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Notes on the Bienergy and Biangle of a Moving Particle Lying in a Surface of Euclidean Space

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ABSTRACT: In this study, we research bienergy and biangles of moving particles lying in the surface of Euclidean space by using their energy and angle values. We present a geometrical understanding of a bienergy of a particle in Darboux vector fields depending on surface. We also give a relation between bienergy of the curve corresponding to a moving particle in space and bienergy of the elastica assuming the curve that has the elastic feature. We conclude our results by providing a bienergy-curve graphics for different cases.

Key Words: Bienergy, Darboux vector fields, surface, space curves.

Contents

1 Introduction **1**

2 Preliminaries **2**

3 Bienergy and biangle of a particle lying in a surface of Euclidean space **3**

4 Application **7**

1. Introduction

The idea of elasticity is a phenomenon that can be evaluated from many different perspectives, especially in the physical and mathematical sciences. For instance, all substances such as cloths, flexible metals, rubber, paper, etc. in the real world are included in the theory of elasticity. Areas where elasticity studies are primarily concentrated are mechanical equilibrium, variational problems, and elliptic integral solutions [1]- [6].

The results obtained in the first studies on elasticity are about equilibrium of moments which establishes the basic principles of statics. Further, the variational problem to minimize bending energy of the elastic curve is solved by elastica. Thereafter, it has been found that there is an equivalency between the pendulum motion and basic differential equation of elastica [4].

Computation energy of a given vector field subject to the structure of space gains such attention in the last couple years. Computations in these forms have various applications in different fields. For example, Wood [1], studied on the unit vector field's energy firstly. Gil-Medrano [2], worked on relation between energy and volume of vector fields. In [3], Kirchhoff, and in [4], Catmull investigated on the energy of distrubutions and corrected energy of distrubutions on Riemannian manifolds. Altin [7], computed energy of a Frenet vector fields for a given nonlightlike curves.

Darboux frame, which is a natural moving frame and corresponds to Frenet frame that implemented to geometry of surface, is used as a bridge since functionals of surface energy are constructed by functionals of energy constructed for curves [8]. Korpinar and Demirkol approached the topic with a different perspective by calculating curvature-based energy for surface curves to investigate the relation between energy of particle on surface and curvature-based bending energy functional. The method they use for computing the energy of Darboux vector fields is that considering a vector field as a map from manifold M to the Riemannian manifold $(TM,\ \rho_s)$, where TM is tangent bundle of a Riemannian manifold and ρ_s is a Sasaki metric induced from TM naturally [9], [12]- [15].

In this study, we first present fundamental definitions and Darboux frame equations for surfaces and curves in Euclidean space. Then we recall the definitions and state interpretation of geometrical meaning

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of the energy for unit vector fields. Based on these relations, we compute the bienergies and the biangles of moving particle's vector fields corresponding to a curve in the space lying in a surface of Euclidean space. Finally, we give examples about particle's energy for different cases by computing their value and drawing thier graph.

2. Preliminaries

Let Ω be a particle moving in a space such that the precise location of the particle is specified by $\Omega = \Omega(t)$, where t is a time parameter. Changing time parameter describes the motion and hereby the trajectory corresponds to a curve ω in the space for a moving particle. It is convenient to remind the arc-length parameter s, which is used to compute the distance traveled by a particle along its trajectory. It is defined by

$$\frac{ds}{dt} = \|\mathbf{v}\|,$$

where $\mathbf{v} = \mathbf{v}(t) = \frac{d\omega}{dt}$ is the velocity vector and $\frac{d\omega}{dt} \neq 0$. In particle dynamics, the arc-length parameter s is considered as a function of t [8]. Thanks to the arc-length, it is also determined Serret-Frenet frame, which allows us to determine characterization of the intrinsic geometrical features of the regular curve. This coordinate system is constructed by three orthonormal vectors $\mathbf{e}^{\mu}_{(\alpha)}$ and the curve ω itself, assuming the curve is sufficiently smooth at each point. The index within the paranthesis is the tetrad index that describes particular member of the tetrad. In particular, $\mathbf{e}^{\mu}_{(0)}$ is the unit tangent vector, $\mathbf{e}^{\mu}_{(1)}$, $\mathbf{e}^{\mu}_{(2)}$ is the unit normal and binormal vector fields of the curve ω, respectively.

For the trajectory of the moving particle which corresponds to a curve ω on the surface, the Darboux vectors $\mathbf{e}^{\mu}_{(0)}, \mathbf{n}, \ \mathbf{P} = \mathbf{e}^{\mu}_{(0)} \times \mathbf{n}$, which are respectively known as curve's unit tangent, surface's unit normal, and tangent's normal, satisfies following equations and properties, [8]

$$\begin{aligned}
&\nabla_{\mathbf{e}^{\mu}_{(0)}} \mathbf{e}^{\mu}_{(0)} = \kappa_g \mathbf{P} + \kappa_n \mathbf{n}, && (1)\\
&\nabla_{\mathbf{e}^{\mu}_{(0)}} \mathbf{P} = -\kappa_g \mathbf{e}^{\mu}_{(0)} + \tau_g \mathbf{n}, && (2)\\
&\nabla_{\mathbf{e}^{\mu}_{(0)}} \mathbf{n} = -\kappa_n \mathbf{e}^{\mu}_{(0)} - \tau_g \mathbf{P}. && (3)
\end{aligned} \tag{2.1}$$

where $\kappa_g, \ \kappa_n, \ \tau_g$ are geodesic curvature, normal curvature and geodesic torsion of the curve, respectively [20]. Since we identify $\mathbf{e}^{\mu}_{(0)}$ as a unit vector which is a tangent to the the curve at each point on the curve, we have $\mathbf{e}^{\mu}_{(0)} = d\Omega^u/ds$, where Ω^u is the point on the trajectory of curve ω. Thus $\mathbf{e}^{\mu}_{(0)}, \mathbf{P}$ and $\mathbf{n}$ generate the Darboux frame and equations $(1-3)$ are known as Darboux equations for each case [8].

Definition 2.1. *For two Riemannian manifolds (M, ρ) and $\left(N, \tilde{h}\right)$, the energy of a differentiable map $f : (M, \rho) \to \left(N, \tilde{h}\right)$ can be defined as*

$$energy\,(f) = \tfrac{1}{2} \int_M \sum_{a=1}^{n} \tilde{h}\left(df\left(e_a\right), df\left(e_a\right)\right) v, \tag{2.2}$$

where $\{\mathbf{e}_a\}$ is a local basis of the tangent space and v is the canonical volume form in M [10]-[11].

Proposition 2.2. *Let $Q : T\left(T^1 M\right) \to T^1 M$ be the connection map. Then following two conditions hold: i) $\omega \circ Q = \omega \circ d\omega$ and $\omega \circ Q = \omega \circ \tilde{\omega}$, where $\tilde{\omega} : T\left(T^1 M\right) \to T^1 M$ is the tangent bundle projection; ii) for $\varrho \in T_x M$ and a section $\xi : M \to T^1 M$; we have*

$$Q\left(d\xi\left(\varrho\right)\right) = D_{\varrho}\xi, \tag{2.3}$$

where D is the Levi-Civita covariant derivative [10]-[11].

Definition 2.3. *For $\varsigma_1, \varsigma_2 \in T_{\xi}\left(T^1 M\right)$, we define*

$$\rho_S\left(\varsigma_1, \varsigma_2\right) = \rho\left(d\omega\left(\varsigma_1\right), d\omega\left(\varsigma_2\right)\right) + \rho\left(Q\left(\varsigma_1\right), Q\left(\varsigma_2\right)\right). \tag{2.4}$$

This yields a Riemannian metric on TM. As known ρ_S is called the Sasaki metric that also makes the projection $\omega : T^1 M \to M$ a Riemannian submersion [10]-[11].

Definition 2.4. *[7] Angle between arbitrary Frenet vectors is given by*

$$angle(\mathbf{V}_i)=\int_0^s \|\nabla_{\mathbf{V}_1}\mathbf{V}_i\|\,du.$$

Definition 2.5. *[16] Bienergy of the vector field* $\mathbf{X}$ *is given by following equation*

$$energy_2\left(\mathbf{X}\right)=\int_0^s \rho_S\left(\nabla_{\mathbf{V}_1}\nabla_{\mathbf{V}_1}\mathbf{X},\nabla_{\mathbf{V}_1}\nabla_{\mathbf{V}_1}\mathbf{X}\right)ds. \tag{2.5}$$

Definition 2.6. *[16] Biangle between arbitrary Frenet vectors is given by*

$$angle_2(\mathbf{V}_i)=\int_\vartheta^s \left\|\nabla^2_{\mathbf{V}_1}\mathbf{V}_i\right\|du. \tag{2.6}$$

3. Bienergy and biangle of a particle lying in a surface of Euclidean space

Theorem 3.1. *Bienergy on the moving particle in tangent vector field by using Sasaki metric is stated by*

$$energy_2\mathbf{e}^{\mu}_{(0)}=\int_0^s\left(\kappa_g^2+\kappa_n^2+(\kappa_g^2+\kappa_n^2)^2+(\kappa_g'-\kappa_n\tau_g)^2+(\kappa_n'+\kappa_g\tau_g)^2\right)ds.$$

Proof. From (2.4) and (2.5) we know

$$energy_2\mathbf{e}^{\mu}_{(0)}=\int_0^s \rho_S\left(\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)},\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}\right)ds.$$

Using Eq. (2.4) we have

$$\begin{aligned}&\rho_S\left(d\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)}),d\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})\right)=\rho(d\omega(\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})),d\omega(\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})))\\&+\rho(Q(\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)})),Q(\mathbf{e}^{\mu}_{(0)}(\mathbf{e}^{\mu}_{(0)}))).\end{aligned}$$

Since $\mathbf{e}^{\mu}_{(0)}$ is a section, we get

$$d(\omega)\circ d(\mathbf{e}^{\mu}_{(0)})=d(\omega\circ\mathbf{e}^{\mu}_{(0)})=d(id_C)=id_{TC}.$$

We also know

$$\begin{aligned}&Q(\mathbf{e}^{\mu}_{(0)}(\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}))=\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}\\&=\kappa_g'\mathbf{P}+\kappa_g(-\kappa_g\mathbf{e}^{\mu}_{(0)}+\tau_g\mathbf{n})+\kappa_n'\mathbf{n}+\kappa_n(-\kappa_n\mathbf{e}^{\mu}_{(0)}-\tau_g\mathbf{P})\\&=-(\kappa_g^2+\kappa_n^2)\mathbf{e}^{\mu}_{(0)}+(\kappa_g'-\kappa_n\tau_g)\mathbf{P}+(\kappa_n'+\kappa_g\tau_g)\mathbf{n}\end{aligned}$$

Thus, we find from (2.4)

$$\begin{aligned}&\rho_S\left(\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)},\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}\right)=\rho\left(\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)},\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}\right)\\&+\rho\left(\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)},\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}\right)\\&=\kappa_g^2+\kappa_n^2+(\kappa_g^2+\kappa_n^2)^2+(\kappa_g'-\kappa_n\tau_g)^2+(\kappa_n'+\kappa_g\tau_g)^2.\end{aligned}$$

So we can easily obtain

$$energy_2\mathbf{e}^{\mu}_{(0)}=\int_0^s\left(\kappa_g^2+\kappa_n^2+(\kappa_g^2+\kappa_n^2)^2+(\kappa_g'-\kappa_n\tau_g)^2+(\kappa_n'+\kappa_g\tau_g)^2\right)ds.$$

This completes the proof. □

Corollary 3.2. *Bienergy on the moving particle in tangent vector field is fixed iff*

$$(\tau_g\kappa_g-\kappa_n')^2+(\kappa_n\kappa_g+\tau_g')^2+(\kappa_n^2+\tau_g^2)^2=0.$$

Theorem 3.3. *Biangle of the moving particle in tangent vector field is*

$$angle_2(\mathbf{e}^{\mu}_{(0)})=\int_0^s((\tau_g\kappa_g-\kappa_n')^2+(\kappa_n\kappa_g+\tau_g')^2+(\kappa_n^2+\tau_g^2)^2)^{\frac{1}{2}}du.$$

Proof. By definition 2.6, we write that

$$angle_2(\mathbf{e}^{\mu}_{(0)}) = \int_0^s \left\| \nabla^2_{\mathbf{e}^{\mu}_{(0)}} \mathbf{e}^{\mu}_{(0)} \right\| du$$
$$= \int_0^s ((\tau_g \kappa_g - \kappa_n')^2 + (\kappa_n \kappa_g + \tau_g')^2 + (\kappa_n^2 + \tau_g^2)^2)^{\frac{1}{2}} du.$$

□

Theorem 3.4. *Bienergy on the moving particle in normal vector field by using Sasaki metric is stated by*

$$energy_2(\mathbf{n}) = \int_0^s \left(\tau_g^2 + \kappa_n^2 + (\tau_g \kappa_g - \kappa_n')^2 + (\kappa_n \kappa_g + \tau_g')^2 + (\kappa_n^2 + \tau_g^2)^2\right) ds.$$

Proof. From (2.4) and (2.5) we arrive

$$energy_2(\mathbf{n}) = \int_0^s \rho_S \left(\nabla_{\mathbf{e}^{\mu}_{(0)}} \nabla_{\mathbf{e}^{\mu}_{(0)}} \mathbf{n}, \nabla_{\mathbf{e}^{\mu}_{(0)}} \nabla_{\mathbf{e}^{\mu}_{(0)}} \mathbf{n} \right) ds.$$

Using Eq. (2.3) in (2.4), we have

$$\rho_S \left(d\mathbf{e}^{\mu}_{(0)}(\mathbf{n}), d\mathbf{e}^{\mu}_{(0)}(\mathbf{n}) \right) = \rho(d\omega(\mathbf{e}^{\mu}_{(0)}(\mathbf{n})), d\omega(\mathbf{e}^{\mu}_{(0)}(\mathbf{n}))) + \rho(Q(\mathbf{e}^{\mu}_{(0)}(\mathbf{n})), Q(\mathbf{e}^{\mu}_{(0)}(\mathbf{n}))).$$

Since $e^{\mu}_{(0)}$ is a section, we get

$$d(\omega) \circ d(\mathbf{n}) = d(\omega \circ \mathbf{n}) = d(id_C) = id_{TC}.$$

We also know

$$Q(\mathbf{e}^{\mu}_{(0)}(\nabla_{\mathbf{e}^{\mu}_{(0)}} \mathbf{n})) = \nabla_{\mathbf{e}^{\mu}_{(0)}} \nabla_{\mathbf{e}^{\mu}_{(0)}} \mathbf{n}$$
$$= (\tau_g \kappa_g - \kappa_n') e^{\mu}_{(0)} - (\kappa_n \kappa_g + \tau_g') P - (\kappa_n^2 + \tau_g^2) n.$$

Thus, we use the Darboux derivative formulas (2.1) in (2.4):

$$\rho_S \left(\nabla_{\mathbf{e}^{\mu}_{(0)}} \nabla_{\mathbf{e}^{\mu}_{(0)}} \mathbf{n}, \nabla_{\mathbf{e}^{\mu}_{(0)}} \nabla_{\mathbf{e}^{\mu}_{(0)}} \mathbf{n} \right) = \rho \left(\nabla_{\mathbf{e}^{\mu}_{(0)}} \mathbf{n}, \nabla_{\mathbf{e}^{\mu}_{(0)}} \mathbf{n} \right)$$
$$+ \rho \left(\nabla_{\mathbf{e}^{\mu}_{(0)}} \nabla_{\mathbf{e}^{\mu}_{(0)}} \mathbf{n}, \nabla_{\mathbf{e}^{\mu}_{(0)}} \nabla_{\mathbf{e}^{\mu}_{(0)}} \mathbf{n} \right)$$
$$= \tau_g^2 + \kappa_n^2 + (\tau_g \kappa_g - \kappa_n')^2 + (\kappa_n \kappa_g + \tau_g')^2 + (\kappa_n^2 + \tau_g^2)^2.$$

So we can easily obtain

$$energy_2(\mathbf{n}) = \int_0^s \left(\tau_g^2 + \kappa_n^2 + (\tau_g \kappa_g - \kappa_n')^2 + (\kappa_n \kappa_g + \tau_g')^2 + (\kappa_n^2 + \tau_g^2)^2\right) ds.$$

This completes the proof. □

Corollary 3.5. *Bienergy on the moving particle in normal vector field is fixed iff*

$$(\tau_g \kappa_g - \kappa_n')^2 + (\kappa_n \kappa_g + \tau_g')^2 + (\kappa_n^2 + \tau_g^2)^2 = 0.$$

Theorem 3.6. *Biangle of the moving particle in normal vector field is*

$$angle_2(\mathbf{n}) = \int_0^s ((\tau_g \kappa_g - \kappa_n')^2 + (\kappa_n \kappa_g + \tau_g')^2 + (\kappa_n^2 + \tau_g^2)^2)^{\frac{1}{2}} du.$$

Proof. By definition 2.6, we write that

$$angle_2(\mathbf{n}) = \int_0^s \left\| \nabla^2_{\mathbf{e}^{\mu}_{(0)}} \mathbf{e}^{\mu}_{(0)} \right\| du$$
$$= \int_0^s ((\tau_g \kappa_g - \kappa_n')^2 + (\kappa_n \kappa_g + \tau_g')^2 + (\kappa_n^2 + \tau_g^2)^2)^{\frac{1}{2}} du.$$

□

Theorem 3.7. *Bienergy on the moving particle in tangent's normal vector field by using Sasaki metric is stated by*

$$energy_2\mathbf{P} = \int_0^s \left(\kappa_g^2 + \tau_g^2 + (\kappa_g' + \kappa_n\tau_g)^2 + (\kappa_g^2 + \tau_g^2)^2 + (\tau_g' - \kappa_n\kappa_g)^2\right) ds.$$

Proof. From (2.4) and (2.5) we have

$$energy_2\left(\mathbf{P}\right) = \int_0^s \rho_S\left(\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{P}, \nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{P}\right) ds.$$

Using Eq. (2.4) we have

$$\rho_S\left(d\mathbf{e}^{\mu}_{(0)}(\mathbf{P}), d\mathbf{e}^{\mu}_{(0)}(\mathbf{P})\right) = \rho(d\omega(\mathbf{e}^{\mu}_{(0)}(\mathbf{P})), d\omega(\mathbf{e}^{\mu}_{(0)}(\mathbf{P}))) + \rho(Q(\mathbf{e}^{\mu}_{(0)}(\mathbf{P})), Q(\mathbf{e}^{\mu}_{(0)}(\mathbf{P}))).$$

Since $\mathbf{e}^{\mu}_{(0)}$ is a section, we get

$$d(\omega) \circ d(\mathbf{P}) = d(\omega \circ \mathbf{P}) = d(id_C) = id_{TC}.$$

We also know

$$Q(\mathbf{e}^{\mu}_{(0)}(\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{P})) = \nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{P} = -(\kappa_g' + \kappa_n\tau_g)\mathbf{e}^{\mu}_{(0)} - (\kappa_g^2 + \tau_g^2)\mathbf{P} + (\tau_g' - \kappa_n\kappa_g)\mathbf{n}.$$

Thus, we substitute (2.1) into (2.4):

$$\begin{aligned}
&\rho_S\left(\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{P}, \nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{P}\right) = \rho\left(\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}, \nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}\right) \\
&+\rho\left(\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}, \nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}\right) \\
&= \kappa_g^2 + \tau_g^2 + (\kappa_g' + \kappa_n\tau_g)^2 + (\kappa_g^2 + \tau_g^2)^2 + (\tau_g' - \kappa_n\kappa_g)^2.
\end{aligned}$$

So we can easily obtain

$$energy_2\mathbf{P} = \int_0^s \left(\kappa_g^2 + \tau_g^2 + (\kappa_g' + \kappa_n\tau_g)^2 + (\kappa_g^2 + \tau_g^2)^2 + (\tau_g' - \kappa_n\kappa_g)^2\right) ds.$$

This completes the proof. □

Corollary 3.8. *Bienergy on the moving particle in tangent's normal vector field is fixed iff*

$$(\kappa_g' + \kappa_n\tau_g)^2 + (\kappa_g^2 + \tau_g^2)^2 + (\tau_g' - \kappa_n\kappa_g)^2 = 0.$$

Theorem 3.9. *Biangle of the moving particle in tangent's normal vector field is*

$$angle_2(\mathbf{P}) = \int_0^s((\kappa_g' + \kappa_n\tau_g)^2 + (\kappa_g^2 + \tau_g^2)^2 + (\tau_g' - \kappa_n\kappa_g)^2)^{\frac{1}{2}} du.$$

Proof. By definition 2.6, we write that

$$\begin{aligned}
angle_2(\mathbf{P}) &= \int_0^s \left\|\nabla^2_{\mathbf{e}^{\mu}_{(0)}}\mathbf{e}^{\mu}_{(0)}\right\| du \\
&= \int_0^s((\kappa_g' + \kappa_n\tau_g)^2 + (\kappa_g^2 + \tau_g^2)^2 + (\tau_g' - \kappa_n\kappa_g)^2)^{\frac{1}{2}} du.
\end{aligned}$$

□

Theorem 3.10. *Bienergy on the moving particle in the vector field*

$$\mathbf{X} = \alpha_1\mathbf{e}^{\mu}_{(0)} + \alpha_2\mathbf{P} + \alpha_3\mathbf{n},$$

where $\alpha_i = \alpha_i(s), i = 1, 2, 3$ are smooth functions, by using Sasaki metric is stated by

$$\begin{aligned}
&energy_2\mathbf{X} = \int_0^s[\{\alpha_1' - \alpha_2\kappa_g - \alpha_3\kappa_n\}^2 + \{\alpha_2' + \alpha_1\kappa_g - \alpha_3\tau_g\}^2 + \{\alpha_3' + \alpha_1\kappa_n + \alpha_2\tau_g\}^2 \\
&+\{\alpha_1'' - (\alpha_2\kappa_g)' - (\alpha_3\kappa_n)' - \alpha_1(\kappa_g^2 + \kappa_n^2) - \alpha_2'\kappa_g - \alpha_2\kappa_n\tau_g - \alpha_3'\kappa_n + \alpha_3\kappa_g\tau_g\}^2 \\
&+\{\alpha_2'' - (\alpha_3\tau_g)' + (\alpha_1\kappa_g)' - \alpha_2(\kappa_g^2 + \tau_g^2) - \alpha_3'\tau_g - \alpha_1\kappa_n\tau_g + \alpha_1\kappa_g - \alpha_3\kappa_n\kappa_g\}^2 \\
&+\{\alpha_3'' + (\alpha_2\tau_g)' + (\alpha_1\kappa_n)' - \alpha_3(\kappa_n^2 + \tau_g^2) + \alpha_2'\tau_g - \alpha_2\kappa_n\kappa_g + \alpha_1\kappa_n + \alpha_1\kappa_g\tau_g\}^2]ds.
\end{aligned}$$

Proof. From (2.4) and (2.5) we know

$$energy_2\mathbf{X}=\int_0^s \rho_S\left(\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{X},\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{X}\right)ds.$$

Using Eq. (2.4) we have

$$\rho_S\left(d\mathbf{e}^{\mu}_{(0)}(\mathbf{X}),d\mathbf{e}^{\mu}_{(0)}(\mathbf{X})\right)=\rho(d\omega(\mathbf{e}^{\mu}_{(0)}(\mathbf{X})),d\omega(\mathbf{e}^{\mu}_{(0)}(\mathbf{X})))+\rho(Q(\mathbf{e}^{\mu}_{(0)}(\mathbf{X})),Q(\mathbf{e}^{\mu}_{(0)}(\mathbf{X}))).$$

Since $\mathbf{e}^{\mu}_{(0)}$ is a section, we get

$$d(\omega)\circ d(\mathbf{e}^{\mu}_{(0)})=d(\omega\circ\mathbf{e}^{\mu}_{(0)})=d(id_C)=id_{TC}.$$

We also know

$$\begin{aligned}
&Q(\mathbf{e}^{\mu}_{(0)}(\nabla_{\mathbf{e}^{\boldsymbol{\mu}}_{(\mathbf{0})}}\mathbf{X}))=\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{X}\\
&=\{\alpha_1''-(\alpha_2\kappa_g)'-(\alpha_3\kappa_n)'-\alpha_1(\kappa_g^2+\kappa_n^2)-\alpha_2'\kappa_g-\alpha_2\kappa_n\tau_g-\alpha_3'\kappa_n+\alpha_3\kappa_g\tau_g\}\mathbf{e}^{\boldsymbol{\mu}}_{(\mathbf{0})}\\
&+\{\alpha_2''-(\alpha_3\tau_g)'+(\alpha_1\kappa_g)'-\alpha_2(\kappa_g^2+\tau_g^2)-\alpha_3'\tau_g-\alpha_1\kappa_n\tau_g+\alpha_1\kappa_g-\alpha_3\kappa_n\kappa_g\}\mathbf{P}\\
&+\{\alpha_3''+(\alpha_2\tau_g)'+(\alpha_1\kappa_n)'-\alpha_3(\kappa_n^2+\tau_g^2)+\alpha_2'\tau_g-\alpha_2\kappa_n\kappa_g+\alpha_1\kappa_n+\alpha_1\kappa_g\tau_g\}\mathbf{n}.
\end{aligned}$$

Thus, using (2.1) in (2.4), we obtain

$$\begin{aligned}
&\rho_S\left(\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{X},\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{X}\right)=\rho\left(\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{X},\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{X}\right)+\rho\left(\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{X},\nabla_{\mathbf{e}^{\mu}_{(0)}}\nabla_{\mathbf{e}^{\mu}_{(0)}}\mathbf{X}\right)\\
&=\{\alpha_1'-\alpha_2\kappa_g-\alpha_3\kappa_n\}^2+\{\alpha_2'+\alpha_1\kappa_g-\alpha_3\tau_g\}^2+\{\alpha_3'+\alpha_1\kappa_n+\alpha_2\tau_g\}^2\\
&+\{\alpha_1''-(\alpha_2\kappa_g)'-(\alpha_3\kappa_n)'-\alpha_1(\kappa_g^2+\kappa_n^2)-\alpha_2'\kappa_g-\alpha_2\kappa_n\tau_g-\alpha_3'\kappa_n+\alpha_3\kappa_g\tau_g\}^2\\
&+\{\alpha_2''-(\alpha_3\tau_g)'+(\alpha_1\kappa_g)'-\alpha_2(\kappa_g^2+\tau_g^2)-\alpha_3'\tau_g-\alpha_1\kappa_n\tau_g+\alpha_1\kappa_g-\alpha_3\kappa_n\kappa_g\}^2\\
&+\{\alpha_3''+(\alpha_2\tau_g)'+(\alpha_1\kappa_n)'-\alpha_3(\kappa_n^2+\tau_g^2)+\alpha_2'\tau_g-\alpha_2\kappa_n\kappa_g+\alpha_1\kappa_n+\alpha_1\kappa_g\tau_g\}^2.
\end{aligned}$$

So we can easily obtain

$$\begin{aligned}
&energy_2\mathbf{X}=\int_0^s[\{\alpha_1'-\alpha_2\kappa_g-\alpha_3\kappa_n\}^2+\{\alpha_2'+\alpha_1\kappa_g-\alpha_3\tau_g\}^2+\{\alpha_3'+\alpha_1\kappa_n+\alpha_2\tau_g\}^2\\
&+\{\alpha_1''-(\alpha_2\kappa_g)'-(\alpha_3\kappa_n)'-\alpha_1(\kappa_g^2+\kappa_n^2)-\alpha_2'\kappa_g-\alpha_2\kappa_n\tau_g-\alpha_3'\kappa_n+\alpha_3\kappa_g\tau_g\}^2\\
&+\{\alpha_2''-(\alpha_3\tau_g)'+(\alpha_1\kappa_g)'-\alpha_2(\kappa_g^2+\tau_g^2)-\alpha_3'\tau_g-\alpha_1\kappa_n\tau_g+\alpha_1\kappa_g-\alpha_3\kappa_n\kappa_g\}^2\\
&+\{\alpha_3''+(\alpha_2\tau_g)'+(\alpha_1\kappa_n)'-\alpha_3(\kappa_n^2+\tau_g^2)+\alpha_2'\tau_g-\alpha_2\kappa_n\kappa_g+\alpha_1\kappa_n+\alpha_1\kappa_g\tau_g\}^2]ds.
\end{aligned}$$

This completes the proof. □

Corollary 3.11. *Bienergy on the moving particle in the vector field*

$$\mathbf{X}=\alpha_1\mathbf{e}^{\mu}_{(0)}+\alpha_2\mathbf{P}+\alpha_3\mathbf{n},$$

where $\alpha_i=\alpha_i(s),i=1,2,3$ *are smooth functions, is fixed iff*

$$\begin{aligned}
&\{\alpha_1''-(\alpha_2\kappa_g)'-(\alpha_3\kappa_n)'-\alpha_1(\kappa_g^2+\kappa_n^2)-\alpha_2'\kappa_g-\alpha_2\kappa_n\tau_g-\alpha_3'\kappa_n+\alpha_3\kappa_g\tau_g\}^2\\
&+\{\alpha_2''-(\alpha_3\tau_g)'+(\alpha_1\kappa_g)'-\alpha_2(\kappa_g^2+\tau_g^2)-\alpha_3'\tau_g-\alpha_1\kappa_n\tau_g+\alpha_1\kappa_g-\alpha_3\kappa_n\kappa_g\}^2\\
&+\{\alpha_3''+(\alpha_2\tau_g)'+(\alpha_1\kappa_n)'-\alpha_3(\kappa_n^2+\tau_g^2)+\alpha_2'\tau_g-\alpha_2\kappa_n\kappa_g+\alpha_1\kappa_n+\alpha_1\kappa_g\tau_g\}^2=0.
\end{aligned}$$

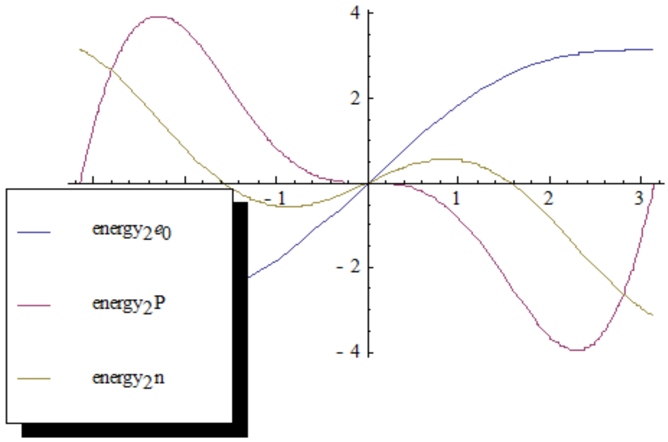

Figure 1:

Theorem 3.12. *Biangle of the moving particle in the vector field*

$$\mathbf{X} = \alpha_1 \mathbf{e}^{\mu}_{(0)} + \alpha_2 \mathbf{P} + \alpha_3 \mathbf{n},$$

where $\alpha_i = \alpha_i(s), i = 1, 2, 3$ *are smooth functions, is*

$$angle_2(\mathbf{X}) = \int_0^s [\{\alpha_1'' - (\alpha_2\kappa_g)' - (\alpha_3\kappa_n)' - \alpha_1(\kappa_g^2 + \kappa_n^2) - \alpha_2'\kappa_g - \alpha_2\kappa_n\tau_g - \alpha_3'\kappa_n + \alpha_3\kappa_g\tau_g\}^2$$
$$+\{\alpha_2'' - (\alpha_3\tau_g)' + (\alpha_1\kappa_g)' - \alpha_2(\kappa_g^2 + \tau_g^2) - \alpha_3'\tau_g - \alpha_1\kappa_n\tau_g + \alpha_1\kappa_g - \alpha_3\kappa_n\kappa_g\}^2$$
$$+\{\alpha_3'' + (\alpha_2\tau_g)' + (\alpha_1\kappa_n)' - \alpha_3(\kappa_n^2 + \tau_g^2) + \alpha_2'\tau_g - \alpha_2\kappa_n\kappa_g + \alpha_1\kappa_n + \alpha_1\kappa_g\tau_g\}^2]^{\frac{1}{2}} du.$$

Proof. From definition 3.6, using the expression

$$angle_2(\mathbf{X}) = \int_0^s \left\| \nabla^2_{\mathbf{e}^{\mu}_{(0)}} \mathbf{e}^{\mu}_{(0)} \right\| du,$$

we have the result desired. □

4. Application

Energy and angle concepts reviewed in optics and geometrical applied physics, [17]- [21]. In this section we conduct our geometric understanding of the bienergy and biangle of particle into graphs for different cases. By doing this practice we have a chance to observe differentiation of the bienergy and biangle of the particle with respect to time and different curves.

Binergy and biangle of Darboux vectors $\{\mathbf{e}^{\mu}_{(0)}, \mathbf{n}, \mathbf{P}\}$ are drawn for helix in Figures 1,2, respectively.

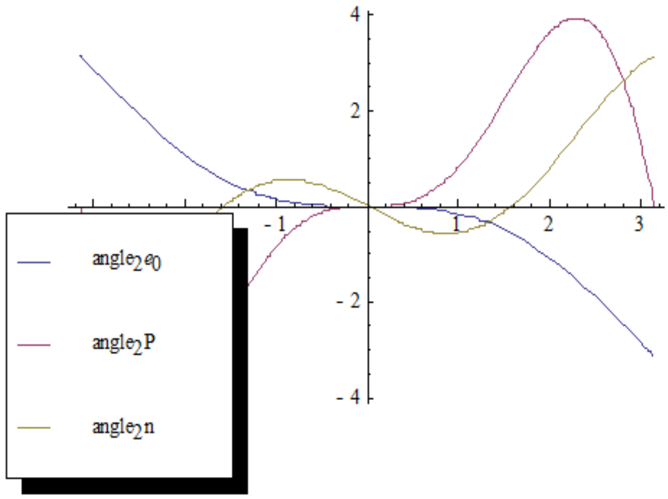

Figure 2:

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