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Higher-order System of $p-$nonlinear Difference Equations Solvable in Closed-form with Variable Coefficients

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ABSTRACT: In this paper, we investigate the solutions of the following system of $p-$nonlinear difference equations

$$x_{n+1}^{(i)}=\frac{a_n x_{n-m}^{(i+1)\bmod(p)}}{b_n+c_n x_{n-m}^{(i+1)\bmod(p)}}, n,m\in\mathbb{N}_0, p\in\mathbb{N}, i\in\{1,...,p\},$$

where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, the sequences (a_n), (b_n), (c_n), are non-zero real numbers and initial values $x_{-j}^{(i)}$, $j\in\{0,...,m\}$, $i\in\{1,...,p\}$, do not equal $-b_n c_n^{-1}$, for all $n\in\mathbb{N}_0$. Also, we investigate the behavior of positive solutions of the above-mentioned system when variable coefficients. Finally, we give some numerical examples which verify our theoretical result.

Key Words: Difference equations, local stability, periodicity, general solution, system of difference equations.

Contents

1. Introduction

In the recent years, there has been a lot of interest in studying nonlinear difference equations and systems. Not surprisingly therefore,

several studies have been published on this topic (see, e.g., [1] - [24], and the related references therein). Besides their theoretical value, most of the recent applications have appeared in many scientific areas such as biology (population dynamics in particular), ecology, physics, engineering and economics (see, e.g. [6], [7], [8], [17]). It is very worthy to find systems belonging to solvable nonlinear difference equations systems in closed-form.

Since the paper by Brand [3], the following one-dimensional nonlinear difference equation of Riccati type,

$$x_{n+1}=\frac{ax_n+d}{cx_n+b}, n\in\mathbb{N}_0, \tag{1.1}$$

where the initial value x_0 is a real number or complex number and the parameters a,b,c and d are the real numbers with the restrictions $c\neq 0$ and $ab\neq cd$, have the most diverse and interesting properties, especially as regards the distribution of their cluster points. This finding, led Stević [19] to study the solutions of the Eq. (1.1).

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In [18] the author presented the solutions of the following two-dimensional nonlinear difference equation generalization of Eq. (1.1):

$$x_{n+1}=\frac{ax_n+d}{cx_n+b},y_{n+1}=\frac{ay_n+d}{cy_n+b},n\in\mathbb{N}_0, \tag{1.2}$$

when $a=b=c=1$ and $d=0$. Its extension with nonconstant coefficients and higher-order is a system of a huge interest. For this reason, system (1.2) can be extended by interchanging the parameters a,b and c with the sequences $(a_n)_{n\geq0}$, $(b_n)_{n\geq0}$ and $(c_n)_{n\geq0}$. More concretely, another extension of system (1.2) is the following system of $p-$dimensional nonlinear difference equations

$$x_{n+1}^{(i)}=\frac{a_nx_{n-m}^{(i+1)\bmod(p)}}{b_n+c_nx_{n-m}^{(i+1)\bmod(p)}},n,m\in\mathbb{N}_0,p\in\mathbb{N},i\in\{1,...,p\}. \tag{1.3}$$

Now, we consider system (1.3) in the case when a $a_n\neq0$ for all $n\in\mathbb{N}_0$. Noticing that in this case, system (1.3) can be written in the form

$$x_{n+1}^{(i)}=\frac{x_{n-m}^{(i+1)\bmod(p)}}{\widetilde{b}_n+\widetilde{c}_nx_{n-m}^{(i+1)\bmod(p)}},n,m\in\mathbb{N}_0,p\in\mathbb{N},i\in\{1,...,p\}.$$

where $\widetilde{b}_n=\dfrac{b_n}{a_n}$ and $\widetilde{c}_n=\dfrac{c_n}{a_n}$, for all $n\in\mathbb{N}_0$, we see that we may assume that $a_n=1$, for all $n\in\mathbb{N}_0$. Hence we consider, without loss of generality, the system

$$x_{n+1}^{(i)}=\frac{x_{n-m}^{(i+1)\bmod(p)}}{b_n+c_nx_{n-m}^{(i+1)\bmod(p)}},n,m\in\mathbb{N}_0,p\in\mathbb{N},i\in\{1,...,p\}. \tag{1.4}$$

using the same notation for coefficients as in (1.3) except for the coefficients a_n, assuming that $a_n=1$, for all $n\in\mathbb{N}_0$.

The remainder of the paper is organized as follows: In section 2, we study the solutions of the given system of the $p-$dimensional nonlinear rational difference equations of first-order. In the next section, the solutions of the Higher-order system of the $p-$dimensional nonlinear rational difference equations are given. In the following section, we investigate the local stability of the equilibrium points and the global behavior of the given system. Numerical examples to illustrate our results are given in Section 5. Section 6 concludes.

2. On the system of first-order difference equations (1.4)

In the following, we give the closed form of the solutions of the first-order system (1.4) :

$$x_{n+1}^{(i)}=\frac{x_{n}^{(i+1)\bmod(p)}}{b_n+c_nx_{n}^{(i+1)\bmod(p)}},n\in\mathbb{N}_0,p\in\mathbb{N},i\in\{1,...,p\}. \tag{2.1}$$

In any such case, we need to define the recursion equation relating to $x_{n+1}^{(1)}$ by using the backward recursion of the system (2.1). So, we replace the right-hand side of the expression of $x_{n+1}^{(p)}$ in the penultimate equation from the system (2.1), we get

$$x_{n+1}^{(p-1)}=\frac{x_n^{(p)}}{b_n+c_nx_n^{(p)}}=\frac{x_{n-1}^{(1)}}{b_nb_{n-1}+(b_nc_{n-1}+c_n)\,x_{n-1}^{(1)}},n\geq1.$$

Similarly, we replace the right-hand side of the expression of $x_{n+1}^{(p-1)}$ in the equation $x_{n+1}^{(p-2)}=x_n^{(p-1)}/\left(b_n+c_nx_n^{(p-1)}\right)$ from the system (2.1), we get

$$x_{n+1}^{(p-2)}=\frac{x_{n-2}^{(1)}}{\left\{\prod_{j=0}^{2}b_{n-j}\right\}+\sum_{k=0}^{2}\left\{\prod_{j=0}^{k-1}b_{n-j}\right\}c_{n-k}x_{n-2}^{(1)}},n\geq2,$$

where $\prod\limits_{j=1}^{l} b_j = 1$ and $\sum\limits_{j=1}^{l} b_j = 0$ if $l < 1$. More generally, by induction, we get the analog for $x_{n+1}^{(2)}$ and $x_{n+1}^{(1)}$,

$$x_{n+1}^{(2)} = \frac{x_{n-(p-2)}^{(1)}}{\left\{\prod\limits_{j=0}^{p-2} b_{n-j}\right\} + \sum\limits_{k=0}^{p-2}\left\{\prod\limits_{j=0}^{k-1} b_{n-j}\right\} c_{n-k} x_{n-(p-2)}^{(1)}}, n \geq p-2,$$

$$x_{n+1}^{(1)} = \frac{x_{n-(p-1)}^{(1)}}{\left\{\prod\limits_{j=0}^{p-1} b_{n-j}\right\} + \sum\limits_{k=0}^{p-1}\left\{\prod\limits_{j=0}^{k-1} b_{n-j}\right\} c_{n-k} x_{n-(p-1)}^{(1)}}, n \geq p-1.$$

In the same context, the system (2.1) can be easily transformed into an equivalent to the following nonlinear difference equation of $p-$order,

$$x_{n+1} = \frac{x_{n-(p-1)}}{\left\{\prod\limits_{j=0}^{p-1} b_{n-j}\right\} + \sum\limits_{k=0}^{p-1}\left\{\prod\limits_{j=0}^{k-1} b_{n-j}\right\} c_{n-k} x_{n-(p-1)}}, n \geq p-1.$$

Indeed, let $x_n(i) = x_{pn+i}, n \in \mathbb{N}_0, i \in \{0, 1, ..., p-1\}$, we get

$$x_{n+1}(i) = \frac{x_n(i)}{\alpha_n^{(i)} + \beta_n^{(i)} x_n(i)}, n \geq p-1, i \in \{0, 1, ..., p-1\},$$

where $\alpha_n^{(i)} = \left\{\prod\limits_{j=1}^{p} b_{p(n+1)+i-j}\right\}$ and $\beta_n^{(i)} = \sum\limits_{k=1}^{p}\left\{\prod\limits_{j=1}^{k-1} b_{p(n+1)+i-j}\right\} c_{p(n+1)+i-k}$. For this purpose, we consider the following nonlinear difference equation of first-order,

$$x_{n+1} = \frac{x_n}{\alpha_n + \beta_n x_n}, n \in \mathbb{N}_0. \tag{2.2}$$

Note that equation (2.2) can be written in the form

$$x_{n+1} = \frac{1}{\beta_n} - \frac{1}{\beta_n}\frac{\alpha_n}{\alpha_n + \beta_n x_n}, n \in \mathbb{N}_0. \tag{2.3}$$

Hence we can use the change of variables $y_n = (\alpha_n + \beta_n x_n)^{-1}$ in (2.3) and obtain

$$y_{n+1} = \frac{\beta_n}{\beta_{n+1}} \frac{1}{\left(1 + \alpha_{n+1}\frac{\beta_n}{\beta_{n+1}}\right) - \alpha_n y_n}, n \in \mathbb{N}_0.$$

If we use the following change of variables $y_n = z_n/z_{n+1}, n \in \mathbb{N}_0$, in the last equation, we get the homogeneous linear second order difference equation with variable coefficients,

$$z_{n+2} - \left(\frac{\beta_{n+1}}{\beta_n} + \alpha_{n+1}\right) z_{n+1} + \left(\frac{\beta_{n+1}}{\beta_n}\alpha_n\right) z_n = 0, n \in \mathbb{N}_0.$$

In other words, we have that

$$z_{n+2} - \alpha_{n+1} z_{n+1} - \frac{\beta_{n+1}}{\beta_n}(z_{n+1} - \alpha_n z_n) = 0, n \in \mathbb{N}_0. \tag{2.4}$$

Using the change of variables $u_n = z_n - \alpha_{n-1} z_{n-1}, n \in \mathbb{N}$, equation (2.4) becomes $u_{n+2} = \frac{\beta_{n+1}}{\beta_n} u_{n+1}, n \in \mathbb{N}_0$, so its solution is $u_n = \frac{\beta_{n-1}}{\beta_0} u_1, n \in \mathbb{N}_0$, thus we have the following linear first-order difference equation with variable coefficients,

$$z_n = \alpha_{n-1} z_{n-1} + \frac{\beta_{n-1}}{\beta_0} (z_1 - \alpha_0 z_0), n \in \mathbb{N}. \tag{2.5}$$

Equation (2.5) can be solved in closed form in many ways, and its general solution is

$$z_n = \left\{ \prod_{j=1}^{n} \alpha_{n-j} \right\} z_0 + \frac{1}{\beta_0} (z_1 - \alpha_0 z_0) \sum_{k=1}^{n} \left\{ \prod_{j=1}^{k-1} \alpha_{n-j} \right\} \beta_{n-k}.$$

Hence, we have

$$y_n = \frac{\left\{ \prod_{j=1}^{n} \alpha_{n-j} \right\} y_0 + \frac{1}{\beta_0} (1 - \alpha_0 y_0) \sum_{k=1}^{n} \left\{ \prod_{j=1}^{k-1} \alpha_{n-j} \right\} \beta_{n-k}}{\left\{ \prod_{j=0}^{n} \alpha_{n-j} \right\} y_0 + \frac{1}{\beta_0} (1 - \alpha_0 y_0) \sum_{k=0}^{n} \left\{ \prod_{j=0}^{k-1} \alpha_{n-j} \right\} \beta_{n-k}}, n \in \mathbb{N}_0.$$

So, we get that the solutions of the difference equation (2.2) is

$$x_n = \frac{x_0}{\left\{ \prod_{j=1}^{n} \alpha_{n-j} \right\} + x_0 \sum_{k=1}^{n} \left\{ \prod_{j=1}^{k-1} \alpha_{n-j} \right\} \beta_{n-k}}, n \in \mathbb{N}_0.$$

From all above mentioned we see that the following corollary holds.

Corollary 2.1. *Let $\{x_n\}_{n \geq -(p-1)}$ be a solution of Equation (2.1). Then for $n \geq -(p-1)$,*

$$x_{pn+i} = \frac{x_i}{\left\{ \prod_{j=1}^{n} \alpha_{n-j}^{(i)} \right\} + x_i \sum_{k=1}^{n} \left\{ \prod_{j=1}^{k-1} \alpha_{n-j}^{(i)} \right\} \beta_{n-k}^{(i)}}, i \in \{0, 1, ..., p-1\}, \tag{2.6}$$

where $\alpha_n^{(i)} = \left\{ \prod_{j=1}^{p} b_{p(n+1)+i-j} \right\}$ *and* $\beta_n^{(i)} = \sum_{k=1}^{p} \left\{ \prod_{j=1}^{k-1} b_{p(n+1)+i-j} \right\} c_{p(n+1)+i-k}$.

Corollary 2.2. *In the constant case, i.e., when the coefficients are constants ($b_n = b$ and $c_n = c$ for all $n \in \mathbb{N}$), in the Corollary (2.1), the solution (2.6) reduces to*

$$x_{pn+i} = \frac{x_i}{b^{pn} + c x_i \sum_{k,l=1}^{n} b^{p(k-1)+l-1}}, i \in \{0, 1, ..., p-1\}, \text{for } n \geq -(p-1).$$

Theorem 2.3. *Let $\left\{ x_n^{(1)}, x_n^{(2)}, ..., x_n^{(p)} \right\}_{n \geq 0}$ be solutions of system (2.1). Then $\left\{ x_{p(n+1)-j}^{(p-t)} \right\}_{n \geq 0}$ is given by the formula for $n \geq p-1$,*

$$x_{p(n+1)-i}^{(p-t)} = \begin{cases} \dfrac{x_0^{(p-i-t)}}{\delta_1(n,i,t) + (\delta_2(n,i,t) + \delta_3(n,i,t)) x_0^{(p-i-t)}}, \text{if } i+t \in \{0,1,...,p-1\} \\ \dfrac{x_0^{(p-s+1)}}{\widetilde{\delta}_1(n,s,t) + \left(\widetilde{\delta}_2(n,s,t) + \widetilde{\delta}_3(n,s,t)\right) x_0^{(p-s+1)}}, \text{if } i+t \in \{p,...,2p-2\} \end{cases}, \tag{2.7}$$

where $t \in \{0,1,...,p-1\}$, $i+t+1 \equiv s \bmod (p)$ *and* $(\delta_u(n,i,t), \widetilde{\delta}_u(n,s,t), u=1,2,3)$ *is the appropriate sequence can easily obtain and uniquely determined by* $\{b_n, c_n\}$, *i.e.,*

$$\delta_1(n,i,t) = \left\{\prod_{j=1}^{n} \alpha_{n-j}^{(p-i-t-1)}\right\}\left\{\prod_{j=1}^{p-i-t-1} b_{p-i-t-j}\right\}\left\{\prod_{j=1}^{t+1} b_{p(n+1)-i-j}\right\},$$

$$\delta_2(n,i,t) = \left\{\prod_{j=1}^{t+1} b_{p(n+1)-i-j}\right\}\left(\sum_{k=1}^{p-i-t-1}\left\{\prod_{j=1}^{n}\alpha_{n-j}^{(p-i-t-1)}\right\}\left\{\prod_{j=1}^{k-1} b_{p-i-t-j}\right\} c_{p-i-t-k}\right.$$
$$\left. + \sum_{k=1}^{n}\left\{\prod_{j=1}^{k-1}\alpha_{n-j}^{(p-i-t-1)}\right\}\beta_{n-k}^{(p-i-t-1)}\right),$$

$$\delta_3(n,i,t) = \sum_{k=1}^{t+1}\left\{\prod_{j=1}^{k-1} b_{p(n+1)-i-j}\right\} c_{p(n+1)-i-k},$$

$$\widetilde{\delta}_1(n,s,t) = \left\{\prod_{j=2}^{n}\alpha_{n-j}^{(p-s)}\right\}\left\{\prod_{j=0}^{p-s-1} b_{p-s-j}\right\}\left\{\prod_{j=0}^{t} b_{pn+t-s-j}\right\},$$

$$\widetilde{\delta}_2(n,s,t) = \left\{\prod_{j=0}^{t} b_{pn+t-s-j}\right\}\left(\sum_{k=0}^{p-s-1}\left\{\prod_{j=2}^{n}\alpha_{n-j}^{(p-s)}\right\}\left\{\prod_{j=0}^{k-1} b_{p-s-j}\right\} c_{p-s-k} + \sum_{k=2}^{n}\left\{\prod_{j=2}^{k-1}\alpha_{n-j}^{(p-s)}\right\}\beta_{n-k}^{(p-s)}\right),$$

$$\widetilde{\delta}_3(n,s,t) = \sum_{k=0}^{t}\left\{\prod_{j=0}^{k-1} b_{pn+t-s-j}\right\} c_{pn+t-s-k}.$$

Proof. By Corollary (2.1), we obtain

$$x_{pn+(p-i)}^{(1)} = \frac{x_{p-i}^{(1)}}{\left\{\prod_{j=1}^{n}\alpha_{n-j}^{(p-i)}\right\} + x_{p-i}^{(1)}\sum_{k=1}^{n}\left\{\prod_{j=1}^{k-1}\alpha_{n-j}^{(p-i)}\right\}\beta_{n-k}^{(p-i)}}, i \in \{0,1,...,p-1\}.$$

Using (2.1), we get

$$x_{p-i}^{(1)} = \frac{x_0^{(p-i+1)}}{\left\{\prod_{j=0}^{p-i-1} b_{p-i-j}\right\} + x_0^{(p-i+1)}\sum_{k=0}^{p-i-1}\left\{\prod_{j=0}^{k-1} b_{p-i-j}\right\} c_{p-i-k}}, i \in \{0,1,...,p-1\},$$

$$\begin{aligned} x_{p(n+1)-i}^{(1)} &= x_0^{(p-i+1)}\left(\left\{\prod_{j=1}^{n}\alpha_{n-j}^{(p-i)}\right\}\left\{\prod_{j=0}^{p-i-1} b_{p-i-j}\right\} + \sum_{k=0}^{p-i-1}\left\{\prod_{j=1}^{n}\alpha_{n-j}^{(p-i)}\right\}\left\{\prod_{j=0}^{k-1} b_{p-i-j}\right\}\right. \\ &\qquad \left. \times c_{p-i-k} x_0^{(p-i+1)} + \sum_{k=1}^{n}\left\{\prod_{j=1}^{k-1}\alpha_{n-j}^{(p-i)}\right\}\beta_{n-k}^{(p-i)} x_0^{(p-i+1)}\right)^{-1}, \end{aligned}$$

for $i \in \{0,1,...,p-1\}$. By using the following recurrence relation

$$x_{n+1}^{(p-t)} = \frac{x_{n-t}^{(1)}}{\left\{\prod_{j=0}^{t} b_{n-j}\right\} + \sum_{k=0}^{t}\left\{\prod_{j=0}^{k-1} b_{n-j}\right\} c_{n-k} x_{n-t}^{(1)}}, t \in \{0,1,...,p-1\},$$

we have

$$x^{(p-t)}_{p(n+1)-i}=\frac{x^{(1)}_{p(n+1)-i-t-1}}{\left\{\prod_{j=1}^{t+1}b_{p(n+1)-i-j}\right\}+\sum_{k=1}^{t+1}\left\{\prod_{j=1}^{k-1}b_{p(n+1)-i-j}\right\}c_{p(n+1)-i-k}x^{(1)}_{p(n+1)-i-t-1}},i,t\in\{0,1,...,p-1\},$$

that is,

$$x^{(p-t)}_{p(n+1)-i}=\frac{x_0^{(p-i-t)}}{\delta_1\left(n,i,t\right)+\left(\delta_2\left(n,i,t\right)+\delta_3\left(n,i,t\right)\right)x_0^{(p-i-t)}},\text{if } i+t\in\left\{0,1,...,p-1\right\},$$

and

$$x^{(p-t)}_{p(n+1)-i}=\frac{x_0^{(p-s+1)}}{\widetilde{\delta}_1\left(n,s,t\right)+\left(\widetilde{\delta}_2\left(n,s,t\right)+\widetilde{\delta}_3\left(n,s,t\right)\right)x_0^{(p-s+1)}},\text{if } i+t\in\left\{p,...,2p-2\right\}.$$

□

Corollary 2.4. *In the constant case, in the Theorem (2.3), the solution (2.7) reduces to*

$$x^{(p-t)}_{p(n+1)-i}=\begin{cases}\dfrac{x_0^{(p-s+1)}}{b^{p(n+1)+1+t-s}+c\left(b^{pn+t}\sum\limits_{k=1}^{p-s}b^k+b^t\sum\limits_{k=0}^{n-1}b^k\sum\limits_{l=1}^{p}b^l+\sum\limits_{k=0}^{t}b^k\right)x_0^{(p-s+1)}}, & \\ \textit{if } i+t=s-1\in\{0,1,...,p-1\} & \\ \dfrac{x_0^{(p-s+1)}}{b^{pn+1+t-s}+c\left(b^{p(n-1)+t}\sum\limits_{k=1}^{p-s}b^k+b^t\sum\limits_{k=0}^{n-2}b^k\sum\limits_{l=1}^{p}b^l+\sum\limits_{k=0}^{t}b^k\right)x_0^{(p-s+1)}}, & \\ \textit{if } i+t=p+s-1\in\{p,...,2p-2\} & \end{cases}.$$

3. On the system of $m-$order difference equations (1.4)

In this section, we focus on the form of system (2.1) which generalizes (1.4) in a very agile way of planning. In other words, we use an appropriate transformation reducing this system to the elegant system of first-order difference equations by which the solution of system (1.4) can be established. We do this as follows, let $x_n^{(t)}\left(i\right)=x^{(t)}_{(m+1)n-i}$, $i\in\{0,...,m\}$, $t\in\{1,...,p\}$, with this notation, we obtain the following system of first-order difference equations

$$x^{(t)}_{n+1}\left(i\right)=\frac{x_n^{(t+1)\bmod(p)}\left(i\right)}{b_{(m+1)(n+1)-i-1}+c_{(m+1)(n+1)-i-1}x_n^{(t+1)\bmod(p)}\left(i\right)},t\in\{1,...,p\},i\in\{0,...,m\}.\tag{3.1}$$

System (3.1) is the same as the system (2.1). We are now in a position to state the following theorem similar to Theorem 2.3.

Theorem 3.1. *Let* $\left\{x_n^{(1)}\left(i\right),x_n^{(2)}\left(i\right),...,x_n^{(p)}\left(i\right)\right\}_{n\geq0}$ *be solutions of system (3.1),* $i\in\{0,...,m\}$*. Then* $\left\{x^{(p-t)}_{p(n+1)-u}\left(i\right)\right\}_{n\geq0}$ *is given by the formula for* $n\geq p-1$,

$$x^{(p-t)}_{p(n+1)-u}\left(i\right)=\begin{cases}\dfrac{x_0^{(p-u-t)}\left(i\right)}{\gamma_1\left(n,u,t\right)+\left(\gamma_2\left(n,u,t\right)+\gamma_3\left(n,u,t\right)\right)x_0^{(p-u-t)}\left(i\right)}, \textit{if } u+t\in\{0,1,...,p-1\}\\ \dfrac{x_0^{(p-s+1)}}{\widetilde{\gamma}_1\left(n,s,t\right)+\left(\widetilde{\gamma}_2\left(n,s,t\right)+\widetilde{\gamma}_3\left(n,s,t\right)\right)x_0^{(p-s+1)}\left(i\right)}, \textit{if } u+t\in\{p,...,2p-2\}\end{cases},\tag{3.2}$$

where $t\in\{0,1,...,p-1\}$, $u+t+1\equiv s\bmod(p)$, $i\in\{0,...,m\}$ *and* $\left(\gamma_v\left(n,u,t\right),\widetilde{\gamma}_v\left(n,s,t\right)\right)_{v=1,2,3}$ *is given to replace* $\{b_n,c_n\}$ *in* $\left(\delta_v\left(n,u,t\right),\widetilde{\delta}_v\left(n,s,t\right)\right)_{v=1,2,3}$ *by* $\left\{b_{(m+1)(n+1)-i-1},c_{(m+1)(n+1)-i-1}\right\}$.

Corollary 3.2. *In the constant case, in the Theorem 3.1, the solution (3.2) reduces to*

$$x^{(p-t)}_{p(n+1)-u}(i) = \begin{cases} \dfrac{x_0^{(p-s+1)}(i)}{b^{p(n+1)+1+t-s} + c\left(b^{pn+t}\sum\limits_{k=1}^{p-s} b^k + b^t \sum\limits_{k=0}^{n-1} b^k \sum\limits_{l=1}^{p} b^l + \sum\limits_{k=0}^{t} b^k\right) x_0^{(p-s+1)}(i)}, \\ \text{if } u+t = s-1 \in \{0,1,...,p-1\} \\ \dfrac{x_0^{(p-s+1)}(i)}{b^{pn+1+t-s} + c\left(b^{p(n-1)+t}\sum\limits_{k=1}^{p-s} b^k + b^t \sum\limits_{k=0}^{n-2} b^k \sum\limits_{l=1}^{p} b^l + \sum\limits_{k=0}^{t} b^k\right) x_0^{(p-s+1)}(i)}, \\ \text{if } u+t = p+s-1 \in \{p,...,2p-2\} \end{cases}.$$

The following theorem due to Theorem 3.1, gives us the main result for the system of p nonlinear difference equations of high-order (1.4).

Theorem 3.3. *Let $\left\{x_n^{(1)}, x_n^{(2)}, ..., x_n^{(p)}\right\}_{n\geq -m}$ be solutions of system (1.4). Then for $n \geq p-1$,*

$$x^{(p-t)}_{(m+1)(p(n+1)-u)-i} = \begin{cases} \dfrac{x_{-i}^{(p-u-t)}}{\gamma_1(n,u,t) + (\gamma_2(n,u,t) + \gamma_3(n,u,t))\, x_{-i}^{(p-u-t)}}, \text{ if } u+t \in \{0,1,...,p-1\} \\ \dfrac{x_{-i}^{(p-s+1)}}{\widetilde{\gamma}_1(n,s,t) + (\widetilde{\gamma}_2(n,s,t) + \widetilde{\gamma}_3(n,s,t))\, x_{-i}^{(p-s+1)}}, \text{ if } u+t \in \{p,...,2p-2\} \end{cases}, \tag{3.3}$$

where $t \in \{0,1,...,p-1\}$, $u+t+1 \equiv s \bmod (p)$, $i \in \{0,...,m\}$.

Corollary 3.4. *In the constant case, in the Theorem (3.3), the solution (3.3) reduces to*

$$x^{(p-t)}_{(m+1)(p(n+1)-u)-i} = \begin{cases} \dfrac{x_{-i}^{(p-s+1)}}{b^{p(n+1)+1+t-s} + c\left(b^{pn+t}\sum\limits_{k=1}^{p-s} b^k + b^t \sum\limits_{k=0}^{n-1} b^k \sum\limits_{l=1}^{p} b^l + \sum\limits_{k=0}^{t} b^k\right) x_{-i}^{(p-s+1)}}, \\ \text{if } u+t = s-1 \in \{0,1,...,p-1\} \\ \dfrac{x_{-i}^{(p-s+1)}}{b^{pn+1+t-s} + c\left(b^{p(n-1)+t}\sum\limits_{k=1}^{p-s} b^k + b^t \sum\limits_{k=0}^{n-2} b^k \sum\limits_{l=1}^{p} b^l + \sum\limits_{k=0}^{t} b^k\right) x_{-i}^{(p-s+1)}}, \\ \text{if } u+t = p+s-1 \in \{p,...,2p-2\} \end{cases}.$$

Remark 3.5. *In this remark we use the formulae in Theorem 3.3 to get solutions of system (1.3), when $a_n \neq 0$ for $n \in \mathbb{N}_0$. So, we replace sequences $(b_n)_{n\in\mathbb{N}_0}$ and $(c_n)_{n\in\mathbb{N}_0}$ in formulas of Theorem 3.3 with sequences $\left(\dfrac{b_n}{a_n}\right)_{n\in\mathbb{N}_0}$ and $\left(\dfrac{c_n}{a_n}\right)_{n\in\mathbb{N}_0}$.*

4. Global stability of positive solutions of (1.3)

In the following, we will study the global stability character of the solutions of system (1.3). Clearly, the unique positive equilibrium of system (1.3) is

$$E = \left(\overline{x^{(1)}}, \overline{x^{(2)}}, ..., \overline{x^{(p)}}\right) = (0,0,...,0).$$

Such as many system of p difference equations, it is usually beneficial to linearized system (1.3) about the equilibrium point E in order to further simplify its study. For this reason, introducing the vectors

$\underline{X}_n' := \left(\underline{X}_n'(1), ..., \underline{X}_n'(p)\right)$ and $\underline{X}_n'(i) = \left(x_n^{(i)}, x_{n-1}^{(i)}, ..., x_{n-m}^{(i)}\right)$, for $1 \leq i \leq p$. With these notations, we obtain the following representation

$$\underline{X}_{n+1} = M_n \underline{X}_n, \tag{4.1}$$

where

$$M_n = \begin{pmatrix} K & \frac{a_n}{b_n}J & O_{(m)} & \cdots & O_{(m)} & O_{(m)} \\ O_{(m)} & K & \frac{a_n}{b_n}J & \cdots & O_{(m)} & O_{(m)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O_{(m)} & O_{(m)} & O_{(m)} & \cdots & K & \frac{a_n}{b_n}J \\ \frac{a_n}{b_n}J & O_{(m)} & O_{(m)} & \cdots & O_{(m)} & K \end{pmatrix}, \ K = \begin{pmatrix} \underline{O}_{(m-1)}' & 0 \\ I_{(m-1)} & \underline{O}_{(m-1)} \end{pmatrix},$$

with $O_{(k,l)}$ denotes the matrix of order $k \times l$ whose entries are zeros, for simplicity, we set $O_{(k)} := O_{(k,k)}$ and $\underline{O}_{(k)} := O_{(k,1)}$, J is $m \times m$ matrix each of whose entries are zeros except a 1 at the $(1, m)$th position and $I_{(m)}$ is the $m \times m$ identity matrix. We summarize the above discussion in the following theorem

Theorem 4.1. *The unique positive equilibrium point E is locally asymptotically stable.*

Proof. After some elementary calculations, the characteristic polynomial of M_n is $P_{M_n}(\lambda) = (-1)^{p(m+1)} P_1(\lambda) + (-1)^m P_{2,n}(\lambda)$, where $P_1(\lambda) = \lambda^{p(m+1)}$ and $P_{2,n}(\lambda) = a_n^p b_n^{-p}$, then $|P_{2,n}(\lambda)| < |P_1(\lambda)|, \forall \lambda : |\lambda| = 1$. So, according to Rouche's Theorem P_1 and $P_1 + P_{2,n}$ have the same number of zeros in the unit disc $|\lambda| < 1$, and since P_1 admits as root $\lambda = 0$ of multiplicity $p(m+1)$, then all the roots of $P_1 + P_{2,n}$ are in the disc $|\lambda| < 1$. Thus, the equilibrium point is locally asymptotically stable. □

Corollary 4.2. *For every well defined solution of system* (1.3)*, if*

$$\lim \left(\min \left(\sum_{v=1}^{3} \widetilde{\gamma}_v(.,.,.), \sum_{v=1}^{3} \gamma_v(.,.,.) \right) \right) = +\infty$$

, we have $\lim x_n^{(p-t)} = 0$*, for each* $0 \leq t \leq p-1$.

Proof. From Theorem 3.1 and under the condition $\lim \left(\min \left(\sum_{v=1}^{3} \widetilde{\gamma}_v(.,.,.), \sum_{v=1}^{3} \gamma_v(.,.,.) \right) \right) = +\infty$, we have $\lim x_n^{(p-t)} = 0$, for each $0 \leq t \leq p-1$. □

The following Corollary is a direct consequence of Theorems 4.1 and Corollary 4.2.

Corollary 4.3. *The unique positive equilibrium point E is globally asymptotically stable.*

5. Numerical Examples

In order to illustrate and support theoretical results of the previous section, we consider several interesting numerical examples in this section.

Example 5.1. *We consider interesting numerical example for the difference equations system* (1.3) *with the initial conditions*

$$\begin{array}{lllll} x_{-2}^{(1)} = \frac{48}{5}, & x_{-2}^{(2)} = \frac{73}{37}, & x_{-2}^{(3)} = \frac{2}{11}, & x_{-2}^{(4)} = \frac{24}{35}, & x_{-2}^{(5)} = \frac{7}{8} \\ x_{-1}^{(1)} = \frac{28}{39}, & x_{-1}^{(2)} = \frac{5}{12}, & x_{-1}^{(3)} = 5, & x_{-1}^{(4)} = \frac{11}{9}, & x_{-1}^{(5)} = \frac{4}{5} \\ x_{0}^{(1)} = \frac{15}{87}, & x_{0}^{(2)} = \frac{31}{35}, & x_{0}^{(3)} = \frac{19}{3}, & x_{0}^{(4)} = \frac{1}{3}, & x_{0}^{(5)} = \frac{5}{3} \end{array}. \tag{5.1a}$$

Moreover, choosing the sequences $a_n = n+1$, $b_n = n^2+2$ *and* $c_n = n+3$, *the system* (1.3) *can be written as follows:*

$$x_{n+1}^{(i)} = \frac{(n+1)\,x_{n-2}^{(i+1)\bmod(5)}}{n^2+2+(n+3)\,x_{n-2}^{(i+1)\bmod(5)}}, n\in\mathbb{N}_0, i\in\{1,2,3,4,5\}, \tag{5.1b}$$

$n=0,1,...$ *The plot of the system* (5.1*b*) *is shown in Figure* 1.

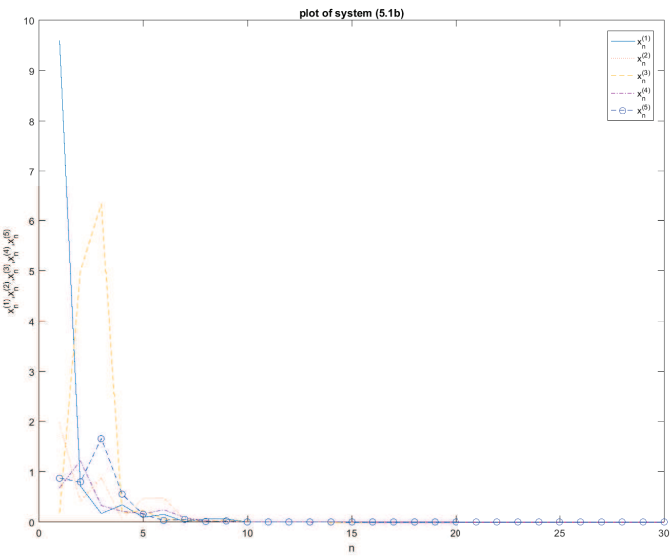

Figure 1: This figure shows the solutions of the system (5.1*b*),when we put the initial conditions (5.1*a*).

Example 5.2. *We consider interesting numerical example for the difference equations system* (1.3) *with the initial conditions*

$$\begin{array}{llllll} x_{-3}^{(1)}=1.6, & x_{-3}^{(2)}=2.3, & x_{-3}^{(3)}=3.9, & x_{-3}^{(4)}=10.1, & x_{-3}^{(5)}=3.4, & x_{-3}^{(6)}=8.3 \\ x_{-2}^{(1)}=5.2, & x_{-2}^{(2)}=4.5, & x_{-2}^{(3)}=8.3, & x_{-2}^{(4)}=1.3, & x_{-2}^{(5)}=3.2, & x_{-2}^{(6)}=3.1 \\ x_{-1}^{(1)}=6.0, & x_{-1}^{(2)}=1.6, & x_{-1}^{(3)}=4.0, & x_{-1}^{(4)}=4.3, & x_{-1}^{(5)}=5.2, & x_{-1}^{(6)}=2.5 \\ x_{0}^{(1)}=2.6, & x_{0}^{(2)}=6.0, & x_{0}^{(3)}=6.4, & x_{0}^{(4)}=2.4, & x_{0}^{(5)}=12.0, & x_{0}^{(6)}=7.0 \end{array}. \tag{5.2a}$$

Moreover, choosing the sequences $a_n = (n+1)^{-1}$, $b_n = n+2$ *and* $c_n = (n+1)^{-1}$, *the system* (1.3) *can be written as follows:*

$$x_{n+1}^{(i)} = \frac{x_{n-3}^{(i+1)\bmod(6)}}{(n+2)\,(n+1)+x_{n-3}^{(i+1)\bmod(6)}}, n\in\mathbb{N}_0, i\in\{1,2,3,4,5,6\}, \tag{5.2b}$$

$n=0,1,...$ *The plot of the system* (5.2*b*) *is shown in Figure* 2.

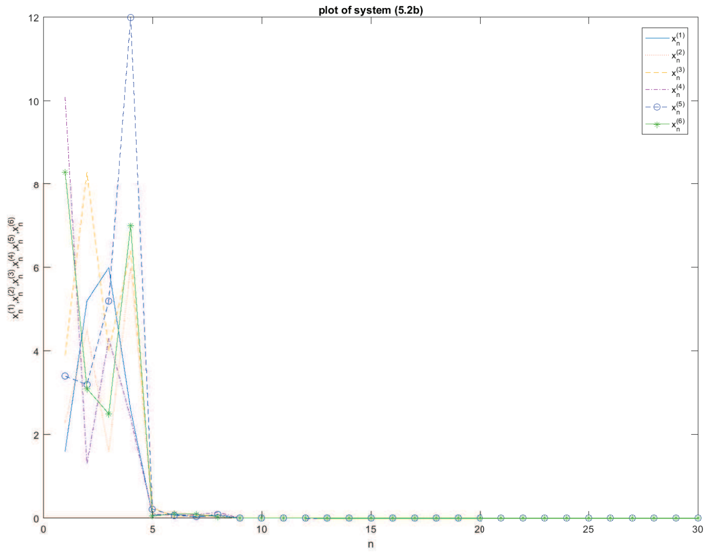

Figure 2: This figure shows the solutions of the system (5.2*b*), when we put the initial conditions (5.2*a*).

Example 5.3. *We consider interesting numerical example for the difference equations system* (1.3) *with the initial conditions*

$$\begin{array}{llllll} x_{-2}^{(1)}=1.0, & x_{-2}^{(2)}=1.3, & x_{-2}^{(3)}=0.1, & x_{-2}^{(4)}=7.0, & x_{-2}^{(5)}=2.1, & x_{-2}^{(6)}=2.3 \\ x_{-1}^{(1)}=0.2, & x_{-1}^{(2)}=5.0, & x_{-1}^{(3)}=3.0, & x_{-1}^{(4)}=8.2, & x_{-1}^{(5)}=11.2, & x_{-1}^{(6)}=5.0 \\ x_{0}^{(1)}=6.0, & x_{0}^{(2)}=0.7, & x_{0}^{(3)}=6.0, & x_{0}^{(4)}=3.5, & x_{0}^{(5)}=2.0, & x_{0}^{(6)}=1.9 \end{array}. \tag{5.3a}$$

Moreover, choosing the sequences $a_n = 2$, $b_n = 4$ *and* $c_n = 3$, *the system* (1.3) *can be written as follows:*

$$x_{n+1}^{(i)} = \frac{2x_{n-2}^{(i+1)\bmod(6)}}{4+3x_{n-2}^{(i+1)\bmod(6)}}, n \in \mathbb{N}_0, i \in \{1,2,3,4,5,6\}, \tag{5.3b}$$

$n = 0, 1, \ldots$ *The plot of the system* (5.3*b*) *is shown in Figure* 3.

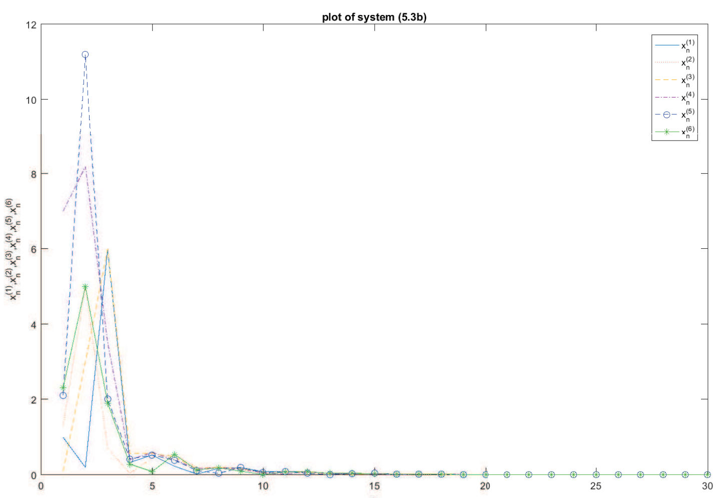

Figure 3: This figure shows the solutions of the system (5.3*b*), when we put the initial conditions (5.3*a*).

Example 5.4. *We consider interesting numerical example for the difference equations system* (1.3) *with the initial conditions*

$$\begin{array}{llll} x_{-1}^{(1)} = 12.0, & x_{-1}^{(2)} = 2.3, & x_{-1}^{(3)} = 42.1, & x_{-1}^{(4)} = 7.0 \\ x_{0}^{(1)} = 4.0, & x_{0}^{(2)} = 22.5, & x_{0}^{(3)} = 9.0, & x_{0}^{(4)} = 17.0 \end{array}. \tag{5.4a}$$

Moreover, choosing the sequences $a_n = e^{-n}$, $b_n = n^2 + 1$ *and* $c_n = \ln(n+2)$, *the system* (1.3) *can be written as follows:*

$$x_{n+1}^{(i)} = \frac{e^{-n} x_{n-1}^{(i+1) \bmod (4)}}{n^2 + 1 + \ln(n+2)\, x_{n-1}^{(i+1) \bmod (4)}}, n \in \mathbb{N}_0, i \in \{1, 2, 3, 4\}, \tag{5.4b}$$

$n = 0, 1, \ldots$ *The plot of the system* (5.4b) *is shown in Figure* 4.

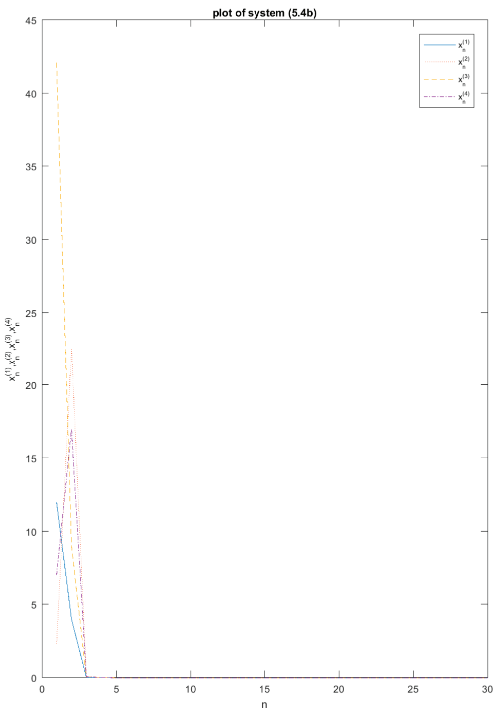

Figure 4: This figure shows the solutions of the system (5.4*b*), when we put the initial conditions (5.4*a*).

Example 5.5. *We consider interesting numerical example for the difference equations system* (1.3) *with the initial conditions*

$$\begin{array}{lllllll} x_{-1}^{(1)} = 1.6, & x_{-1}^{(2)} = 2.3, & x_{-1}^{(3)} = 5.1, & x_{-1}^{(4)} = 11.0, & x_{-1}^{(5)} = 2.1, & x_{-1}^{(6)} = 12.3, & x_{-1}^{(7)} = 1.3 \\ x_{0}^{(1)} = 6.2, & x_{0}^{(2)} = 3.5, & x_{0}^{(3)} = 3.0, & x_{0}^{(4)} = 3.2, & x_{0}^{(5)} = 1.2, & x_{0}^{(6)} = 5.0, & x_{0}^{(7)} = 14.0 \end{array}. \tag{5.5a}$$

Moreover, choosing the sequences $a_n = 0.11$, $b_n = \ln\left(n^2 + 1\right)$ *and* $c_n = 0.25$, *the system* (1.3) *can be written as follows:*

$$x_{n+1}^{(i)} = \frac{0.11 x_{n-1}^{(i+1) \bmod (7)}}{\ln(n^2 + 1) + 0.25 x_{n-1}^{(i+1) \bmod (7)}}, n \in \mathbb{N}_0, i \in \{1, 2, 3, 4, 5, 6, 7\}, \tag{5.5b}$$

$n = 0, 1, \ldots$ *The plot of the system* (5.5b) *is shown in Figure* 5.

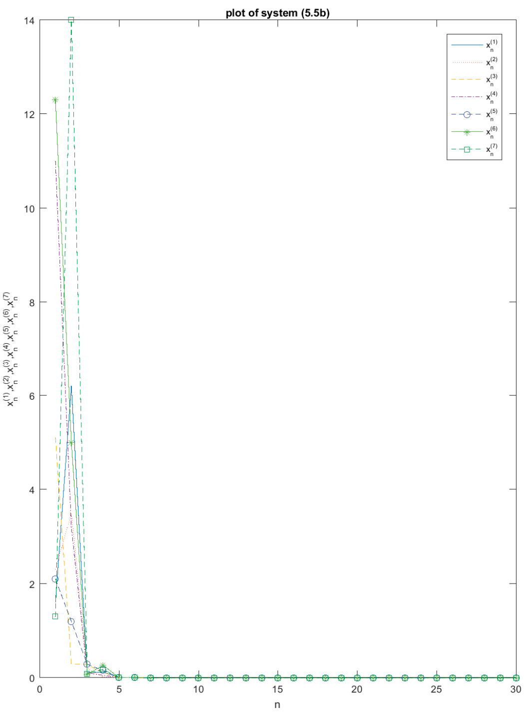

Figure 5: This figure shows the solutions of the system (5.5b), when we put the initial conditions (5.5a).

Example 5.6. *We consider interesting numerical example for the difference equations system* (1.3) *with the initial conditions*

$$\begin{array}{llllll} x_{-3}^{(1)}=1.6, & x_{-3}^{(2)}=2.3, & x_{-3}^{(3)}=3.9, & x_{-3}^{(4)}=10.1, & x_{-3}^{(5)}=3.4, & x_{-3}^{(6)}=8.3 \\ x_{-2}^{(1)}=5.2, & x_{-2}^{(2)}=4.5, & x_{-2}^{(3)}=8.3, & x_{-2}^{(4)}=1.3, & x_{-2}^{(5)}=3.2, & x_{-2}^{(6)}=3.1 \\ x_{-1}^{(1)}=6.0, & x_{-1}^{(2)}=1.6, & x_{-1}^{(3)}=4.0, & x_{-1}^{(4)}=4.3, & x_{-1}^{(5)}=5.2, & x_{-1}^{(6)}=2.5 \\ x_{0}^{(1)}=2.6, & x_{0}^{(2)}=6.0, & x_{0}^{(3)}=6.4, & x_{0}^{(4)}=2.4, & x_{0}^{(5)}=12.0, & x_{0}^{(6)}=7.0 \end{array}. \tag{5.6a}$$

Moreover, choosing the sequences $a_n = n+2$, $b_n = n+1$ *and* $c_n = 1$, *the system* (1.3) *can be written as follows:*

$$x_{n+1}^{(i)} = \frac{(n+2)\, x_{n-3}^{(i+1)\,\mathrm{mod}(6)}}{n+1+x_{n-3}^{(i+1)\,\mathrm{mod}(6)}}, n \in \mathbb{N}_0, i \in \{1,2,3,4,5,6\}, \tag{5.6b}$$

$n = 0, 1, \ldots$ *The plot of the system* (5.6b) *is shown in Figure* 6.

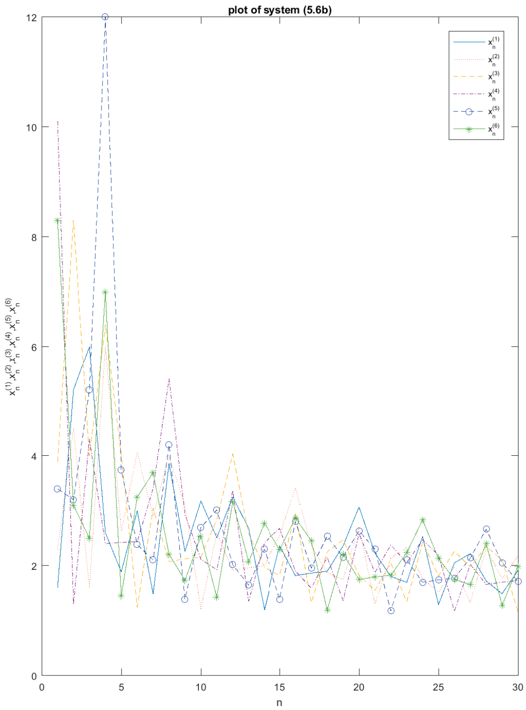

Figure 6: This figure shows the solutions of the system (5.6b), when we put the initial conditions (5.6a).

6. Conclusion

In this paper, we represented the general solutions of $p-$dimensional systems of nonlinear rational difference equations with variable coefficients of $(m+1)-$order. Firstly, we have obtained the closed-form of well-defined solutions of first-order, i.e., $m=0$, which enables us to use an appropriate transformation reducing the equations of our system (1.3) to first-order. In addition, in the case where the coefficients are constant in the system, we have obtained the solutions for this case. Secondly, we presented some results about the general behavior of the solutions of this system. Finally, we will give the following important open problem for system of difference equations theory to researchers. The system (1.3) can extend to equations more general than that in (1.3). For example, the $p-$dimensional system of nonlinear rational difference equations of $(\max(m,k)+1)-$order,

$$x_{n+1}^{(i)}=\frac{a_n^{(i)}x_{n-k}^{(i+1)\bmod(p)}}{b_n^{(i)}+c_n^{(i)}x_{n-m}^{(i+1)\bmod(p)}}, n,m,k\in\mathbb{N}_0, p\in\mathbb{N}, i\in\{1,...,p\},$$

where $\mathbb{N}_0=\mathbb{N}\cup\{0\}$, the sequences $\left(a_n^{(i)}\right)$, $\left(b_n^{(i)}\right)$, $\left(c_n^{(i)}\right)$, are non-zero real numbers and initial values $x_{-j}^{(i)}$, $j\in\{0,...,\max(m,k)\}$, $i\in\{1,...,p\}$, do not equal $-b_n^{(i)}\left(c_n^{(i)}\right)^{-1}$, for all $n\in\mathbb{N}_0$.

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