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On *semi**-$\mathcal{J}$-open Sets, *pre**-$\mathcal{J}$-open Sets and e-$\mathcal{J}$-open Sets in Ideal Topological Spaces

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Abstract: In this paper we introduce and investigate some properties of $semi^*$-$\mathcal{J}$-open sets, pre^*-$\mathcal{J}$-open sets and e-$\mathcal{J}$-open sets in ideal topological spaces. Moreover, some relationships among $semi^*$-$\mathcal{J}$-open sets, e-$\mathcal{J}$-open sets and pre^*-$\mathcal{J}$-open sets in ideal topological spaces are established. Finally, we obtain the decompositions of continuity.

Key Words: Ideal topological space, e-$\mathcal{J}$-open, $semi^*$-$\mathcal{J}$-open, pre^*-$\mathcal{J}$-open.

Contents

1. Introduction

$semi^*$-$\mathcal{J}$-open sets, pre^*-$\mathcal{J}$-open sets and e-$\mathcal{J}$-open sets in ideal topological spaces were studied by [5], [4] and [2,12], respectively. In this paper, some properties of $semi^*$-$\mathcal{J}$-open sets, pre^*-$\mathcal{J}$-open sets and e-$\mathcal{J}$-open sets in ideal topological spaces are investigated. Some relationships among pre^*-$\mathcal{J}$-open sets, $semi^*$-$\mathcal{J}$-open sets and e-$\mathcal{J}$-open sets in ideal topological spaces are discussed. Furthermore, decompositions of continuous functions have been introduced.

An ideal $\mathcal{J}$ on a nonempty set X is a nonempty collection of subsets of X which satisfies the following conditions: $A \in \mathcal{J}$ *and* $B \subset A$ implies $B \in \mathcal{J}$; $A \in \mathcal{J}$ *and* $B \in \mathcal{J}$ implies $A \cup B \in \mathcal{J}$ [8]. Applications to various fields were further investigated by Jankovic and Hamlett [6]; Mukherjee et al. [9]; Arenas et al. [1]; Nasef and Mahmoud [10], etc. Given a topological space (X, τ) with an ideal $\mathcal{J}$ on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^* : \wp(X) \to \wp(X)$, called a local function [8] of A with respect to τ and $\mathcal{J}$ is defined as follows: for $A \subseteq X$,

$$A^*(\mathcal{J}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{J} \text{ for every } U \in \tau(x)\},$$

where $\tau(x) = \{U \in \tau \mid x \in U\}$. Furthermore $Cl^*(A) = A \cup A^*(\mathcal{J}, \tau)$ defines a Kuratowski closure operator for the topology τ^*, called the $*$-topology, finer than τ. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{J}, \tau)$. X^* is often a proper subset of X. By a space, we always mean a topological space (X, τ) with no separation properties assumed. If $A \subset X$, $Cl(A)$ and $Int(A)$ will denote the closure and interior of A in (X, τ), respectively.

A topological space (X, τ) with an ideal $\mathcal{J}$ is called an ideal topological space and is denoted by $(X, \tau, \mathcal{J})$. A subset A of an ideal space (X, τ) is said to be R-$\mathcal{J}$-open (resp. R-$\mathcal{J}$-closed) [13] if $A = Int(Cl^*(A))$ $(resp. A = Cl^*(Int(A))$. A point $x \in X$ is called a δ-$\mathcal{J}$-cluster point of A if $Int(Cl^*(U)) \cap A \neq \emptyset$ for each open set V containing x. The family of all δ-$\mathcal{J}$-cluster points of A is called the δ-$\mathcal{J}$-closure of A and is denoted by $\delta Cl_{\mathcal{J}}(A)$. The set δ-$\mathcal{J}$-interior of A is the union of all R-$\mathcal{J}$-open sets of X contained in A and its denoted by $\delta Int_{\mathcal{J}}(A)$. A is said to be δ-$\mathcal{J}$-closed if $\delta Cl_{\mathcal{J}}(A) = A$ [13]. δ-$\mathcal{J}$-open sets form a topology τ_δ-$\mathcal{J}$ and that it is coarser than τ.

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In this paper, we define and study some properties of *semi**-$\mathfrak{I}$-open sets, *pre**-$\mathfrak{I}$-open sets and e-$\mathfrak{I}$-open sets in ideal topological spaces and investigate some of their properties. Moreover, some relationships among *semi**-$\mathfrak{I}$-open sets, e-$\mathfrak{I}$-open sets and *pre**-$\mathfrak{I}$-open sets in ideal topological spaces are established. Several interesting properties and characterizations are introduced and discussed. Finally, we obtain the decompositions of continuity.

2. Preliminaries

Definition 2.1. *A subset U of an ideal topological space $(X, \tau, \mathfrak{I})$ is said to be*

1. *semi**-$\mathfrak{I}$-*open* [5] *if* $U \subset Cl(\delta Int_{\mathfrak{I}}(U))$.
2. *pre**-$\mathfrak{I}$-*open* [4] *if* $U \subseteq Int(\delta Cl_{\mathfrak{I}}(U))$.
3. e-$\mathfrak{I}$-*open* [2] *if* $U \subset Cl(\delta Int_{\mathfrak{I}}(U)) \cup Int(\delta Cl_{\mathfrak{I}}(U))$.
4. *a* $\mathcal{B}G_{\mathfrak{I}^*}$-*set* [3] *if* $U = V \cap C$, *where* V *is* $\delta_{\mathfrak{I}}$-*open and* C *is* e-$\mathfrak{I}$-*closed.*
5. *weakly* $\delta_{\mathfrak{I}}$-*local closed* [7] *if* $U = V \cap C$, *where* V *is an open set and* C *is a* $\delta_{\mathfrak{I}}$-*closed set in* X.

The class of all *semi**-$\mathfrak{I}$-open (resp. *pre**-$\mathfrak{I}$-open, $\delta\alpha$-$\mathfrak{I}$-open) sets of $(X, \tau, \mathfrak{I})$ is denoted by $S^*\mathfrak{I}O(X)$ (resp. $P^*\mathfrak{I}O(X)$, $\delta\alpha\mathfrak{I}O(X)$) [5,4]. The complement of a *semi**-$\mathfrak{I}$-open (resp. *pre**-$\mathfrak{I}$-open, e-$\mathfrak{I}$-open) set is said to be *semi**-$\mathfrak{I}$-closed (resp. *pre**-$\mathfrak{I}$-closed, e-$\mathfrak{I}$-closed).

The e-$\mathfrak{I}$-interior [2,11] (resp. *semi**-$\mathfrak{I}$-interior [5], *pre**-$\mathfrak{I}$-interior [4]) of U is denoted by $Int^*_e(U)$ (resp. $s\delta Int_{\mathfrak{I}}(U)$, $P^*\mathfrak{I}Int(U)$) is defined by the union of all e-$\mathfrak{I}$-open [2](resp. *semi**-$\mathfrak{I}$-open [5], *pre**-$\mathfrak{I}$-open [4]) sets contained in U. The intersection of all e-$\mathfrak{I}$-closed (resp. *semi**-$\mathfrak{I}$-closed [5], *pre**-$\mathfrak{I}$-closed [4]) sets containing U is called the e-$\mathfrak{I}$-closure (resp. *semi**-$\mathfrak{I}$-closure [5], *pre**-$\mathfrak{I}$-closure [4]) of U and is denoted by $Cl^*_e(U)$ (resp. $s\delta Cl_{\mathfrak{I}}(U)$, $P^*\mathfrak{I}Cl(U)$).

Theorem 2.2. [5] *Let Q be a subset of an ideal space $(X, \tau, \mathfrak{I})$. Then*

1. $s\delta Cl_{\mathfrak{I}}(Q) = Q \cup Int(\delta Cl_{\mathfrak{I}}(Q))$ *and* $P^*\mathfrak{I}Cl(Q) = Q \cup Cl(\delta Int_{\mathfrak{I}}(Q))$,
2. $s\delta Int_{\mathfrak{I}}(Q) = Q \cap Cl(\delta Int_{\mathfrak{I}}(Q))$ *and* $P^*\mathfrak{I}Int(Q) = Q \cap Int(\delta Cl_{\mathfrak{I}}(Q))$.

Theorem 2.3. [3] *Let Q be a subset of an ideal topological space $(X, \tau, \mathfrak{I})$. Then Q is a $\mathcal{B}G_{\mathfrak{I}^*}$-set if and only if $Q = F \cap Cl^*_e(Q)$ for a $\delta_{\mathfrak{I}}$-open set F in X.*

Definition 2.4. [3] *An ideal topological space $(X, \tau, \mathfrak{I})$ is said to be $\delta\mathfrak{I}$-extremally disconnected if $\delta Cl_{\mathfrak{I}}(Q) \in \tau$ for each $Q \in \tau$.*

Lemma 2.5. [3] *An ideal topological space $(X, \tau, \mathfrak{I})$ is $\delta\mathfrak{I}$-extremally disconnected if and only if it is extremally disconnected.*

Theorem 2.6. [3] *For an ideal topological space $(X, \tau, \mathfrak{I})$, the following properties are equivalent:*

1. *X is $\delta_{\mathfrak{I}}$-extremally disconnected,*
2. *$\delta Int_{\mathfrak{I}}(Q)$ is closed for every closed subset Q of X,*
3. *$\delta Cl_{\mathfrak{I}}(Int(Q)) \subset Int(\delta Cl_{\mathfrak{I}}(Q))$ for every subset Q of X.*
4. *Every semi**-$\mathfrak{I}$-*open set is pre**-$\mathfrak{I}$-*open.*

3. $semi^*$-$\mathfrak{I}$-open sets, pre^*-$\mathfrak{I}$-open sets and e-$\mathfrak{I}$-open sets in ideal topological spaces

Lemma 3.1. *Let $(X, \tau, \mathfrak{I})$ be an ideal topological space. A subset Q is weakly $\delta_{\mathfrak{I}}$-locally closed if and only if $Q = K \cap \delta Cl_{\mathfrak{I}}(Q)$, where K is an open set.*

Proof. Let Q be weakly $\delta_{\mathfrak{I}}$-locally closed. Then $Q = K \cap C$, where K is open and C is $\delta_{\mathfrak{I}}$-closed. We have $Q \subseteq C$ and $\delta Cl_{\mathfrak{I}}(Q) \subseteq \delta_{\mathfrak{I}}(C) = C$. Hence $Q \subseteq V \cap \delta Cl_{\mathfrak{I}}(Q) \subseteq V \cap C = Q$. Therefore, $Q = K \cap \delta Cl_{\mathfrak{I}}(Q)$, where K is an open set. □

Theorem 3.2. *Let $(X, \tau, \mathfrak{I})$ be a $\delta\mathfrak{I}$-extremally disconnected ideal space and $Q \subset X$, then the following properties are equivalent:*

1. *Q is an open set,*
2. *Q is pre^*-$\mathfrak{I}$-open and weakly $\delta_{\mathfrak{I}}$-local closed,*
3. *Q is e-$\mathfrak{I}$-open and weakly $\delta_{\mathfrak{I}}$-local closed.*

Proof. (1) ⇒ (2) ⇒ (3): the proof is obvious.
(3) ⇒ (1) : Suppose that Q is an e-$\mathfrak{I}$-open set and a weakly $\delta_{\mathfrak{I}}$-local closed set in X. It follows that $Q \subset Cl(\delta Int_{\mathfrak{I}}(Q)) \cup Int(\delta Cl_{\mathfrak{I}}(Q))$. Since Q is a weakly $\delta_{\mathfrak{I}}$-local closed set, by Lemma 3.1 there exists an open set K such that $Q = K \cap \delta Cl_{\mathfrak{I}}(Q)$. It follows from Theorem 2.6 that

$$\begin{aligned}.Q \subseteq & K \cap [Cl(\delta Int_{\mathfrak{I}}(Q)) \cup Int(\delta Cl_{\mathfrak{I}}(Q))] \\ =& (K \cap Cl(\delta Int_{\mathfrak{I}}(Q))) \cup (K \cap Int(\delta Cl_{\mathfrak{I}}(Q))) \\ \subseteq& (K \cap Int(\delta Cl_{\mathfrak{I}}(Q))) \cup (K \cap Int(\delta Cl_{\mathfrak{I}}(Q))) \\ =& Int(K \cap \delta Cl_{\mathfrak{I}}(Q)) \\ =& Int(Q).\end{aligned}$$

Thus, $Q \subseteq Int(Q)$ and hence Q is an open set in X. □

Theorem 3.3. *The following properties hold for a subset Q of an ideal topological space $(X, \tau, \mathfrak{I})$:*

1. *If Q is a pre^*-$\mathfrak{I}$-open set, then $s\delta Cl_{\mathfrak{I}}(Q) = Int(\delta Cl_{\mathfrak{I}}(Q))$.*
2. *If Q is a $semi^*$-$\mathfrak{I}$-open set, then $P^*\mathfrak{I}Cl(Q) = Cl(\delta Int_{\mathfrak{I}}(Q))$.*

Proof. (1) : Suppose that Q is a pre^*-$\mathfrak{I}$-open set in X. Then we have $Q \subseteq Int(\delta Cl_{\mathfrak{I}}(Q))$. By Theorem 2.2 $s\delta Cl_{\mathfrak{I}}(Q) = Q \cup Int(\delta_I(Q) = Int(\delta Cl_{\mathfrak{I}}(Q))$.

(2) : Let Q be a $semi^*$-$\mathfrak{I}$-open set in X. It follows that $Q \subseteq Cl(\delta Int_{\mathfrak{I}}(Q))$. By Theorem 2.2, we have

$P^*\mathfrak{I}Cl(Q) = Q \cup Cl(\delta Int_{\mathfrak{I}}(Q)) = Cl(\delta Int_{\mathfrak{I}}(Q))$. □

Remark 3.4. *The converse of these implications of Theorem 3.3 are not true in general as shown as shown by the following examples:*

Example 3.5. *Let $X = \{x, y, z, e\}$,$\tau = \{\emptyset, X, \{x\}, \{y, z\}, \{x, y, z\}\}$ and $\mathfrak{I} = \{\emptyset, \{x\}, \{e\}, \{x, e\}\}$. Then $s\delta Cl_{\mathfrak{I}}(Q) = \delta Int_{\mathfrak{I}}(Cl(Q))$ for the subset $Q = \{y, e\}$ but Q is not pre^*-$\mathfrak{I}$-open. Moreover, $P^*\mathfrak{I}Cl(Q) = Cl(\delta Int_{\mathfrak{I}}(Q))$ for the subset $R = \{a, d\}$ but R is not $semi^*$-$\mathfrak{I}$-open.*

Theorem 3.6. *Let $(X, \tau, \mathfrak{I})$ be an ideal topological space and $Q \subseteq X$, then the following properties hold:*

1. *If Q is a $semi^*$-$\mathfrak{I}$-closed set, then $P^*\mathfrak{I}Int(Q) = Int(\delta Cl_{\mathfrak{I}}(Q))$.*
2. *If Q is a pre^*-$\mathfrak{I}$-closed set, then $s\delta Int_{\mathfrak{I}}(Q) = Cl(\delta Int_{\mathfrak{I}}(Q))$.*

Proof. (1) : Suppose that Q is a *semi**-$\mathfrak{J}$-closed set. We have $Int(\delta Cl_{\mathfrak{J}}(Q)) \subseteq Q$. Hence, $P^*\mathfrak{J}Int(Q) = Q \cap Int(\delta Cl_{\mathfrak{J}}(Q)) = Int(\delta Cl_{\mathfrak{J}}(Q))$.

(2) : Let Q be a *pre**-$\mathfrak{J}$-closed set. Then $Cl(\delta Int_{\mathfrak{J}}(Q)) \subseteq Q$. This implies that $s\delta Int_{\mathfrak{J}}(Q) = Q \cap Cl(\delta Int_{\mathfrak{J}}(Q)) = Cl(\delta Int_{\mathfrak{J}}(Q))$. □

Theorem 3.7. *For a subset Q of an ideal topological space $(X, \tau, \mathfrak{J})$, Q is an e-$\mathfrak{J}$-closed set if and only if $Q = P^*\mathfrak{J}Cl(Q) \cap s\delta Cl_{\mathfrak{J}}(Q)$.*

Proof. (⇒) Suppose that Q is an e-$\mathfrak{J}$-closed set in X. This implies $Int(\delta Cl_{\mathfrak{J}}(Q)) \cap Cl(\delta Int_{\mathfrak{J}}(Q)) \subseteq Q$. We have

$$\begin{aligned} .P^*\mathfrak{J}Cl(Q) \cap s\delta Cl_{\mathfrak{J}}(Q) &= (Q \cup Cl(\delta Int_{\mathfrak{J}}(Q))) \cap (Q \cup Int(\delta Cl_{\mathfrak{J}}(Q))) \\ &= Q \cup (Cl(\delta Int_{\mathfrak{J}}(Q)) \cap Int(\delta Cl_{\mathfrak{J}}(Q))) \\ &= Q. \end{aligned}$$

Thus, $Q = P^*\mathfrak{J}Cl(Q) \cap s\delta Cl_{\mathfrak{J}}(Q)$.

(⇐) Let $Q = P^*\mathfrak{J}Cl(Q) \cap s\delta Cl_{\mathfrak{J}}(Q)$. Then we have

$$\begin{aligned} .Q =& P^*\mathfrak{J}Cl(Q) \cap s\delta Cl_{\mathfrak{J}}(Q) \\ =& (Q \cup Cl(\delta Int_{\mathfrak{J}}(Q))) \cap (Q \cup Int(\delta Cl_{\mathfrak{J}}(Q))) \\ \supseteq& Cl(\delta Int_{\mathfrak{J}}(Q)) \cap Int(\delta Cl_{\mathfrak{J}}(Q)). \end{aligned}$$

This implies that $Cl(\delta Int_{\mathfrak{J}}(Q)) \cap Int(\delta Cl_{\mathfrak{J}}(Q)) \subseteq Q$. Then, Q is an e-$\mathfrak{J}$-closed set in X. □

Corollary 3.8. *For a subset Q of an ideal topological space $(X, \tau, \mathfrak{J})$, $Cl_e^*(Q) = P^*ICl(Q) \cap s\delta Cl_{\mathfrak{J}}(Q)$.*

Proof. In general, $Cl_e^*(Q) \subset P^*ICl(Q) \cap s\delta Cl_{\mathfrak{J}}(Q) \subset P^*ICl(Cl_e^*(Q)) \cap s\delta Cl_{\mathfrak{J}}(Cl_e^*(Q))$. Since $Cl_e^*(Q)$ is e-$\mathfrak{J}$-closed, by Theorem 3.7,$Cl_e^*(Q) = P^*ICl(Cl_e^*(Q)) \cap s\delta Cl_{\mathfrak{J}}(Cl_e^*(Q))$. Therefore, we obtain $Cl_e^*(Q) = P^*\mathfrak{J}Cl(Q) \cap s\delta Cl_{\mathfrak{J}}(Q)$. □

Corollary 3.9. *Let $(X, \tau, \mathfrak{J})$ be an ideal topological space and $Q \subseteq X$. If Q is pre*-$\mathfrak{J}$-open and semi*-$\mathfrak{J}$-open, then $Cl_e^*(Q) = Cl(\delta Int_{\mathfrak{J}}(Q)) \cap Int(\delta Cl_{\mathfrak{J}}(Q))$.*

Proof. By Theorem 3.3 and Corollary 3.8,

$$Cl_e^*(Q) = P^*ICl(Q) \cap s\delta Cl_{\mathfrak{J}}(Q) = Cl((\delta Int_{\mathfrak{J}}(Q)) \cap Int(\delta Cl_{\mathfrak{J}}(Q)).$$

□

Remark 3.10. *The converse of Corollary 3.9 is not true in general as shown in the following example:*

Example 3.11. *Let $X = \{x, y, z, e\}$,$\tau = \{\emptyset, X, \{x\}, \{y, z\}, \{x, y, z\}\}$ and $\mathfrak{J} = \{\emptyset, \{x\}, \{e\}, \{x, e\}\}$. Take $Q = \{y, z, e\}$. Then $Cl_e^*(Q) = Cl(\delta Int_{\mathfrak{J}}(Q)) \cap Int(\delta Cl_{\mathfrak{J}}(Q))$ but Q is not pre*-$\mathfrak{J}$-open.*

Theorem 3.12. *Let $(X, \tau, \mathfrak{J})$ be an ideal topological space and $Q \subseteq X$. If Q is pre*-$\mathfrak{J}$-closed and semi*-$\mathfrak{J}$-closed , then $Int_e^*(Q) = Cl(\delta Int_{\mathfrak{J}}(Q)) \cup Int(\delta Cl_{\mathfrak{J}}(Q))$.*

Proof. Suppose that Q is a *pre**-$\mathfrak{J}$-closed set and a *semi**-$\mathfrak{J}$-closed set. By Theorem 3.6, we have $s\delta Int_{\mathfrak{J}}(Q) = Cl(\delta Int_{\mathfrak{J}}(Q))$ and $P^*\mathfrak{J}Int(Q) = Int(\delta Cl_{\mathfrak{J}}(Q))$. Thus, $Int_e^*(Q) = P^*\mathfrak{J}Int(Q) \cup s\delta Int_{\mathfrak{J}}(Q) = Int(\delta Cl_{\mathfrak{J}}(Q)) \cup Cl(\delta Int_{\mathfrak{J}}(Q))$. □

Lemma 3.13. *For a subset Q of an ideal topological space $(X, \tau, \mathfrak{J})$, the following properties hold:*

1. $Cl^*_e(Int(Q)) \subseteq Int(\delta Cl_{\mathcal{I}}(Int(Q)))$.
2. $Cl(P^*\mathcal{I}Int(Q)) \subseteq Cl(Int(\delta Cl_{\mathcal{I}}(Q)))$.
3. $Int(s\delta Cl_{\mathcal{I}}(Q)) = Int(\delta Cl_{\mathcal{I}}(Q))$.

Proof. 1): We have

$$\begin{aligned}&Cl^*_e(Int(Q))\\ =&P^*\mathcal{I}Cl(Int(Q)) \cap s\delta Cl_{\mathcal{I}}(Int(Q))\\ =&(Int(Q) \cup Cl(\delta Int_{\mathcal{I}}(Int(Q)))) \cap (Int(Q) \cup Int(\delta Cl_{\mathcal{I}}(Int(Q))))\\ =&\subseteq Int(Q) \cup Int(\delta Cl_{\mathcal{I}}(Int(Q)))\\ =&Int(\delta Cl_{\mathcal{I}}(Int(Q))).\end{aligned}$$

This implies that $Cl^*_e(Int(Q)) \subset Int(\delta Cl_{\mathcal{I}}(Int(Q)))$.
2): We have

$$\begin{aligned}&Cl(P^*\mathcal{I}Int(Q))\\ =&Cl(Q \cap Int(\delta Cl_{\mathcal{I}}(Q)))\\ \subseteq&Cl(Int(\delta Cl_{\mathcal{I}}(Q))).\end{aligned}$$

Hence, we have $Cl(P^*\mathcal{I}Int(Q)) \subseteq Cl(Int(\delta Cl_{\mathcal{I}}(Q)))$.

3): We have

$$\begin{aligned}&Int(s\delta Cl_{\mathcal{I}}(Q))\\ =&Int(Q \cup Int(\delta Cl_{\mathcal{I}}(Q)))\\ \supseteq&Int(Q) \cup Int(\delta Cl_{\mathcal{I}}(Q))\\ =&Int(\delta Cl_{\mathcal{I}}(Q)).\end{aligned}$$

□

As the previous version, conversely we have four lines

Corollary 3.14. *For a subset Q of an ideal topological space $(X, \tau, \mathcal{I})$, the following properties hold:*

1. $Int^*_e(Cl(Q)) \supseteq Cl(\delta Int_{\mathcal{I}}(Cl(Q)))$.
2. $Int(P^*\mathcal{I}Cl(Q)) \supseteq Int(Cl(\delta Int_{\mathcal{I}}(Q)))$.
3. $Cl(s\delta Int_{\mathcal{I}}(Q)) = Cl(\delta Int_{\mathcal{I}}(Q))$.

Proof. It follows from Lemma 3.13. □

Theorem 3.15. *For a subset Q of an ideal topological space $(X, \tau, \mathcal{I})$, the following properties hold:*

1. $Int(Cl^*_e(Q)) = Int(Cl(\delta Int_{\mathcal{I}}(Q)))$.
2. $Cl(Int^*_e(Q)) = Cl(Int(\delta Cl_{\mathcal{I}}(Q)))$.

Proof. (1) : We have

$$\begin{aligned}&Int(Cl^*_e(Q))\\ =&Int(P^*\mathcal{I}Cl(Q) \cap s\delta Cl_{\mathcal{I}}(Q))\\ =&Int(P^*\mathcal{I}Cl(Q)) \cap Int(s\delta Cl_{\mathcal{I}}(Q))\\ =&Int(P^*\mathcal{I}Cl(Q)) \cap Int(\delta Cl_{\mathcal{I}}(Q))\\ =&Int(P^*\mathcal{I}Cl(Q))\\ =&Int(Cl(\delta Int_{\mathcal{I}}(Q))).\end{aligned}$$

by Lemma 3.13. Thus, $Int(Cl_e^*(Q)) = Int(Cl(\delta Int_{\mathfrak{I}}(Q)))$.
(2) : It follows from (1). □

Theorem 3.16. *For a subset Q of an ideal topological space $(X, \tau, \mathfrak{I})$, the following properties hold:*

1. $P^*\mathfrak{I}Cl(s\delta Int_{\mathfrak{I}}(Q)) \subseteq Cl(\delta Int_{\mathfrak{I}}(Q))$.
2. $P^*\mathfrak{I}Int(s\delta Cl_{\mathfrak{I}}(Q)) \supseteq Int(\delta Cl_{\mathfrak{I}}(Q))$.

Proof. (1) : By Theorem 3.3, we have
$P^*\mathfrak{I}Cl(s\delta Int_{\mathfrak{I}}(Q)) = Cl(\delta Int_{\mathfrak{I}}(s\delta Int_{\mathfrak{I}}(Q))) \subseteq Cl(\delta Int_{\mathfrak{I}}(Q))$.
This implies
$P^*\mathfrak{I}Cl(s\delta Int_{\mathfrak{I}}(Q)) \subseteq Cl(\delta Int_{\mathfrak{I}}(Q))$.
(2) : This follows from (1). □

Theorem 3.17. *For a subset Q of an ideal topological space $(X, \tau, \mathfrak{I})$, the following properties hold:*

1. $Cl_e^*(s\delta Int_{\mathfrak{I}}(Q)) \subseteq s\delta Int_{\mathfrak{I}}(Q) \cup \delta Int_{\mathfrak{I}}(Cl(Int(Q)))$.
2. $P^*\mathfrak{I}Int(Cl_e^*(Q)) \supseteq P^*\mathfrak{I}Cl(Q) \cap Int(\delta Cl_{\mathfrak{I}}(Q))$.
3. $s\delta Int_{\mathfrak{I}}(Cl_e^*(Q)) \supseteq s\delta Cl_{\mathfrak{I}}(Q) \cap Cl(\delta Int_{\mathfrak{I}}(Q))$.

Proof. (1) : By Theorem 3.16 and Corollary 3.14, we have

$$\begin{aligned}
Cl_e^*(s\delta Int_{\mathfrak{I}}(Q)) &= P^*\mathfrak{I}Cl(s\delta Int_{\mathfrak{I}}(Q)) \cap s\delta Cl_{\mathfrak{I}}(s\delta Int_{\mathfrak{I}}(Q)) \\
&\subseteq Cl(\delta Int_{\mathfrak{I}}(Q)) \cap [s\delta Int_{\mathfrak{I}}(Q) \cup Int(\delta Cl_{\mathfrak{I}}(Int(Q)))] \\
&\subseteq s\delta Int_{\mathfrak{I}}(Q) \cup Int(\delta Cl_{\mathfrak{I}}(Int(Q))) \\
&= s\delta Int_{\mathfrak{I}}(Q) \cup Int(Cl(Int(Q))) \\
&= s\delta Int_{\mathfrak{I}}(Q) \cup \delta Int_{\mathfrak{I}}(Cl(Int(Q))).
\end{aligned}$$

Then, $Cl_e^*(s\delta Int_{\mathfrak{I}}(Q)) \subseteq s\delta Int_{\mathfrak{I}}(Q) \cup \delta Int_{\mathfrak{I}}(Cl(Int(Q)))$.
(2) : We have

$$\begin{aligned}
&P^*Int(Cl_e^*(Q)) \\
&= Cl_e^*(Q) \cap Int(\delta Cl_{\mathfrak{I}}(Cl_e^*(Q))) \\
&= [P^*\mathfrak{I}Cl(Q) \cap s\delta Cl_{\mathfrak{I}}(Q)] \cap Int(\delta Cl_{\mathfrak{I}}(Cl_e^*(Q))) \\
&= P^*\mathfrak{I}Cl(Q) \cap [Q \cup Int(\delta Cl_{\mathfrak{I}}(Q))] \cap Int(\delta Cl_{\mathfrak{I}}(Cl_e^*(Q))) \\
&\supset P^*\mathfrak{I}Cl(Q) \cap Int(\delta Cl_{\mathfrak{I}}(Q)) \cap Int(\delta Cl_{\mathfrak{I}}(Cl_e^*(Q))) \\
&\supset P^*\mathfrak{I}Cl(Q) \cap Int(\delta Cl_{\mathfrak{I}}(Q)).
\end{aligned}$$

This implies $P^*\mathfrak{I}Int(Cl_e^*(Q)) \supseteq P^*\mathfrak{I}Cl(Q) \cap Int(\delta Cl_{\mathfrak{I}}(Q))$.
(3): We have

$$\begin{aligned}
&s\delta Int_{\mathfrak{I}}(Cl_e^*(Q)) \\
&= Cl_e^*(Q) \cap Cl(\delta Int_{\mathfrak{I}}(Cl_e^*(Q))) \\
&= [P^*\mathfrak{I}Cl(Q) \cap s\delta Cl_{\mathfrak{I}}(Q)] \cap Cl(\delta Int_{\mathfrak{I}}(Cl_e^*(Q))) \\
&\supset Cl(\delta Int_{\mathfrak{I}}(Q)) \cap s\delta Cl_{\mathfrak{I}}(Q) \cap Cl(\delta Int_{\mathfrak{I}}(Q)) \\
&= s\delta Cl_{\mathfrak{I}}(Q) \cap Cl(\delta Int_{\mathfrak{I}}(Q)).
\end{aligned}$$

Hence, $s\delta Int_{\mathfrak{I}}(Cl_e^*(Q)) \supseteq s\delta Cl_{\mathfrak{I}}(Q) \cap Cl(\delta Int_{\mathfrak{I}}(Q))$. □

Corollary 3.18. *For a subset Q of an ideal topological space $(X, \tau, \mathcal{I})$, the following properties hold:*

1. $Int^*_e(s\delta Cl_{\mathcal{I}}(Q)) \supseteq s\delta Cl_{\mathcal{I}}(Q) \cap \delta Cl_{\mathcal{I}}(Int(Cl(Q)))$.
2. $P^*\mathcal{I}Cl(Int^*_e(Q)) \subseteq P^*\mathcal{I}Int(Q) \cup Cl(\delta Int_{\mathcal{I}}(Q))$.
3. $s\delta Cl_{\mathcal{I}}(Int^*_e(Q)) \subseteq s\delta Int_{\mathcal{I}}(Q) \cup Int(\delta Cl_{\mathcal{I}}(Q))$.

Proof. The proof follows from Theorem 3.17. □

4. Further Properties and Decompositions of Continuity

Definition 4.1. *A function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is said to be weakly $\delta_{\mathcal{I}}$-locally-continuous if $f^{-1}(Q)$ is weakly $\delta_{\mathcal{I}}$-locally closed for each open set Q in Y.*

Definition 4.2. *A function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ is said $\delta\alpha$-$\mathcal{I}$-continuous [5] (resp. semi$*$-$\mathcal{I}$-continuous [4] (resp. pre*-$\mathcal{I}$-continuous [4], e-$\mathcal{I}$-continuous [2]) if $f^{-1}(Q)$ is semi$*$-$\mathcal{I}$-open (rep. pre*-$\mathcal{I}$-open, e-$\mathcal{I}$-open) for each open set Q in Y.*

Theorem 4.3. *For a function $f : (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma)$ where $(X, \tau, \mathcal{I})$ is a $\delta\mathcal{I}$-extremally disconnected ideal space, the following properties are equivalent:*

1. *f is continuous,*
2. *f is pre*-$\mathcal{I}$-continuous and weakly $\delta_{\mathcal{I}}$-locally-continuous,*
3. *f is e-$\mathcal{I}$-continuous and weakly $\delta_{\mathcal{I}}$-locally-continuous.*

Proof. It follows from Theorem 3.3. □

Definition 4.4. *A subset Q of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be*

1. *generalized e-$\mathcal{I}$-open (gE$\mathcal{I}$-open) if $H \subseteq Int^*_e(Q)$ whenever $K \subseteq Q$ and H is a closed set in X.*
2. *generalized e-$\mathcal{I}$-closed (gE$\mathcal{I}$ -closed) if and only if $X \setminus Q$ is a gE$\mathcal{I}$-open in X.*

Theorem 4.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $Q \subseteq X$. Then Q is a e-$\mathcal{I}$-closed set iff Q is a $\mathcal{B}G_{\mathcal{I}^*}$-set and a gE$\mathcal{I}$-closed set in X.*

Proof. Let Q be a $\mathcal{B}G_{\mathcal{I}^*}$-set and a $gE\mathcal{I}$-closed set in X. By Theorem 2.3, $Q = F \cap Cl^*_e(Q)$ for a $\delta_{\mathcal{I}}$-open set F in X. Since $Q \subseteq F$ and Q is $gE\mathcal{I}$-closed, then we have $Cl^*_e(Q) \subseteq F$. Thus, $Cl^*_e(Q) \subseteq F \cap Cl^*_e(Q) = Q$ and hence Q is e-$\mathcal{I}$-closed.
Conversely, it follows from the fact that any e-$\mathcal{I}$-closed set is a $\mathcal{B}G_{\mathcal{I}^*}$ -set and a $gE\mathcal{I}$-closed.

□

Theorem 4.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $Q \subseteq X$. Then Q is a gE$\mathcal{I}$-closed set iff $Cl^*_e(Q) \subseteq F$ whenever $Q \subseteq F$ and F is an open set in X.*

Proof. Let Q be a $gE\mathcal{I}$-closed set in X. Suppose that $Q \subseteq F$ and F is an open set in X. This implies that $X \setminus Q$ is a $gE\mathcal{I}$-open set and $X \setminus F$ is a closed set. Since $X \setminus Q$ is a $gE\mathcal{I}$-open set, then $X \setminus F \subseteq Int^*_e(X \setminus Q)$. Since $Int^*_e(X \setminus Q) = X \setminus Cl^*_e(Q)$, then we have $Cl^*_e(Q) = X \setminus Int^*_e(X \setminus Q) \subseteq F$. Thus, $Cl^*_e(Q) \subseteq F$. The proof of converse same. □

Theorem 4.7. *For a subset Q of an ideal topological space $(X, \tau, \mathcal{I})$, if Q is a $\mathcal{B}G_{\mathcal{I}^*}$-set in X, then $Cl^*_e(Q) \setminus Q$ is a e-$\mathcal{I}$-closed set and $Q \cup (X \setminus Cl^*_e(Q))$ is a e-$\mathcal{I}$-open set in X.*

Proof. Suppose that Q is a $\mathcal{B}G_{\mathfrak{I}^*}$-set in X. By Theorem 2.3, we have $Q = F \cap Cl_e^*(Q)$ for a $\delta_{\mathfrak{I}}$-open set F. This implies

$$\begin{aligned} Cl_e^*(Q) \setminus Q &= Cl_e^*(Q) \setminus (F \cap Cl_e^*(Q)) \\ &= Cl_e^*(Q) \cap (X \setminus (F \cap Cl_e^*(Q))) \\ &= Cl_e^*(Q) \cap ((X \setminus F) \cup (X \setminus Cl_e^*(Q))) \\ &= (Cl_e^*(Q) \cap (X \setminus F)) \cup (Cl_e^*(Q) \cap (X \setminus Cl_e^*(Q))) \\ &= Cl_e^*(Q) \cap (X \setminus F). \end{aligned}$$

Consequently, $Cl_e^*(Q) \setminus Q$ is e-$\mathfrak{I}$-closed. On the other hand, since $Cl_e^*(Q) \setminus Q$ is a e-$\mathfrak{I}$-closed set, then $X \setminus (Cl_e^*(Q) \setminus Q)$ is a e-$\mathfrak{I}$-open set. Since $X \setminus (Cl_e^*(Q) \setminus Q) = X \setminus (Cl_e^*(Q) \cap (X \setminus Q)) = (X \setminus Cl_e^*(Q)) \cup Q$, then $Q \cup (X \setminus Cl_e^*(Q))$ is a e-$\mathfrak{I}$-open set □

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