Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–5.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.50824

On the logarithmic summability $(L, 1)$ of integrals on $[1, \infty)$

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ABSTRACT: Móricz [Analysis (Munich) 18(1) (1998), 1-8] characterized summability $(C, 1)$ of integrals by convergence of another integral. In this work, we extend this result to logarithmic summability $(L, 1)$ of integrals.

Key Words: Tauberian theorems, logarithmic summability $(L, 1)$, improper integrals.

Contents

1. Introduction

Let $f : [0, \infty) \to \mathbb{C}$ be a Lebesgue integrable function on every bounded interval $[0, t]$ for $0 < t$. In this case we write $f \in L^1_{loc}[0, \infty)$. We define

$$k(t) := \int_0^t f(x)dx \quad \text{and} \quad \sigma(t) := \frac{1}{t}\int_0^t k(u)du.$$

The integral

$$\int_0^\infty f(x)dx \tag{1.1}$$

is called to be summable $(C, 1)$ (or Cesàro summable of first order) to a finite complex number l if

$$\lim_{t\to\infty} \sigma(t) = l. \tag{1.2}$$

Let $f : [1, \infty) \to \mathbb{C}$ be such that $f \in L^1_{loc}[1, \infty)$ and $s \in L^1_{loc}[1, \infty)$. We define

$$s(t) := \int_1^t f(x)dx \quad \text{and} \quad \tau(t) := \frac{1}{\log t}\int_1^t \frac{s(u)}{u}du,$$

where the logarithm is to the naturel base e. The integral

$$\int_1^\infty f(x)dx \tag{1.3}$$

is called to be summable $(L, 1)$ (or logarithmic summable of first order) to a finite complex number l if

$$\lim_{t\to\infty} \tau(t) = l. \tag{1.4}$$

We note that if the integral (1.3) is summable $(C, 1)$, then it is summable $(L, 1)$ to the same limit, but the converse is not satisfied in general (see [3]).

2010 *Mathematics Subject Classification:* 40A10, 40E05.
Submitted November 01, 2019. Published June 09, 2021

Typeset by $\mathcal{B}^S\mathcal{P}_\mathcal{M}$ style.
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Given $f \in L^1_{loc}[1,\infty)$, we define $a(t)$ as follows:

$$a(t) := \int_t^\infty \frac{f(x)}{t\log x}dx := \lim_{u\to\infty}\int_t^u \frac{f(x)}{t\log x}dx, \tag{1.5}$$

provided that the limit exists for some $t>1$. One can easily see that (1.5) exists and $a(t)$ is continuous at any $t>1$, $a \in L^1_{loc}[1,\infty)$ and

$$a(t)\to 0 \quad \text{as} \quad t\to\infty. \tag{1.6}$$

Hardy [1] characterized summability $(C,1)$ of series by convergence of another series. As an integral analogue to a corresponding theorem on series proved by Hardy [1], Móricz [2] extended this result to locally integrable functions over $[0,\infty)$ and characterized summability $(C,1)$ of integrals by convergence of another integral. In this work, we extend this result to locally integrable functions over $[1,\infty)$ and characterize logarithmic summability $(L,1)$ of integrals by convergence of another integral.

Our main theorem is as follows:

Theorem 1.1. *Suppose that $f \in L^1_{loc}[1,\infty)$. Then the integral (1.3) is summable $(L,1)$ to a finite complex number l and*

$$\frac{s(t)}{\log t}\to 0 \quad as \quad t\to\infty \tag{1.7}$$

if and only if $a(t)$ exists for $t>1$ and

$$\int_1^\infty a(t)dt := \lim_{u\to\infty}\int_1^u a(t)dt = l. \tag{1.8}$$

2. Auxiliary results

For the proof of our main theorem, we need the following lemmas.

Lemma 2.1. *Suppose that $f \in L^1_{loc}[1,\infty)$. If the integral (1.3) is summable $(L,1)$ to a finite complex number l and condition (1.7) holds, then $a(t)$ defined in (1.5) exists for $t>1$.*

Proof. Let $1<t<u<\infty$. If we apply integrating by parts twice, we obtain

$$\begin{aligned}\int_t^u \frac{f(x)}{\log x}dx &= \frac{s(u)}{\log u}-\frac{s(t)}{\log t}+\int_t^u \frac{s(x)}{x\log^2 x}dx\\ &= \frac{s(u)}{\log u}-\frac{s(t)}{\log t}+\frac{\tau(u)}{\log u}-\frac{\tau(t)}{\log t}+2\int_t^u \frac{\tau(x)}{x\log^2 x}dx.\end{aligned} \tag{2.1}$$

Condition (1.7) implies

$$\frac{\tau(t)}{\log t}\to 0 \quad \text{as} \quad t\to\infty \tag{2.2}$$

by regularity of the logarithmic summability method. Taking (1.7) and (2.2) into account, we get

$$a(t) = -\frac{s(t)}{t\log t}-\frac{\tau(t)}{t\log t}+\frac{2}{t}\int_t^u \frac{\tau(x)}{x\log^2 x}dx, \tag{2.3}$$

where $u\to\infty$ in (2.1) for $t>1$. Since the integral on the right exists in Lebesgue's sense by (1.4), we see that $a(t)$ defined in (1.5) exists for $t>1$. □

Remark 2.2. *Under conditions of Lemma 2.1, we obtain that*

$$\log t\int_t^\infty \frac{s(x)-l}{x\log^2 x}dx\to 0 \quad as \quad t\to\infty.$$

It follows from (2.1) that

$$\int_t^u \frac{s(x)}{x\log^2 x}dx = \frac{\tau(u)}{\log u}-\frac{\tau(t)}{\log t}+2\int_t^u \frac{\tau(x)}{x\log^2 x}dx.$$

Lemma 2.3. *Suppose that $f \in L^1_{loc}[1,\infty)$. If $a(t)$ defined in (1.5) exists for $t>1$, then condition (1.7) holds.*

Proof. Let $1<t<u<\infty$. We obtain

$$\begin{aligned}\int_t^u f(x)dx &= \int_t^u \frac{f(x)\log x}{\log x}dx \\ &= \log u \int_1^t \frac{f(y)}{\log y}dy + \log u \int_t^u \frac{f(y)}{\log y}dy - \log t \int_1^t \frac{f(y)}{\log y}dy - \int_t^u \frac{dx}{x}\int_1^x \frac{f(y)}{\log y}dy \\ &= \log u \int_t^u \frac{f(y)}{\log y}dy - \int_t^u \frac{dx}{x}\left(\int_1^x \frac{f(y)}{\log y}dy - \int_1^t \frac{f(y)}{\log y}dy\right)\end{aligned}$$

by applying integrating by parts. Since

$$\int_1^u f(x)dx - \int_1^t f(x)dx = \log u \int_t^u \frac{f(y)}{\log y}dy - \int_t^u \frac{dx}{x}\int_t^x \frac{f(y)}{\log y}dy,$$

we get

$$\frac{s(u)}{\log u} = \frac{s(t)}{\log u} - \int_t^u \frac{dx}{x}\int_t^x \frac{f(y)}{\log y}dy + \int_t^u \frac{f(y)}{\log y}dy. \tag{2.4}$$

The first term on the right of the last equality tends to 0 as $u\to\infty$. The second term on the right is logarithmic mean of the third term, except the coefficient $(-(\log u - \log t)/\log u)$, which tends to (-1) as $u\to\infty$. By regularity, we have

$$\lim_{u\to\infty} \frac{1}{\log u - \log t}\int_t^u \frac{dx}{x}\int_t^x \frac{f(y)}{\log y}dy = \lim_{u\to\infty}\int_t^u \frac{f(y)}{\log y}dy.$$

Thus, we conclude that condition (1.7) is satisfied by (2.4). □

3. Proof of Theorem 1.1

Necessity. Assume that the integral (1.1) is $(L,1)$ summable to a finite complex number l and condition (1.7) is satisfied. The function $a(t)$ defined in (1.5) exists for $t>1$ by Lemma 2.1. We apply Fubini's theorem because of the integral $\displaystyle\int_u^v \frac{f(x)}{\log x}dx$ in the lower limit u belongs to $L^1_{loc}(1,v)$ for any $v>1$. For $1<t<v$, we obtain

$$\int_1^t \frac{du}{u}\int_u^v \frac{f(x)}{\log x}dx = \int_1^t\int_1^x \frac{f(x)}{u\log x}dudx + \int_t^v\int_1^t \frac{f(x)}{u\log x}dudx = s(t) + \log t\int_t^v \frac{f(x)}{\log x}dx. \tag{3.1}$$

Keeping t fixed as $v\to\infty$, we get

$$\frac{1}{\log t}\int_1^t a(u)du = \frac{s(t)}{\log t} + ta(t) \tag{3.2}$$

for $t>1$. Taking (2.3) and (3.2) into account, we have

$$\int_1^t a(u)du = -\tau(t) + 2\log t\int_t^\infty \frac{\tau(x)}{x\log^2 x}dx \tag{3.3}$$

and thus, we obtain

$$\int_1^t a(u)du - l = -(\tau(t)-l) + 2\log t\int_t^\infty \frac{\tau(x)-l}{x\log^2 x}dx. \tag{3.4}$$

We conclude that relation (1.8) is satisfied by (1.4).

Sufficiency. Assume that $a(t)$ defined in (1.5) exists for $t > 1$ and (1.8) is satisfied. Condition (1.7) is satisfied by Lemma 2.3. We just have to prove (1.4).

Taking into account (1.7) and (2.2) gives

$$\tau'(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.5}$$

If we set

$$\eta(t) := \log^2 t \int_t^{\infty} \frac{\tau(x) - l}{x \log^2 x} dx, \ \ t > 0, \tag{3.6}$$

we have

$$2\frac{\eta(t)}{\log t} - (\tau(t) - l)) \to 0 \quad \text{as} \quad t \to \infty \tag{3.7}$$

by (1.8) and (3.4).

It follows from (3.6) that

$$\begin{aligned} t\left(\eta(t) - \eta(t-1)\right) &= \frac{t\left(\log^2 t - \log^2(t-1)\right)}{2\log t}\left(2\frac{\eta(t)}{\log t} - (\tau(t) - l)\right) \\ &+ \frac{t\left(\log^2 t - \log^2(t-1)\right)}{2\log t}(\tau(t) - l) - t\log^2(t-1)\int_{t-1}^{t} \frac{\tau(x) - l}{x\log^2 x}dx \\ &= \frac{t\left(\log^2 t - \log^2(t-1)\right)}{2\log t}\left(2\frac{\eta(t)}{\log t} - (\tau(t) - l)\right) \\ &+ \frac{t(\log(t-1)(\log t - \log(t-1))}{2}\int_{t-1}^{t} \frac{\tau(x) - l}{x\log^2 x}dx \\ &+ t\log^2(t-1)\int_{t-1}^{t} \frac{\tau(t) - \tau(x)}{x\log^2 x}dx. \end{aligned}$$

The first term on the right hand side of the equality above tends to 0 as $t \to \infty$ by (3.7). By (2.2), (3.5) and the mean value theorem, we obtain that the second and third term on the right hand side of the equality above also tend to 0 as $t \to \infty$. This implies that

$$\frac{\eta(t)}{\log t} \to 0 \quad \text{as} \quad t \to \infty. \tag{3.8}$$

We have (1.4) by (3.7) and (3.8).

Acknowledgment

This paper is presented at the 3rd International Conference of Mathematical Sciences (ICMS 2019), Maltepe University, İstanbul, Turkey.

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