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Pendant and Isolated Vertices of Comaximal Graphs of Modules

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ABSTRACT: A comaximal graph $\Gamma(M)$ is an undirected graph with vertex set as the collection of all submodules of a module M and any two vertices A and B are adjacent if and only if $A + B = M$. We discuss characteristics of pendant vertices in $\Gamma(M)$. We also observe features of isolated vertices in a special spanning subgraph in $\Gamma(M)$.

Key Words: Comaximal graph, pendant vertex, isolated vertex, pendant homomorphism, isolated homomorphism.

Contents

1. Introduction

After the introduction of Istvan Beck's most motivating insight zero divisor graph of commutative ring in 1988 [8], the concept of graphical aspects of algebraic structures are marvelously expanded in various dimensions. Enormous number of research article's publication reveals its utility prodigiously. Some profound ideas, which make co-relation in between graph theoretic characteristics with ring theoretic characteristics, are comaximal graph [19] , total graph [2], unit graph [3], intersection graph [9] etc. In [19], Sharma and Bhatwadekar defined co-maximal graph, $\Gamma(R)$, of a commutative ring R with unity, such that the vertices as elements of R, where two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. Using the concept of clique, they interpreted the coloring of comaximal graph. The comaximal graph concept is also developed by many researchers, Wang [20], Maimani et al. [16], Akbari et al. [1] are some of them.

A comaximal graph $\Gamma(M)$ is an undirected graph with vertex set as the collection of all submodules of a unital module M over a ring R and for $A, B \in V(\Gamma(M))$, A and B are adjacent if and only if $A + B = M$, we write A *adj* B. If A and B are not adjacent, we write A *nadj* B. Henceforth, $\Gamma(M)$ is the comaximal graph of M. Obviously $\Gamma(M)$ is a connected graph. Recollect that a small or superfluous submodule A of M is a submodule of M whose sum with a proper submodule of M is proper. Thus, $A + B$ is a proper submodule of M whenever B is so. Equivalently, $A + B = M$ implies $B = M$. It is noticed that the small submodules of M play the role of pendant vertex in $\Gamma(M)$. It is clear that 0 is always a pendant vertex of $\Gamma(M)$. Also, if $M \neq 0$ and A is a pendant vertex of $\Gamma(M)$, then $A \neq M$. The importance of semi small submodules of modules are seen in [17]. We depict this concept of small submodule as isolated vertex of a particular spanning subgraph of $\Gamma(M)$. In this paper, some features of pendant and isolated vertex in comaximal graphs of modules are discussed. We introduce pendant and isolated homomorphism. Some stimulating results are obtained by employing these. In the last part of this discussion, an interesting idea is developed from the concept of monotonic chain conditions, namely Artinian and Noetherian comaximal graph. Throughout our discussion, unless otherwise specified, all modules are defined over R. We simply say, M is a module over R, rather than left or right module over R.

Any undefined terminology can be obtained in [4-5,7,10-14,21].

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2. Pendant Vertex of $\Gamma(M)$

The core theme of this paper is to analyze the small and semi small submodules of modules in graphical representation. The characteristics of such representations enable us to visualize its importance and future prospects. In this section, the features of small submodules of modules in the diagrammatic representations are observed. We first start this section with the definitions of pendant homomorphism and complemented comaximal graph.

Definition 2.1. Let $f : A \longrightarrow B$ be a module homomorphism. Then f is called pendant if kernel of f $(Ker(f))$ is a pendant vertex of $\Gamma(A)$. A module epimorphism which also pendant is called a pendant epimorphism. Hence K is pendant in $\Gamma(M)$ if and only if the canonical projection $M \longrightarrow M/K$ is a pendant epimorphism.

Definition 2.2. $\Gamma(M)$ is said to be complemented graph if for every $A \in V(\Gamma(M))$, there is a $B \in V(\Gamma(M))$ with $A + B = M$ and $A \cap B = 0$. We say A and B are complements of each other. Notice that if $\Gamma(M)$ is complemented then 0 is the only pendant vertex.

We now remember the definition of local ring from [13]. A ring, which satisfies the following equivalent properties is called a local ring.

Lemma 2.3. Let A be the set of all non-invertible elements of R, then the following properties are equivalent:
(i) A is additively closed.
(ii) A is a two-sided ideal.
(iiir) A is the largest proper right ideal.
(iiil) A is the largest proper left ideal.
(ivr) In R there exists a largest proper right ideal.
(ivl) In R there exists a largest proper left ideal.
(vr) For every $r \in R$ either r or $1 - r$ is right invertible.
(v1) For every $r \in R$ either r or $1 - r$ is left invertible.
(vi) For every $r \in R$ either r or $1 - r$ is invertible.

If R is a local ring, but not a skew field and A is the two-sided ideal consisting of the non-invertible elements of R. Then A is non-zero and is the largest proper right, left or two sided ideal of R . Thus A is a pendant vertex in R_R. Consider the local ring $R = \frac{\mathbb{Z}}{p^n\mathbb{Z}}$, where $\mathbb{Z}$ is the set of integers, p is a prime number, n is a positive integer. Here $A = \frac{p\mathbb{Z}}{p^n\mathbb{Z}}$ is the two sided ideal consisting of the non-invertible elements of R.

Before going to our next discussion, we interpret the concept of free module. A set $E \subseteq M$ is a basis for M if:
1. E is a generating set for M; that is to say, every element of M is a finite sum of elements of E multiplied by coefficients in R;
2. E is linearly independent, that is, if $r_1e_1 + r_2e_2 + \cdots + r_ne_n = 0$ for $e_1, e_2, \ldots, e_n$ distinct elements of E, then $r_1 = r_2 = \cdots = r_n = 0_R$ (where 0 is the zero element of M and 0_R is the zero element of R). A free module is a module with a basis i.e. with a linearly independent generating set.

Theorem 2.4. *Let M be a free $\mathbb{Z}-$module. Then only* 0 *is a pendant vertex of* $\Gamma(M)$.

Proof. Let $M = \bigoplus x_i\mathbb{Z}$ with basis $\{x_i | i \in I,\ I\ is\ an\ index\ set\}$. Consider a vertex A in $\Gamma(M), a \in A$. Let $a = x_{i_1}z_1 + x_{i_2}z_2 + \ldots x_{i_m}z_m\ z_i \in \mathbb{Z}$ with $z_1 \neq 0$. Let n be relatively prime with $z_1, n > 1$. Suppose $U = \bigoplus_{i \in I, i \neq i_1} +x_{i_1}n\mathbb{Z}$. Then it follows that $a\mathbb{Z} + U = M$. Thus $A + U = M$ with $U \neq M$. Hence 0 is the only pendant vertex of $\Gamma(M)$.

□

Remark 2.5. In particular, the only pendant vertex of $\Gamma(\mathbb{Z})$ is 0. On the other hand, any nil (left) ideal I of R is pendant in $\Gamma(R)$. Assume B *adj* U for some $L \in V(\Gamma(R))$. Then $1 = i + l$ for suitable $i \in I, l \in L$, and hence, for some $k \in \mathbb{N}$, we get $0 = i^k = (1-l)^k = 1+l'$ for some $l' \in L$, so that $1 \in L = R$.

Theorem 2.6. *An epimorphism $f : M \longrightarrow N$ is pendant if and only if every (mono) morphism $h : L \longrightarrow M$ with fh epimorphism is epimorphism.*

Proof. If fh is epimorphism and $m \in M$, then there exists $l \in L$ with $f(m) = (fh)(l)$, which means $m = h(l) + (m - h(l)) \in Im(h) + Ker(f)$ and hence $Im(h)$ *adj* $Ker(f)$. Now $Ker(f)$ is pendant implies $M = Imh$.

Conversely, suppose that $Ker(f)$ *adj* L for some $L \in V(\Gamma(M))$. With the inclusion $i : L \longrightarrow M$ the map fi is an epimorphism. By the given property, i has to be an epimorphism, i.e. $L = M$.

Theorem 2.7. *Every finitely generated submodule of $M = \mathbb{Q}_{\mathbb{Z}}$ is a pendant vertex of $\Gamma(M)$.*

Proof. Let $q_1, q_2, \ldots, q_n \in \mathbb{Q}$ and let U be a vertex of $\Gamma(M)$ with $q_1\mathbb{Z} + q_2\mathbb{Z} + \ldots + q_n\mathbb{Z} + U = \mathbb{Q}$. Then $\{q_1, q_2, \ldots, q_n\} \cup U$ is a generating set of $\mathbb{Q}$. Consequently from 2.3.7.[13] is already a generating set of $\mathbb{Q}$. Thus we have $U = \mathbb{Q}$. Hence the result. □

Theorem 2.8. *If $A \in V(\Gamma(B)), M \in V(\Gamma(N))$ and B is a pendant vertex of $\Gamma(M)$, then A is a pendant vertex of $\Gamma(N)$.*

Proof. Let A *adj* U with $U \in V(\Gamma(N))$. Then B *adj* U in $\Gamma(N)$. By Modular law, $B + (U \cap M) = M \cap (B + U) = M$, as $B \subseteq M$ i.e. B *adj* $U \cap M$ in $\Gamma(M)$. But B is a pendant vertex of $\Gamma(M)$, therefore $U \cap M = M$. And so $M \in V(\Gamma(U))$. Since by our assumption $A \in V(\Gamma(M))$, we deduce that $U = A + U = N$. Hence the result □

Remark 2.9. If $K \in V(\Gamma(L)), L \in V(\Gamma(M))$, then L is pendant in $\Gamma(M)$ if and only if K and L/K are pendant in $\Gamma(M)$ and $\Gamma(M/K)$ respectively. Also if $K \in V(\Gamma(L)), L \in V(\Gamma(M))$ and L has a complement in $\Gamma(M)$, then K is pendant in $\Gamma(M)$ if and only if K is pendant in $\Gamma(L)$.

Theorem 2.10. *If A_i is a pendant vertex of $\Gamma(M)$, $i = 1, 2, \ldots, n$, then so is $A_1 + A_2 + \ldots + A_n$.*

Proof. The result is proved by induction on n. By assumption, the statement is true for $n = 1$. Assume that $A = A_1 + A_2 + \ldots + A_{n-1}$ is a pendant vertex of $\Gamma(M)$. Suppose, we have $(A + A_n)$ *adj* U, for $U \in V(\Gamma(M))$. Then A_n *adj* U, since A is a pendant vertex of $\Gamma(M)$. Again, as A_n is a pendant vertex of $\Gamma(M)$, and thus $U = M$. Hence the result.

Theorem 2.11. *If A is a pendant vertex of $\Gamma(M)$ and $\phi : M \longrightarrow N$ is a homomorphism, then $\phi(A)$ is a pendant vertex of $\Gamma(N)$.*

Proof. Assume that $\phi(A)$ *adj* U for $U \in V(\Gamma(N))$. Then for $m \in M$ $\phi(m) = \phi(a) + u$ with $a \in A$, $u \in U$. This gives $\phi(m - a) = u$ and so $m - a \in \phi^{-1}(U)$, thus $m \in (A + \phi^{-1}(U))$. Therefore $A + \phi^{-1}(U) = M$, but A is a pendant vertex of $\Gamma(M)$ and so $M = \phi^{-1}(U)$. From this we get $\phi(M) = U \cap Im(\phi)$. Thus $\phi(A) \in V(\Gamma(\phi(M)))$, and $\phi(M) \in V(\Gamma(U))$ and so $U = \phi(A) + U = N$. Hence the result. □

Theorem 2.12. *If $\alpha : A \longrightarrow B$, $\beta : B \longrightarrow C$ are pendant epimorphisms, then so is $(\beta\alpha) : A \longrightarrow C$ and conversely.*

Proof. Suppose that $Ker(\beta\alpha)$ *adj* U, for $U \in V(\Gamma(A))$. Then, as $Ker(\beta\alpha) = \alpha^{-1}(Ker\beta)$, it follows that $\alpha(Ker(\beta\alpha)) + \alpha(U) = Ker(\beta) + \alpha(U) = \alpha(A) = B$ i.e. $Ker(\beta)$ *adj* $\alpha(U)$. Since by assumption we have $Ker(\beta)$ is a pendant vertex of $\Gamma(B)$, we obtain $\alpha(U) = B$ and consequently $Ker(\alpha)$ *adj* U. As $Ker(\alpha)$ is a pendant vertex of $\Gamma(A)$ and so $U = A$. The converse part follows from 19.3[21]. This completes the proof. □

Now, we remember the definition of maximal submodule. A maximal submodule A of M is a submodule $A \neq M$ for which for any other submodule N, if $A \subseteq N \subseteq M$ then $N = A$ or $N = M$. Utilizing the concept of this maximal submodule, we define maximal vertex in $\Gamma(M)$.

Definition 2.13. Let $A \in V(\Gamma(M))$. Then A is a maximal vertex of $\Gamma(M)$ if A is a maximal submodule of M.

Employing the idea of maximal vertex, we establish the following result.

Theorem 2.14. *For $a \in M$, we have aR is not a pendant vertex of $\Gamma(M)$ if and only if there is a maximal vertex C of $\Gamma(M)$ with $a \notin C$.*

Proof. If C is a maximal vertex of $\Gamma(M)$ with $a \notin C$, then we have aR *adj* C. Thus aR is not a pendant vertex of $\Gamma(M)$. The converse part is proved by using Zorn's lemma. Let $\mathfrak{F} = \{B | B \in V(\Gamma(M)), B \neq M, aR \text{ adj } B\}$. Since aR is not a pendant vertex of $\Gamma(M)$, so $\mathfrak{F} \neq \phi$. By using Zorn's lemma, we can show that so $\mathfrak{F}$ contains a maximal member C. We claim that C is in fact a maximal vertex of $\Gamma(M)$. Let $C \in V(\Gamma(U)), C \neq U, U \in V(\Gamma(M))$, then it follows that $U \notin \mathfrak{F}$, since C is a maximal in $\mathfrak{F}$. From $M = aR + C \in V(\Gamma(aR + U)), (aR + U) \in V(\Gamma(M))$, we get aR *adj* U and $U \notin \mathfrak{F}$. This must give $U = M$. The proof is complete. □

3. Isolated Vertex of Subgraph of $\Gamma(M)$

The prime motive of this section is in the characterization of semi small submodules of the unital module M over a commutative ring R in diagrammatic representation. Accordingly we define a special type of spanning subgraph of $\Gamma(M)$. We denote it by H. A vertex A of H is adjacent to a vertex B if and only if B is a primary submodule of M, i.e. $A + B = M$ if and only if B is a primary submodule of M. Remember that a proper submodule B of M is called primary if whenever $rm \in B$ for $r \in R$ and $m \in M$, either $m \in B$ or $r^n \in (B : M)$ for some positive integer n, where $(B : M) = \{r \in R | rM \subseteq B\}$, [15]. Also an ideal I in R is called a primary ideal in R if $xy \in I$, where $x, y \in R$, then either $x^n \in I$ or $y^k \in I$ for some positive integers n and k [17]. In our discussion we emphasize in isolated vertex of H and its characteristics and relations with pendant vertex of $\Gamma(M)$.

Remark 3.1. Evidently, 0 is an isolated vertex of H. Also if A is a pendant vertex of $\Gamma(M)$ then A is an isolated vertex of H. Again, if M is a semi-simple module then 0 is the only isolated vertex of H. It is easy to observe that each finitely generated submodule of $M = \mathbb{Q}_{\mathbb{Z}}$ is an isolated vertex of H. In a free $\mathbb{Z}$-module M only 0 is the isolated vertex of H.

Theorem 3.2. *Let M be a finitely generated module. A is an isolated vertex of H if and only if A is a pendant vertex of M.*

Proof. Assume that A is an isolated vertex of H. Consider $B(\neq M) \in V(\Gamma(M))$ with A *adj* B. As M is finitely generated, so there is a maximal $L \in V(\Gamma(M)$ such that $B \in V(\Gamma(L)$. Thus A *adj* L. But L is a primary submodule of M and A is an isolated vertex of H and so $L = M$. Hence a contradiction. Thus A *nadj* B for each $B(\neq M) \in V(\Gamma(M))$. This concludes that A is a pendant vertex of $\Gamma(M)$. The converse part follows immediately. Hence the theorem. □

We just state the following theorem as its proof is simple.

Theorem 3.3. *If every proper submodule of M is primary, then A is an isolated vertex of H if and only if A is a pendant vertex of $\Gamma(M)$.*

Theorem 3.4. *Let N be an isolated vertex of H and $A \in V(\Gamma(N))$. Then A is an isolated vertex of H.*

Proof. Let A *adj* B, for some primary submodule B of M. Since $A \in V(\Gamma(N))$, so N *adj* B. Again N is an isolated vertex H. Therefore $B = M$, a contradiction. Thus A is an isolated vertex of $\Gamma(M)$. □

The converse of the above theorem is not true in general. Consider $M = Z_{12}$ and $R = \mathbb{Z}$. The primary submodules of M are $I_1 = \{0, 2, 4, 6, 8, 10\}, I_2 = \{0, 3, 6, 9\}, I_3 = \{0, 4, 8\}, I_4 = \{0, 6\}$. Here I_4 is an isolated vertex of H, but I_2 is not an isolated vertex of H as I_2 *adj* I_3.

Remark 3.5. We notice that intersection of any two isolated vertex of H is also an isolated vertex of H. In general, if $N \in V(\Gamma(M))$ and A is an isolated vertex of H, $A \cap N$ is an isolated vertex of H. Also, if A_i is an isolated vertex of H, for $i = 1, 2, \ldots, n$, so is $\bigcap_{i=1}^{n} A_i$. Again, $A_i \in V(\Gamma(M)), i = 1, 2, \ldots, n$ and A_j is an isolated vertex of H for some $j, 1 \leq j \leq n$, then $\bigcap_{i=1}^{n} A_i$ is an isolated vertex of H.

Theorem 3.6. *If $A \in V(\Gamma(M))$ and N is an isolated vertex of H, then $N + A$ is an isolated vertex of H if and only if A is an isolated vertex of H.*

Proof. Assume that $N + A$ is an isolated vertex of H and B is a primary submodule of M. Then $(N + A)$ *nadj* B. It is easy to observe that N *nadj* $(A + B)$. From this we get A *nadj* B and consequently, A is an isolated vertex of H. For the converse part, suppose A is an isolated vertex of H and B is a primary submodule of M. Then A *nadj* B. Also since N is an isolated vertex of H, N *nadj* $(A + B)$. This gives $(N + A)$ *nadj* B. Hence $N + A$ is an isolated vertex of H. □

Now we introduce a generalization for pendant homomorphism concept namely isolated homomorphism.

Definition 3.7. A module homomorphism $\theta : M \longrightarrow N$ is called an isolated homomorphism, if $\theta(M)$ is an isolated vertex of K, where K is the analogous concept of H of $\Gamma(M)$ in $\Gamma(\theta(M))$. Henceforth, we use K for the analogous concept of H in the concept of homomorphism.

Theorem 3.8. *Let $\theta : M \longrightarrow N$ be a module epimorphism with $Ker(\theta) \subseteq B$, for each primary submodule B of M. If A is an isolated vertex of K, then so is $\theta^{-1}(A)$ of H.*

Proof. Assume that $\theta^{-1}(A))$ *adj* B, for some primary submodule B of M. Then A *adj* $\theta(B)$ in $\Gamma(N)$, θ is an epimorphism. By Lemma 1.13 [17], $\theta(B)$ is a primary submodule of N, $Ker(\theta) \subseteq B$. Since A is an isolated vertex of K, thus $\theta(B) = N$, a contradiction. It follows that $\theta^{-1}(A))$ *adj* B in $\Gamma(M)$ for each primary submodule B of M. Hence $\theta^{-1}(A)$ is an isolated vertex of H. □

Theorem 3.9. *Let $\theta : M \longrightarrow N$ be a module epimorphism. If A is an isolated vertex of H such that $Ker(\theta) \subseteq A$, for each primary submodule B of M, then so is $\theta(A)$ of K.*

Proof. Let $\theta(A)$ *adj* B in K, for some primary submodule B of N. Then $\theta^{-1}(\theta(A))$ *adj* $\theta^{-1}(B)$ and then $\theta^{-1}(B)$ is a primary submodule M by Lemma 1.15 [17]. But $Ker(\theta) \subseteq A$, so $\theta^{-1}(\theta(A)) = A + Ker(\theta) = A$. Therefore A *adj* $\theta^{-1}(B)$, which is a contradiction. This contradiction follows that $\theta(A)$ *nadj* B, for each primary submodule B of M. Hence $\theta(A)$ is an isolated vertex of K. □

Remark 3.10. If $A \in V(\Gamma(M))$ then A is an isolated vertex of H if and only if the inclusion function $i : A \longrightarrow M$ is an isolated monomorphism.

It is needed to be mentioned here that the term 'isolated vertex' is used to specify in the usual sense. That is whenever we employ the term 'isolated vertex' in a comaximal graph $\Gamma(M)$, we mean that it is an isolated vertex of the spanning subgraph whose adjacency relation deals with the notion $A + B = M; A, B \in V(\Gamma(M))$, where B is a primary submodule of M.

Theorem 3.11. *Let $A, N \in V(\Gamma(M))$ such that $A \in (\Gamma(N))$ and $A \in V(\Gamma(B))$ for each primary submodule B of M. If A is an isolated vertex of $\Gamma(M)$ then $\frac{N}{A}$ is an isolated vertex of $\Gamma(\frac{M}{A})$ if and only if N is an isolated vertex of $\Gamma(M)$.*

Proof. Assume $\frac{N}{A}$ is an isolated vertex of $\Gamma(\frac{M}{A})$ and N *adj* B in $\Gamma(M)$ for some primary submodule B of M. Then $\frac{N}{A}$ *adj* $\frac{B}{A}$ in $\Gamma(\frac{M}{A})$, a contradiction, as $\frac{B}{A}$ is a primary submodule of $\frac{M}{A}$ and $\frac{N}{A}$ is an isolated vertex of $\Gamma(\frac{M}{A})$. It follows that N *nadj* B for each primary submodule B of M. This concludes that N is an isolated vertex of $\Gamma(M)$.

Conversely, suppose that N is an isolated vertex of $\Gamma(M)$ and $\pi : M \longrightarrow \frac{M}{A}$ is a natural projective function. Let $\frac{N}{A}$ *adj* $\overline{B}$ for some primary submodule $\overline{B}$ of $\frac{M}{A}$. This provides a primary submodule P of M such that $P = \pi^{-1}(\overline{B})$. Thus $\pi(P) = \overline{B} = \frac{P}{A}$ is a primary submodule of $\frac{M}{A}$. Hence $\frac{N}{A}$ *adj* $\frac{P}{A}$ in $\Gamma(\frac{M}{A})$ and so $\frac{N+P}{A} = \frac{M}{A}$ and consequently N *adj* P, which is a contradiction, since N is an isolated vertex of $\Gamma(M)$. This implies that $\frac{N}{A}$ *nadj* $\overline{B}$ for every primary submodule $\overline{B}$ of $\frac{M}{A}$. Thus $\frac{N}{A}$ is an isolated vertex of $\Gamma(\frac{M}{A})$. Hence the theorem.

□

Recall that M is a multiplication module if every submodule N of M is of the form BM, for some ideal B of R [6]. And M is called a faithful module if $ann(M) = 0$ [5].

In the following theorem, we give a condition under which a vertex of $\Gamma(M)$, M, a faithful multiplication module, is an isolated vertex of $\Gamma(M)$. We assign a condition on a finitely generated faithful multiplication module M to find a relation between isolated vertex of $\Gamma(M)$ and isolated vertex of $\Gamma(R)$.

Theorem 3.12. *Let M be a finitely generated faithful multiplication module and let $A = IM$ be a vertex of $\Gamma(M)$ with $A \neq M$. Then I is an isolated vertex $\Gamma(R)$ if and only if A is an isolated vertex of $\Gamma(M)$.*

Proof. Suppose that I is an isolated vertex $\Gamma(R)$. Take A *adj* B for some primary submodule B of M. By Lemma 2.1 [17], this will give a primary ideal E in R, such that $B = EM$, as M is a multiplication module. From this we have IM *adj* EM. But M is a multiplication module, so I *adj* E, which is absurd. Hence A *nadj* B for every primary submodule B of M. Consequently, A is an isolated vertex of $\Gamma(M)$.

Conversely, assume that A is an isolated vertex of $\Gamma(M)$. Toward a contradiction, consider the case I *adj* E for some primary ideal E in R. Again, as M is a multiplication module and by the help of Lemma 2.1 [17], we assert that A *adj* EM. This is a contradiction to the fact that A is an isolated vertex of $\Gamma(M)$. Therefore I is an isolated vertex $\Gamma(R)$. Hence the theorem.

Immediately we see the following corollary.

Corollary 3.13. *Let M be a finitely generated faithful multiplication module and let $A \in V(\Gamma(M)), A \neq M$. Then A is an isolated vertex of $\Gamma(M)$ if and only $(A : M)$ is an isolated vertex of $\Gamma(R)$.*

Remark 3.14. Next we observe that isolated vertex concept and pendant vertex concept are equivalent in the class of multiplication module. The proof of the result is same with Theorem 3.2, thus we just state it. Let M be a multiplication module and let $A \in V(\Gamma(M))$. Then A is isolated if and only if A is pendant in $\Gamma(M)$.

Recollect, M is said to be satisfy the ascending chain if each ascending chain of submodule of M terminate. Moreover, M is called Noetherian module if and only if M satisfies Acc. And M is said to be satisfy the descending chain condition (Dcc) if each descending chain of submodules of M terminates. Also, M is called Artinian module if and only if M satisfies Dcc.

Now we introduce an interesting generalized comaximal graph's concept. Let $A_j \in V(\Gamma(M))$ with $A_j \in V(\Gamma(A_{j+1})), \forall j \in \mathbb{N} = \{1, 2, \cdots\}$. Then $\Gamma(M)$ is called a Noetherian graph if $\Gamma(A_n) = \Gamma(A_{n+1}) = \Gamma(A_{n+2}) = \cdots$ for some $n \in \mathbb{N}$. A Noetherian graph $\Gamma(M)$ is called an isolated Noetherian if each $A_j \in V(\Gamma(M))$ is isolated with $A_j \in V(\Gamma(A_{j+1})), \forall j \in \mathbb{N} = \{1, 2, \cdots\}$. Observe that $\Gamma(M)$ is Noetherian if and only if M is Noetherian. Analogously, $\Gamma(M)$ is called an Artinian graph if $\Gamma(A_n) = \Gamma(A_{n+1}) = \Gamma(A_{n+2}) = \cdots$ for some $n \in \mathbb{N}$, whenever $A_{j+1} \in V(\Gamma(A_j)), \forall j \in \mathbb{N} = \{1, 2, \cdots\}$. An Artinian graph $\Gamma(M)$ is called an isolated Artinian if each $A_j \in V(\Gamma(M))$ is isolated with $A_j \in V(\Gamma(A_{j+1})), \forall j \in \mathbb{N} = \{1, 2, \cdots\}$. Equivalently, $\Gamma(M)$ is Artinian if and only if M is Artinian.

Theorem 3.15. In a finitely generated faithful multiplication module M, $\Gamma(R)$ is isolated Noetherian if and only if $\Gamma(M)$ is isolated Noetherian.

Proof. Let $\Gamma(R)$ be isolated Noetherian. Assume that $A_j \in V(\Gamma(A_{j+1})), \forall j \in \mathbb{N} = \{1, 2, \cdots\}$ with $A_j \in V(\Gamma(M))$ is isolated. As M is a multiplication module, thus for each j we have $A_j = B_jM$, for some isolated vertex B_j in $\Gamma(R)$. This provides $B_jM \in V(\Gamma(B_{j+1}M)), \forall j \in \mathbb{N} = \{1, 2, \cdots\}$. But M is a finitely generated faithful module, so $B_j \in V(\Gamma(B_{j+1})), \forall j \in \mathbb{N} = \{1, 2, \cdots\}$ [18]. By assumption, $\Gamma(R)$ is isolated Noetherian and thus $\Gamma(B_n) = \Gamma(B_{n+1}) = \Gamma(B_{n+2}) = \cdots$ for some $n \in \mathbb{N}$. This automatically implies that $\Gamma(A_n) = \Gamma(A_{n+1}) = \Gamma(A_{n+2}) = \cdots$ for some $n \in \mathbb{N}$.

Conversely, suppose that $\Gamma(M)$ is isolated Noetherian. Now, consider $B_j \in V(\Gamma(B_{j+1})), \forall j \in \mathbb{N} = \{1, 2, \cdots\}$ with $B_j \in V(\Gamma(R))$ is isolated. Then, by Theorem 3.12, we get $B_jM \in V(\Gamma(B_{j+1}M)), \forall j \in \mathbb{N} = \{1, 2, \cdots\}$ with $B_jM \in V(\Gamma(M))$ is isolated. Since $\Gamma(M)$ is isolated Noetherian and moreover M is finitely generated faithful multiplication module, and hence from this, it is easy to establish that $\Gamma(R)$ is isolated Noetherian. This completes the proof.

Theorem 3.16. In a finitely generated faithful multiplication module M, the following are equivalent :
(i) $\Gamma(M)$ is isolated Noetherian.
(ii) $\Gamma(R)$ is isolated Noetherian.
(iii) $\Gamma(N)$ is isolated Noetherian, where $N = End(M)$.
(iv) $\Gamma(M')$ is isolated Noetherian, where M' is an N-module.

Proof. It is easily obtainable by using Theorem 2.7 [17]

Finally we state the following results for Artinian graph as its proofs follow the same way as in Noetherian graph.

Theorem 3.17. In a finitely generated faithful multiplication module M, $\Gamma(R)$ is isolated Artinian if and only if $\Gamma(M)$ is isolated Artinian.

Theorem 3.18. In a finitely generated faithful multiplication module M the following are equivalent :
(i) $\Gamma(M)$ is isolated Artinian.
(ii) $\Gamma(R)$ is isolated Artinian.
(iii) $\Gamma(N)$ is isolated Artinian, where $N = End(M)$.
(iv) $\Gamma(M')$ is isolated Artinian, where M' is an N-module.

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