B) = \left(x\sqrt{2A} + 1 - \hat{D}_z^{-1}\right){}_{gL}G_n^m(x,y,z;A,B) + m!\binom{n}{m-1}By$ $\times {}_{gL}G_{n-m+1}^m(x,y,z;A,B) + \frac{1}{2}\sum_{k=1}^{n}\binom{n}{k}G_k\ {}_{gL}G_{n-k}^m(x,y,z;A,B).$
II.	*Shift operators*	${}_x\mathcal{L}_n^- := \frac{1}{n\sqrt{2A}}\hat{D}_x.$ ${}_z\mathcal{L}_n^- := \frac{-1}{n}\hat{D}_z.$ ${}_y\mathcal{L}_n^- := \frac{(\sqrt{2A})^{m-1}}{nB}\hat{D}_x^{1-m}\hat{D}_y.$ ${}_x\mathcal{L}_n^+ := x\sqrt{2A} + 1 - \hat{D}_z^{-1} + mBy(\sqrt{2A})^{1-m}\hat{D}_x^{m-1} + \frac{1}{2}\sum_{k=1}^{n}\frac{G_k}{k!}(\sqrt{2A})^{-k}\hat{D}_x^k.$ ${}_z\mathcal{L}_n^+ := x\sqrt{2A} + 1 - \hat{D}_z^{-1} + mBy(-1)^{m-1}\hat{D}_z^{m-1} + \frac{1}{2}\sum_{k=1}^{n}\frac{G_k}{k!}(-1)^k\hat{D}_z^k.$ ${}_y\mathcal{L}_n^+ := x\sqrt{2A} + 1 - \hat{D}_z^{-1} + mB^{2-m}y(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{-(m-1)^2}\hat{D}_y^{m-1}$ $+\frac{1}{2}\sum_{k=1}^{n}\frac{G_k}{k!}(\sqrt{2A})^{k(m-1)}\hat{D}_x^{k(1-m)}\hat{D}_y^k.$
III.	*Matrix differential equation*	$\left((x\sqrt{2A}+1)\hat{D}_z + mBy(-1)^{m-1}\hat{D}_z^m + \frac{1}{2}\sum_{k=1}^{n}\frac{G_k}{k!}(-1)^k\hat{D}_z^{k+1} + (n-1)\right){}_{gL}G_n^m(x,y,z;A,B) = 0.$
IV.	*Matrix integro differential equations*	$\Big((x\sqrt{2A} + 1 - \hat{D}_z^{-1})\hat{D}_x + mBy(\sqrt{2A})^{1-m}\hat{D}_x^m + \frac{1}{2}\sum_{k=1}^{n}\frac{G_k}{k!}(\sqrt{2A})^{-k}\hat{D}_x^{k+1}$ $-n\sqrt{2A}\Big){}_{gL}G_n^m(x,y,z;A,B) = 0.$ $\Big((x\sqrt{2A} + 1 - \hat{D}_z^{-1})\hat{D}_y + mB^{2-m}(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{-(m-1)^2}\hat{D}_y^{m-1}$ $+myB^{2-m}(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{-(m-1)^2}\hat{D}_y^m + \frac{1}{2}\sum_{k=1}^{n}\frac{G_k}{k!}(\sqrt{2A})^{k(m-1)}B^{-k}\hat{D}_x^{-k(m-1)}\hat{D}_y^{k+1}$ $-(n+1)B(\sqrt{2A})^{1-m}\hat{D}_x^{m-1}\Big){}_{gL}G_n^m(x,y,z;A,B) = 0.$
V.	*Matrix partial differential equations*	$\Big((x\sqrt{2A}+1)\hat{D}_x\hat{D}_z^n - \hat{D}_z^{n-1}\hat{D}_x + mBy(\sqrt{2A})^{1-m}\hat{D}_x^m\hat{D}_z^n + \frac{1}{2}\sum_{k=1}^{n}\frac{G_k}{k!}(\sqrt{2A})^{-k}\hat{D}_x^{k+1}\hat{D}_z^n$ $-n\sqrt{2A}\hat{D}_z^n\Big){}_{gL}G_n^m(x,y,z;A,B) = 0.$ $\Big((x\sqrt{2A} + 1 - \hat{D}_z^{-1})\hat{D}_y\hat{D}_x^{(m-1)(n-1)} + (m-1)(n-1)\hat{D}_x^{(m-1)(n-1)-1}\sqrt{2A}\hat{D}_y + m(\sqrt{2A})^{(m-1)^2}$ $\times B^{2-m}\hat{D}_x^{(n-m)(m-1)}\hat{D}_y^{m-1} + myB^{2-m}(\sqrt{2A})^{(m-1)^2}\hat{D}_x^{(n-m)(m-1)}\hat{D}_y^m + \frac{1}{2}\sum_{k=1}^{n}\frac{G_k}{k!}B^{-k}$ $\times(\sqrt{2A})^{k(m-1)}\hat{D}_x^{(m-1)(n-1-k)}\hat{D}_y^{k+1} - (n+1)B(\sqrt{2A})^{1-m}\hat{D}_x^{n(m-1)}\Big){}_{gL}G_n^m(x,y,z;A,B) = 0.$

The problems arising in different areas of science and engineering are usually expressed in terms of differential equations, which in most of the cases have special functions as their solutions. During the past three decades, the development of nonlinear analysis, dynamical systems and their applications to science and engineering has stimulated renewed enthusiasm for the theory of differential equations. The differential equations satisfied by these hybrid type special matrix polynomials may be useful in solving various problems arising in certain branches of science and engineering.

Conflict of interest: The authors declare that they have no conflict of interest.

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