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The Canonical Form of Multiplication Modules

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ABSTRACT: Let R be a commutative ring with unit. An R-module M is called a multiplication module if for every submodule N of M, there is an ideal I of R such that $N = IM$. M is called also a CF-module if there is a chain of ideals $I_1 \subseteq ... \subseteq I_n$ of R such that $M \simeq R/I_1 \bigoplus R/I_2 \bigoplus ... \bigoplus R/I_n$. In this paper, we use some new results about $\mu_R(M)$ the minimal number of generators of M to show that a finitely generated multiplication module is a CF-module if and only if it is a cyclic module.

Key Words: Multiplication modules, canonical forms of modules, minimal number of generators.

Contents

1. Introduction

Let R be a commutative nonzero ring with unit. An R-module M is called a multiplication module if for every submodule N of M, there is an ideal I of R such that $N = IM$. The multiplication modules are largely used to study the prime modules like in [1] and [2]. On the other hand, the classification of R-modules is always wanted for commutative algebra, K-theory, representation theory... Then the invariant factors was often investigated. When R is a principal ideal domain (PID), the classification of R-modules can be obtained by studying the elementary divisors (see [3] and [4]). This advantage of PIDs was generalized for every ring R by the CF-modules, that are the R-modules which have a canonical form: there exists a chain of ideals $I_1 \subseteq ... \subseteq I_n$ of R such that $M \simeq R/I_1 \bigoplus R/I_2 \bigoplus ... \bigoplus R/I_n$ (see [4], [5], [11] and [12] ...).

In this paper, we investigate when a finitely generated multiplication module will be a CF-module. We use for that: $\mu_R(M)$ the minimal number of generators. This parameter $\mu_R(M)$ has its particular importance for example to minimize the size of syzygies [6] and the size of the matrix representation of R-modules [7]. It was studied by Gilmer and Heinzer [8] and Kumar [9]... in some particular cases. We show a constructive approach introduced by Charkani and Akharraz [5] to study $\mu_R(M)$ by using the Fitting ideals. Recall that $F_k(M)$ the k-th Fitting ideal of M is the $(n-k)$-th determinantal ideal where n is the number of generators of M (not necessary the minimal number of generators), for more informations please see the beautiful work done by Brown [10]. Recall that $F_0(M) \subseteq F_1(M) \subseteq ... \subseteq F_{\mu_R(M)}(M) = R$ [10] and we set $\nu_R(M) = min\{k \in \mathbb{N} \mid F_k(M) = R\}$.

In the second section we show some properties in commutative algebra as preliminaries and we define a μ-module. Then, in the third section we prove our main result:

Theorem 2.1. Let R be a commutative ring with unit and M be a nonzero finitely generated R-module. Then the following statements are equivalents:

1. M is a multiplication CF-module.
2. M is a cyclic R-module.
3. $\mu_R(M) = 1$.
4. $\nu_R(M) = 1$ and M is a μ-module.

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2. Preliminaries

We start this section by some properties about the localization:

Lemma 2.1. *Let R be a commutative ring with unit, x be an element of R and $max(R)$ the set of maximal ideals of R. Then:*

$$[(\forall m \in max(R)) : \ (xR)_m = 0R_m] \Leftrightarrow x = 0$$

Proof. Let m be a maximal ideal $(xR)_m = 0$ implies that there exist $t_m \notin m$ such that $t_m x = 0$. Then, $Ann(x)$ is not contained in m. This is true for every $m \in max(R)$. Then, $Ann(x)$ is not contained in any maximal ideal. Therefore, $Ann(x) = R$. Hence, $x = 0$ (Because $1 \in Ann(x)$) □

Corollary 2.2. *Let R be a commutative ring with unit, I be an ideal of R and $max(R)$ the set of maximal ideals of R. Then:*

$$[(\forall m \in max(R)) : \ I_m = 0R_m] \Leftrightarrow I = 0$$

Proof. Let $x \in I$, then $(xR)_m \subseteq I_m = 0R_m$. Then, $(xR)_m = 0R_m$. By 2.1, this implies that $x = 0$. Thus, $I = (0)_R$. □

Corollary 2.3. *Let R be a commutative ring with unit, I and J be two ideals of R and $max(R)$ the set of maximal ideals of R. Then:*

$$[(\forall m \in max(R)) : \ I_m = J_m] \Leftrightarrow I = J$$

Proof. In R/I: $J_m/I_m = (0)_{R_m}$. Since $J_m/I_m = (J_m + I_m)/I_m = ((J+I)/I)_m$, $(J+I)/I = 0_{R/I}$ (By 2.2). Thus, $J + I = I$. Hence $J \subseteq I$.
As far as in R/J we get $I \subseteq J$. Therefore, $I = J$. □

It is known in general that $\mu_R(M) \geq \nu_R(M)$, we show now one particular case when $\mu_R(M) \leq \nu_R(M)$:

Lemma 2.4. *Let R be a commutative ring with unit and M be a free R-module. Then, $\mu_R(E) = \nu_R(M)$.*

Proof. When M is a free R-module of rank $r = \mu_R(M)$ then the k-th determinantal ideal $I_k(M) = 0$ for all $k \in \{1, ..., r\}$ [10]. Thus, the first Fitting ideal that is equal to R is the $\mu_R(M)$-th Fitting ideal. Hence $\mu_R(M) = \nu_R(M)$. □

Finally we define the μ-module:

Definition 2.5. *Let R be a commutative ring with unit. an R-module is called a μ-module if $\mu_R(M) = max\{\mu_{R_m}(M_m) \mid m$ is a maximal ideal of $R\}$.*

Example 2.6. *If R is a local ring it is obvious that all R-module is μ-module.*
If M is a finitely generated multiplication module not cyclic. Then, M is not a μ-module. Indeed M is locally cyclic [13] and not cyclic: $\mu_R(M) \neq 1$ and $max\{\mu_{R_m}(M_m) \mid m$ is a maximal ideal of $R\} = 1$.

3. Main result

Theorem 3.1. *Let R be a commutative ring with unit and M be a nonzero finitely generated R-module. Then the following statements are equivalents:*

1. *M is a multiplication CF-module.*
2. *M is a cyclic R-module.*
3. *$\mu_R(M) = 1$.*
4. *$\nu_R(M) = 1$ and M is a μ-module.*

Proof. Let M be a nonzero finitely generated R-module.
If M is a CF-module, then there is a chain of ideals $I_1 \subseteq ... \subseteq I_n$ of R such that

$$M \simeq R/I_1 \bigoplus R/I_2 \bigoplus ... \bigoplus R/I_n.$$

By [5], for any $k \in \{0, ..., n-1\}$, $F_k(M) = I_{k+1}.I_{k+2}...I_n$. Thus, $F_0(M) = I_1...I_n$ and $F_1(M) = I_2...I_n$. Further, if M is a multiplication R-module, then it is locally cyclic (there is an equivalence proved in [Prop. 4, [13]]). Then for any maximal ideal m of R, $F_1(M_m) = R_m$. By [13.38, p.161, [10]], for each multiplicative closed set S of R, $S^{-1}F_k(M) \simeq F_k(S^{-1}M)$. Then, $F_1(M_m) = (F_1(M))_m = R_m$ for each maximal ideal m of R. Therefore, by 2.3 $F_1(M) = R$. Thus, $I_2...I_n = R$. Hence $I_2 = ... = I_n = R$ and $F_0(M) = I_1$. Namely $M \simeq R/I_1$. So that, there is an isomorphism: $\varphi : M \to R/I_1$, let $x \in M$ which verify $\varphi(x) = \overline{1}$, then $M = xR$.
Conversely, if M is cyclic then there exists $x \in M$ such that $M = xR$. It is obvious that M is a multiplication module and $M \simeq R/Ann(x)$ is a CF-module.
It is also obvious that $\mu_R(M) = 1$ if and only if M is cyclic.
Further, we have proved that $F_1(M) = R$ and $F_0(M) = I_1$. If $I_1 = R$ we get $M = 0$ but we have assumed that M is nonzero. Then $\nu_R(M) = 1$.
Conversely, if $\nu_R(M) = 1$, then $F_1(M) = R$. Thus, for any maximal ideal m of R, $(F_1(M))_m = F_1(M_m) = R_m$. On the other hand, $F_1(M_m/mM_m) = F_1(R_m/mR_m \bigotimes M_m) = F_1(M_m).R_m/mR_m$ [Cor. 20.5, p.498, [14]]. Thus, $F_1(M_m/mM_m) = R_m/mR_m$. Since M_m/mM_m is a R_m/mR_m-vector space, by 2.4 $\mu_{R_m}(M_m/mM_m) = 1$. Since R_m is local, M_m is generated by the same number of generators of the quotient M_m/mM_m. That is $\mu_{R_m}(M_m) = 1$ for any maximal ideal m of R. Therefore, M is finitely generated locally cyclic module that is M is a multiplication module [13]. Then $max\{\mu_R(M_m) \mid m$ is a maximal ideal of $R\} = 1$. When M is μ-module, $\mu_R(M) = 1$. □

References

1. El-Bast,Z.A., Smith, P.F., *Multiplication modules.* Comm. Algebra 16,755-779, (1988)
2. Azizi,A., Shiraz, *Weak multiplication modules.* Czechoslovak mathematical journal 53(128), 529-534, (2003)
3. Lang, S., *Algebra.* 3rd Edition, Addison-Wesley, (1993).
4. Kaplansky, I., *Elementary Divisors and Modules.* Tran. Ame. Math. Soc. 66, 153-169, (1949).
5. M.E. Charkani, M.E., Akharraz, I., *Fitting ideals and cyclic decomposition of finitely generated modules.* Arabian Journal for Science and Engineering 25(2), 151-156, (2000).
6. Cayley, A., *On the theory of involution in geometry*,Cambridge Math. 11, 52-61, (1847).
7. Lombardi, H., Quitte,C., *Algebre Commutative, Methodes Constructives: Modules Projectifs de Type Fini*, Calvage et Mounet, (2016).
8. Gilmer, R., Heinzer, W., *On the Number of Generators of an Invertible Ideal*, Journal of Algebra 14, 139-151, (1970).
9. Kumar, N.M., *On Two Conjectures About Polynomial Rings*, Inventiones Math. 46, 225-236, (1978).
10. Brown, W.C., *Matrices over commutative rings.* Marcel Decker Inc, New York, 149-175, (1993).
11. Shores, T., Wiegand, R., *Rings whose finitely generated modules are direct sums of cyclics.* Journal of Algebra 32, 152-172, (1974).
12. Lafond, J.P., *Anneaux locaux commutatifs sur lesquels tout module de type fini est somme directe de modules monogènes.* Journal of Algebra 17, 575 - 591, (1971).
13. Barnard, A., *Multiplication modules.* Journal of Algebra 71, 174-178, (1981).
14. Eisenbud, D., *Commutative Algebra with a view towards Algebraic Geometry.* GTM 150, Springer Verlag, New York (1994).

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