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On the Divergence of Two Subseries $\sum \frac{1}{p}$, and Theorems of De La Vallée Poussin and Landau-Walfisz

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ABSTRACT: Let $K = Q(\sqrt{d})$ be a quadratic field with discriminant d. It is shown that $\sum\limits_{(\frac{d}{p})=+1, p\ prime} \frac{1}{p}$ and $\sum\limits_{(\frac{d}{q})=-1, q\ prime} \frac{1}{q}$ are both divergent. Two different approaches are given to show the divergence: one using the Dedekind Zeta function and the other by Tauberian methods. It is shown that these two divergences are equivalent. It is shown that the divergence is equivalent to $L_d(1) \neq 0$(de la Vallée Poussin's Theorem).We prove that the series $\sum\limits_{(\frac{d}{p})=+1, p\ prime} \frac{1}{p^s}$ and $\sum\limits_{(\frac{d}{q})=-1, q\ prime} \frac{1}{q^s}$ have singularities on all the imaginary axis(analogue of Landau-Walfisz theorem)

Key Words: Legendre symbol, Dedekind Zeta function, density of primes, L-series, infinite products.

Contents

1. Introduction

Let $K = Q(\sqrt{d})$ be a quadratic field with discriminant d. There are an infinite number of primes p with Legendre symbol $(\frac{d}{p}) = +1$ (respectively -1) ([10], p91, p98). We ask : is $\sum\limits_{(\frac{d}{p})=+1} \frac{1}{p} = \infty$ (respectively $\sum\limits_{(\frac{d}{q})=-1} \frac{1}{q} = \infty$)?. The series are both divergent as we show in this article. We present two approaches

(i) using the divergence of the Dedekind Zeta function ζ_K at $s = 1$

(ii) using the value $\frac{1}{2}$ of the density of primes of symbol +1 (respectively -1).

2. Summary of our results

The accompanying diagram shows the logical implications. Of course, our motivation is to consider the two subseries of the divergent series $\sum\limits_{p} \frac{1}{p}$(Euler)

(i) Using $\zeta_K(1) = \infty$

Recall the Dedekind Zeta function in Euler product form

$$\zeta_K(s) = \prod_{\mathbb{P}} (1 - \frac{1}{N(\mathbb{P})^s})^{-1} \ (Re\ s > 1)$$

(here $\mathbb{P}$ ranges over all prime ideals of the integer ring of K and $N(\mathbb{P})$ denotes the norm of the ideal $\mathbb{P}$) ([10], p89)

Lemma 2.1. *([9], p40), $\sum\limits_{\mathbb{P}} \frac{1}{N(\mathbb{P})^s} = \sum\limits_{(\frac{d}{p})=0} \frac{1}{p^s} + \sum\limits_{(\frac{d}{p})=+1} \frac{1}{p^s} + \sum\limits_{(\frac{d}{q})=-1} \frac{1}{q^{2s}}$. Hence the series on the left diverges for $s = 1$ iff $\sum\limits_{(\frac{d}{p})=+1} \frac{1}{p} = \infty$.*

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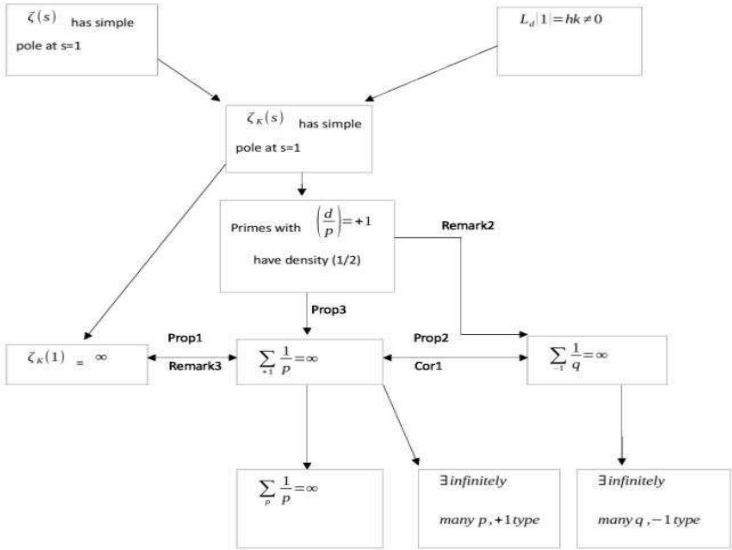

Figure 1:

Proof. This is by the standard criterion for norms: depending on the value of the symbol it is either a prime number p or the square of a prime number, q^2 ([10], p63, [9], p40). This decomposition into three sums is valid for Re $s > 1$ by absolute convergence of the series on the left.
Now the first series is finite and the last convergent for Re $s > \frac{1}{2}$. Hence taking limits as $s \to 1^+$, the LHS series is convergent iff $\sum\limits_{+1} \frac{1}{p} < \infty$. This proves the Lemma □

Recall that if $\prod\limits_{n=1}^{\infty}(1 - a_n)$ is an infinite product with $0 \le a_n < 1$ for all n then the product converges to a non zero number iff $\sum a_n$ converges; the product is 0 iff $\sum\limits_{n=1}^{\infty} a_n = \infty$. Likewise $\prod\limits_{n=1}^{\infty}(1 + a_n) < \infty$ iff $\sum\limits_{n=1}^{\infty} a_n < \infty$

Dedekind's Theorem :([10], p89) $Lim_{s\to 1^+}(s-1)\zeta_K(s) = L_d(1) \neq 0$.
Here the L-series is defined by

$$L_d(s) = \prod_p (1 - \frac{(\frac{d}{p})}{p^s})^{-1} = \sum_{n=1}^{\infty} \frac{(\frac{d}{n})}{n^s}$$

for Re $s > 1$. The last series converges for Re $s > 0$ and defines L_d as an analytic function for Re $s > 0$ and $Lim_{s\to 1^+} L_d(s) = L_d(1) \neq 0$

$$\zeta_K(s) = \zeta(s) L_d(s) \ (Re\ s > 1)$$

for the Riemann Zeta function $\zeta(s)$ ([10], p90).

Remark 2.2. *We have the logical equivalences, by the above:*

$$\sum_{+1} \frac{1}{p} = \infty \Longleftrightarrow \sum_{\mathbb{P}} \frac{1}{N(\mathbb{P})} = \infty$$

$$\Longleftrightarrow \prod_{\mathbb{P}} (1 - \frac{1}{N(\mathbb{P})}) = 0$$

$$\Longleftrightarrow \zeta_K(1) = \infty$$

Proposition 2.3. $\sum\limits_{+1} \frac{1}{p} = \infty$

Proof. By Dedekind, $Lim_{s\to 1^+}\zeta_K(s)(s-1) \neq 0$ so that $\zeta_K(1)$ cannot be finite, or else the limit would be zero. Now the above train of equivalence yields the divergence of our series. □

Proposition 2.4. $\sum\limits_{-1} \frac{1}{q} = \infty$

Proof. The value of $(\frac{d}{p}) = 0,\ +1\ or\ -1$ ([10], p63) according as $p|d$, p splits in K or p is prime in K. The corresponding products give, by rearrangement using absolute convergence for Re $s > 1$,

$$L_d(s) = \frac{1}{\prod\limits_{p,+1}(1-\frac{1}{p^s})\prod\limits_{q,-1}(1+\frac{1}{q^s})} = \frac{1}{\prod\limits_{p}(1-\frac{(\frac{d}{p})}{p^s})}$$

So $(\prod\limits_{p,+1}(1-\frac{1}{p^s}))L_d(s) = \frac{1}{\prod\limits_{q,-1}(1+\frac{1}{q^s})}$ Let $s \to 1^+$ and use the divergence of $\sum\limits_{+1}\frac{1}{p}$ above: the LHS has limit$=(0)(L_d(1)) = 0$. Hence the product on RHS cannot converge to a non zero number. So $\sum\limits_{-1}\frac{1}{q} = \infty$ □

Corollary 2.5. *If* $\sum\limits_{-1}\frac{1}{q} = \infty$*, then* $\sum\limits_{+1}\frac{1}{p} = \infty$

Proof. By the argument in the above proof of Prop2, we may cross multiply

$$L_d(s)\prod_{q,-1}(1+\frac{1}{q^s}) = \frac{1}{\prod\limits_{p,+1}(1-\frac{1}{p^s})}\ (Re\ s > 1)$$

Now as $s \to 1^+$ the limit on the left is ∞ so that the product on the right cannot converge to a non zero value as $s \to 1^+$. Hence $\sum\limits_{p,+1}\frac{1}{p}$ cannot converge.

(ii) Using Density of Splitting Primes

The Dirichlet density $d(A)$ for a subset A of the set of primes is defined as

$$d(A) = Lim_{s\to 1^+}\frac{\sum\limits_{p\in A}\frac{1}{p^s}}{log(\frac{1}{s-1})}$$

We are interested in the case $A = \{p|(\frac{d}{p}) = +1\}$.

We have then, $d(A) = Lim_{x\to\infty}\frac{|\{p|p\in A, p\le x\}|}{|\{p|p\le x\}|}$= natural density of A ([8], p76) □

Theorem 2.6. *If A is the set of primes p of symbol* $(\frac{d}{p}) = +1$*, then the density* $d(A) = \frac{1}{2}$ *([3], p163, [8], p75).*

We show the divergence of $\sum\limits_{+1}\frac{1}{p}$ using the equality

$$\prod_{p,+1}(1-\frac{1}{p^s}) = \sum_{n'}\frac{\mu(n')}{(n')^s}$$

where n' ranges over the positive integers whose prime factors are all of symbol $+1$, μ is Moebius function.

Proposition 2.7. $\sum\limits_{n'} \frac{\mu(n')}{n'} = 0$ *(*$= Lim_{s\to 1^+} \sum\limits_{n'} \frac{1}{(n')^s} = Lim_{s\to 1^+} \prod\limits_{p,+1}(1-\frac{1}{p^s}))$ *so that* $\prod\limits_{p,+1}(1-\frac{1}{p}) = 0$ *and* $\sum\limits_{+1} \frac{1}{p} = \infty$

Proof. We show in Lemma 2 below that $\sum\limits_{n'\le x} \mu(n') = \circ(x)$. In Lemma 3 we use a result in Hlawka et al [4], p200 to conclude that $\sum\limits_{n'} \frac{\mu(n')}{n'} = 0$. □

Lemma 2.8. $Lim_{x\to\infty} \frac{1}{x}(\sum\limits_{n'\le x} \mu(n')) = 0$

Proof.

$$\begin{aligned}
|Lim_{x\to\infty} \frac{1}{x} \sum_{n'\le x} \mu(n')| &\le Lim_{x\to\infty} \frac{1}{x}(\sum_{n'\le x} |\mu(n')|) \\
&\le Lim_{x\to\infty}\{\frac{(\sum\limits_{n'\le x} |\mu(n')|)}{\sum\limits_{n\le x} |\mu(n)|} \frac{\sum\limits_{n\le x} |\mu(n)|}{x}\} \\
&= Lim_{x\to\infty}\{\frac{\sum\limits_{n'\le x} |\mu(n')|}{\sum\limits_{n\le x} |\mu(n)|}\}.\frac{1}{\zeta(2)}
\end{aligned}$$

Now we claim the ratio $\frac{(\sum\limits_{n'\le x} |\mu(n')|)}{(\sum\limits_{n\le x} |\mu(n)|)}$ tends to 0 as $x\to\infty$ because it can be rewritten as the ratio of the number of squarefree integers formed from the first t primes of symbol +1: $\frac{|\{p_{i_1}p_{i_2}\dots p_{i_m} | i_j \le t, (\frac{d}{p_{i_j}})=+1\}|}{|\{p_{i_1}p_{i_2}\dots p_{i_m} | i_j \le t\}|}$ t depending on x.

But counting the "favourable cases" and using the above Theorem on density, this ratio is asymptotically $(\frac{3}{4})^t$ which tends to 0 as $t\to\infty$ or $x\to\infty$. This proves Lemma2.

We cite the result in Hlawka et al [5], p200 as.

□

Theorem 2.9. *Let* $f(n)$ *be an arithmetic function whose mean is 0 i.e* $\sum\limits_{n\le x} f(n) = \circ(x)$*. Then*

$$\sum_{n\le x} \frac{f(n)}{n} = \frac{1}{x}\{\sum_{n\le x}(\sum_{k|n} f(k))\} + \circ(1)$$

.

We apply this result with the choice of $f(n)$:

$$f(n) = \begin{cases} \mu(n') & if\ n = n' \\ 0 & \text{otherwise} \end{cases}$$

Indeed, by Lemma2 the hypothesis of Theorem2 is satisfied and $\sum\limits_{n\le x} \frac{f(n)}{n} = \sum\limits_{n'\le x} \frac{\mu(n')}{n'} = \frac{1}{x}\{\sum\limits_{n\le x}(\sum\limits_{k|n} f(k))\} + \circ(1)$

Now $\sum\limits_{k|n} f(k)$ is evaluated as follows. Write $n = n'n''$ where n' gives the sub product of the prime factors of symbol +1 and n'' the product of the prime factors of symbol -1 or 0. So

$$\sum_{k|n} f(k) = \sum_{k|n'n''} f(k) = \sum_{k|n'} \mu(k) = \begin{cases} 1 & if\ n' = 1 \\ 0 & if\ n' > 1 \end{cases}$$

Hence $\frac{1}{x}(\sum_{n\leq x}(\sum_{k|n} f(k))) = \frac{1}{x}(\sum_{n=n''\leq x} 1)$ as $x \to \infty$ this last ratio tends to 0 because asymptotically this fraction is $(\frac{1}{2})^{\omega(m)} \sim (\frac{1}{2})^{log\ logm}$ by Hardy-Ramanujan's estimate ([4], p 50). This shows $\sum \frac{\mu(n')}{n'} = 0$ and proves Proposition3.

Remark 2.10. *We may show* $\sum_{-1} \frac{1}{q} = \infty$ *by a simple adaptation of the proof of Proposition3, exchanging the roles of* n' *and* n''*. As we have deduced this divergence from that of* $\sum_{+1} \frac{1}{p}$ *in Prop2 above, we do not rewrite the proof of Prop3.*

Remark 2.11. *We obtain an independent proof of the divergence of* ζ_K *at* $s = 1$ *in view of Remark 1 above. Indeed* $\sum_{+1} \frac{1}{p^s}$ *is a subseries of* $\sum_{\mathbb{I}\ ideal} \frac{1}{N(\mathbb{I})^s} = \zeta_K(s)$*.*

(iii)Equivalence with de la Vallée Poussin's Theorem

We showed that $\sum_{(\frac{d}{p})=+1} \frac{1}{p} = \infty$ if d is a squarefree integer ($d \neq 1$). Here we deduce that this divergence is equivalent to $L_d(1) \neq 0$. Recall the definition

$$L_d(s) = \prod_p (1 - \frac{(\frac{d}{p})}{p^s})^{-1} = \sum_{n=1}^{\infty} \frac{(\frac{d}{n})}{n^s}$$

This is done by showing that the divergence is equivalent to "$\psi(1) = \infty$" (simple pole) for

$$\psi(s) = \frac{L_d(s)L(s,\chi_0)}{L(2s,\chi_0)} \ (\star)$$

where $L(s,\chi_0) = (\prod_{p|d}(1 - \frac{1}{p^s}))\zeta(s)$. This behaviour of ψ is the basis of de la Valle'e Poussin's proof that $L_d(1) \neq 0$ ([6], p260).

Proposition 2.12. *Let* $\psi(s)$ *be defined by* $(\star)$*. Then* $Lim_{s\to 1^+}\psi(s)$ *is not finite.Hence* $L_d(1) \neq 0$

Proof. As in [6], p260 we expand for Re $s > 1$

$$\begin{aligned}
\psi(s) &= \prod_{(\frac{d}{p})=+1} \frac{(1+p^{-s})}{(1-p^{-s})} \\
&= \prod_{(\frac{d}{p})=+1} \frac{(1+p^{-s})(1-p^{-s})}{(1-p^{-s})(1-p^{-s})} \\
&= \prod_{(\frac{d}{p})=+1} \frac{(1-p^{-2s})}{(1-p^{-s})^2} \\
&= \frac{\prod_{(\frac{d}{p})=+1} (1-p^{-2s})}{\prod_{(\frac{d}{p})=+1} (1-p^{-s})^2}
\end{aligned}$$

when $s = 1$ the numerator is $\frac{1}{\zeta(2)} = \frac{6}{\pi^2}$. We examine the denominator for $s = 1$.

Now $\prod_{(\frac{d}{p})=+1} (1-p^{-1})^2 < \prod_{(\frac{d}{p})=+1} (1-p^{-1}) = 0$

since $\sum_{(\frac{d}{p})=+1} \frac{1}{p} = \infty$, using the standard criterion for infinite product:

$\prod_n (1-a_n)$ *converges* $\Longleftrightarrow \sum a_n$ *converges and* $\sum |a_n|^2$ converges ($\star\star$)

Indeed the L-functions are analytic for $\{Re\ s > 0\}$ ([5]) and so if $L_d(1) = 0$ then $\psi(1)$ is finite, since this zero cancels the pole of $\zeta(1)$. But the above gives $\psi(1)$ infinite so that $L_d(1) \neq 0$.

Conversely, if $L_d(1) \neq 0$ then $\psi(s)$ has a pole at $s = 1$ and so the product $\prod_{(\frac{d}{p})=+1} (1 - \frac{1}{p})$ cannot converge(to a nonzero number)
i.e., $\prod_{(\frac{d}{p})=+1} (1-\frac{1}{p}) = 0$ and $\sum_{(\frac{d}{p})=+1} \frac{1}{p} = \infty$.

□

Remark 2.13. *An important consequence of $L_d(1) \neq 0$ is that the full infinite product*

$$\prod_p (1 - \frac{(\frac{d}{p})}{p})^{-1}$$

converges. Further, the value is $L_d(1)$ since L_d is continuous. Thus the divergence of the subproduct implies the convergence of the full Euler product. Though this convergence is known the standard textbooks like [2] do not give a proof. Here we sketch a proof based on the results in Chs 6 and 7 of [2].

Lemma 2.14. $\sum_{p \le x} \frac{(\frac{d}{p}) logp}{p} = \bigcirc(1)$

Proof. See [2], pp 149-152. □

Corollary 2.15. $\sum_p \frac{(\frac{d}{p})}{p} = 0$

Proof. Let

$$a_n = \begin{cases} \frac{(\frac{d}{p})}{p} & if\ n = p,\ prime \\ 0 & \text{if n not prime} \end{cases}$$

Thus $\sum_{p \le x} a_n = \sum_{p \le x} \frac{(\frac{d}{p})}{p} = \bigcirc(1)$ by Lemma 3.

Now consider $\sum_{p<n\le x} \frac{a_n}{logn}$. Applying Abel summation

$\sum_{1<n\le x} \frac{a_n}{logn} = \sum_{p \le x} \frac{(\frac{d}{p})}{p} = \bigcirc(\frac{1}{logx})$ ([2], Th 4.2)

Letting $x \to \infty$ we have the series $\sum_p \frac{(\frac{d}{p})}{p} = 0$. □

Corollary 2.16. $\prod_p (1 - \frac{(\frac{d}{p})}{p})$ *is convergent (to $L_d(1) \neq 0$)*

Proof. By the standard criterion ($\star\star$) the product converges if and only if the series $\sum_p \frac{(\frac{d}{p})}{p}$ converges and $\sum |\frac{(\frac{d}{p})}{p}|^2 < \infty$. This is ensured by Cor 2 and the convergence of $\sum_p \frac{1}{p^2}$. Since $L_d(s)$ is continuous (indeed analytic) for Re $s > 0$, the value of the product agrees with the limit as $s \to 1^+$ of $L_d(s)$ which is $L_d(1)$.
□

(iv)An analogue of Landau-Walfisz Theorem

We prove the following analogues of the Landau-Walfisz theorem on analytic continuation of $\sum_{p\ prime} \frac{1}{p^s}$. Recall that this states that this imaginary axis is a natural boundary of the meromorphic continuation of this series([9], p40, [8])

Proposition 2.17.

(a) *Let d be a squarefree integer. Then the series* $\sum_{(\frac{d}{p})=+1} \frac{1}{p^s}$ *and* $\sum_{(\frac{d}{q})=-1} \frac{1}{q^s}$ *have the imaginary axis as natural boundary.*

(b) *Consider the quadratic field* $Q(\sqrt{d})$. *The analogous series over prime ideals* $\sum_{P} \frac{1}{N(P)^s}$ *has the imaginary axis as natural boundary(Andrade's result, special case)*

Before giving the proof of the proposition1 we mention the following relevant results.

Kurokawa's Theorem: The Euler products $\zeta_+(s) = \prod_{(\frac{d}{p})=+1} (1-\frac{1}{p^s})^{-1}$ and $\zeta_-(s) = \prod_{(\frac{d}{q})=-1} (1-\frac{1}{q^s})^{-1}$ have analytic continuation in $\{Re\ s > 1\}$ and natural boundary $\{Re\ s = 0\}$

Proposition 2.18.

(a) $\sum_{p} \frac{1}{p^s} = \sum_{n} \frac{\mu(n)}{n} log\zeta(ns)$ *($\zeta(s)$=Riemann Zeta function)*

(b) $\sum_{P} \frac{1}{N(P)^s} = \sum_{n} \frac{\mu(n)}{n} log\zeta_k(ns)$ *($\zeta_k(s)$=Dedekind Zeta function)*

6(a) is due to Landau-Walfisz ([11], p12) and 6(b) was derived in ([9])

Proof of Proposition1:

(a) One has

$$\sum_{(\frac{d}{p})=+1} \frac{1}{p^s} = \sum_{n} \frac{\mu(n)}{n} log\zeta_+(ns)$$

imitating the derivation in Titchmarsh ([11],p12). For each n, the function $f_n(s) = \frac{\mu(n)}{n} log\zeta_+(ns)$ is analytic on $\{Re\ s > 0\}$ except for singularities which accumulate in each interval of the imaginary axis ([1]). Now if $m \neq n$ the singularities of $f_m(s)$ are distinct from those of $f_n(s)$. Hence the function $F_N(s) = \sum_{n=1}^{N} f_n(s)$ has the same behaviour as the individual $f_n(s)$ with singularity set= union of singularity sets of $f_1, f_2, ..., f_N$. Letting $N \to \infty$ we have that each point of the imaginary axis is a singularity for $f(s) = Lim_{N\to\infty} f_N(s)$.

An identical argument applies for $\sum_{(\frac{d}{q})=-1} \frac{1}{q^s}$.

(b) Note that due to the splitting behaviour of primes ([10],p63)

$$\begin{aligned}
\sum_{P}\frac{1}{N(P)^s} &= \sum_{+1}\frac{1}{p^s}+\sum_{-1}\frac{1}{q^{2s}}\\
&= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}log\zeta_{+}(ns)+\sum_{n=1}^{\infty}\frac{\mu(n)}{n}log\zeta_{-}(2ns)\\
&= \sum_{n=1}^{\infty}\frac{\mu(n)}{n}log(\zeta_{+}(ns)\zeta_{-}(2ns))
\end{aligned}$$

But

$$\begin{aligned}
\zeta_{+}(ns)\zeta_{-}(2ns) &= \prod_{+1}(1-\frac{1}{p^s})^{-1}\prod_{-1}(1-\frac{1}{q^{2s}})^{-1}\\
&= \prod_{+1}(1-\frac{1}{p^s})^{-1}\prod_{-1}(1-\frac{1}{q^{ns}})^{-1}\prod_{-1}(1+\frac{1}{q^{sn}})^{-1}\\
&= \frac{\zeta(ns)}{\prod_{p|n}(1-\frac{1}{p^{ns}})^{-1}}\prod_{-1}(1+\frac{1}{q^{sn}})^{-1}\\
&= \{E(s)\}\frac{\prod_{-1}(1-\frac{1}{q^{ns}})^{-1}}{\prod_{-1}(1-\frac{1}{q^{2ns}})^{-1}}\\
&= E(s)\frac{\zeta_{-}(ns)}{\zeta_{-}(2sn)}
\end{aligned}$$

Now $E(s)$ is entire except for a pole at $s=\frac{1}{n}$ and the ratio gives a unique meromorphic function. Now the argument in (a) applies to give singularity at every point of the imaginary axis.

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