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A New Spectral Approach In The Matrix Algebra: Trace Pseudospectrum

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ABSTRACT: We work with the notion of trace pseudospectra for an element in the matrix algebra. Many new interesting properties of the trace pseudospectrum have been discovered. In addition, we show an analogue of the spectral mapping theorem for trace pseudospectrum in the matrix algebra. Among other things, we illustrate the applicability of this concepts by a considerable number of examples.

Key Words: Pseudospectrum, condition pseudospectrum, trace pseudospectrum, determinant pseudospectrum.

Contents

1. Introduction

Let $\mathcal{M}_n(\mathbb{C})$ ($\mathcal{M}_n(\mathbb{R})$) denote the algebra of all $n \times n$ complex (real) matrices, I denote the $n \times n$ identity matrix and the transpose of T is denoted by T^*. We denote by Tr, (resp. Det) the trace (resp. determinant) map on $\mathcal{M}_n(\mathbb{C})$. The set of all eigenvalues of $T \in \mathcal{M}_n(\mathbb{C})$ is denoted by $\sigma(T)$ and is defined as

$$\sigma(T) = \big\{\lambda \in \mathbb{C} : \ \lambda - T \text{ is not invertible}\big\},$$

and its spectral radius by

$$r(T) = \sup\Big\{|\lambda| : \lambda \in \sigma(T)\Big\}.$$

Let $\mathrm{s}_n(T) \leq \ldots \leq \mathrm{s}_2(T) \leq \mathrm{s}_1(T)$ be the singular values of $T \in \mathcal{M}_n(\mathbb{C})$, where $\mathrm{s}_1(T)$ the smallest and $\mathrm{s}_n(T)$ largest singular values of the matrix T. For a $n \times n$ complex matrix T and $\varepsilon > 0$, the pseudospectrum of T is defined as the following closed set in the complex plane

$$\sigma_\varepsilon(T) := \{\lambda \in \mathbb{C} : \ \mathrm{s}_n(\lambda - T) \leq \varepsilon\}.$$

Let $T \in \mathcal{M}_n(\mathbb{C})$ and $0 < \varepsilon < 1$. The condition pseudospectrum of T is denoted by $\Sigma_\varepsilon(T)$ and is defined as

$$\Sigma_\varepsilon(T) := \left\{\lambda \in \mathbb{C} : \frac{\mathrm{s}_n(\lambda - T)}{\mathrm{s}_1(\lambda - T)} \leq \varepsilon\right\}.$$

Recall that the usual condition pseudospectral radius $r_\varepsilon(T)$ of $T \in \mathcal{M}_n(\mathbb{C})$ is defined by

$$r_\varepsilon(T) := \sup\Big\{|\lambda| : \lambda \in \Sigma_\varepsilon(T)\Big\}.$$

Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. The determinant spectrum of T is denoted by $\mathrm{d}_\varepsilon(T)$ and is defined as

$$\mathrm{d}_\varepsilon(T) = \big\{\lambda \in \mathbb{C} : |\mathrm{Det}(\lambda I - T)| \leq \varepsilon\big\}.$$

For more information on various details on the above concepts, properties and applications of pseudospectrum see [1,4,7,9,15] and condition spectrum see [2,3,5,6,13]. In [12], Krishna Kumar. G, introduced the concept of the determinant spectrum for an element in the matrix algebra.

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In this paper, we are interested by a another generalization of eigenvalues called trace pseudospectrum for an element in the matrix algebra to give more information about T. For $\varepsilon > 0$, the trace pseudospectrum of $T \in \mathcal{M}_n(\mathbb{C})$ is denoted by $\mathrm{Tr}_\varepsilon(T)$ and is defined as

$$\mathrm{Tr}_\varepsilon(T) = \sigma(T) \bigcup \big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda I - T)| \leq \varepsilon\big\}.$$

The trace pseudoresolvent of T is denoted by $\mathrm{Tr}\rho_\varepsilon(T)$ and is defined as

$$\mathrm{Tr}\rho_\varepsilon(T) = \rho(T) \bigcap \big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda I - T)| > \varepsilon\big\}$$

while the trace pseudospectral radius of T is defined as

$$\mathrm{Trr}_\varepsilon(T) := \sup \Big\{|\lambda| : \lambda \in \mathrm{Tr}_\varepsilon(T)\Big\}.$$

The singular values of a matrix play an important role in diagonalization also for their utility in a variety of applications. Since $\mathrm{Tr}_\varepsilon(|T|)$ use all the singular values of $\lambda I - T$ to get defined, it is expected to give more information about T than eigenvalues, pseudospectrum and condition spectrum. In fact, we have

$$\mathrm{Tr}(|T|) = \sum_{i=1}^{n} \mathrm{s}_i(T)$$

where, $|T| = \sqrt{T^*T}$ is the unique positive semidefinite square root of T^*T. The following is theoretically equivalent to the definition of trace pseudospectrum of $|T|$,

$$\mathrm{Tr}_\varepsilon(|T|) = \Big\{\lambda \in \mathbb{C} : \sum_{i=1}^{n} \mathrm{s}_i(\lambda I - T) \leq \varepsilon\Big\}.$$

Moreover, to study eigenvalues of perturbations of T, we can look at the sets like

$$\Big\{\lambda \in \mathbb{C} : \mathrm{Tr}(|(\lambda I - T)^{-1}|) \geq \frac{1}{\varepsilon}\Big\} = \Big\{\lambda \in \mathbb{C} : \sum_{i=1}^{n} \frac{1}{\mathrm{s}_i(\lambda I - T)} \geq \frac{1}{\varepsilon}\Big\}.$$

Since $\mathrm{s}_n(\lambda I - T) = 0$ when, $\lambda \in \sigma(T)$. The theory of eigenvalues and the generalized eigenvalues (trace pseudospectrum, see [8]) of a matrices are established in different fields of mathematics and their applications. These approaches are useful in studying for an element in the matrix algebra, and have attracted a lot of interest of many authors in the last few years (see [11,10]). Applications of this concept can be found in perturbation theory, generalized eigenvalue problems, numerical analysis, system theory, and dilation theory.

The main contributions of this paper are as follows. In Section 2, we introduce and study the trace pseudospectrum for an element in the matrix algebra. We begin by the definition, also we focus on the characterization of trace pseudospectrum (Theorems 2.3 and 2.4). We will prove some results and properties of the trace pseudospectrum (Theorems 2.7 and 2.6) and then we investigate the connection between trace pseudospectrum and algebraic multiplicity of the eigenvalues (Theorem 2.15). In Section 3, we give an analogue of the spectral mapping theorem for the trace pseudospectrum in the matrix algebra (Theorems 3.2 and 3.3).

2. Trace pseudospectrum

In this section, we describe some basic properties of the trace pseudospectrum. For $\varepsilon > 0$, $T \in \mathcal{M}_n(\mathbb{C})$ is said to be invertible with respect to the trace pseudospectrum, if $0 \notin \mathrm{Tr}_\varepsilon(T)$, that is, T is invertible and $|\mathrm{Tr}(T)| > \varepsilon$. Obviously,

$$\Omega := \big\{T \in \mathcal{M}_n(\mathbb{C}) : \ |\mathrm{Tr}(T)| > \varepsilon\big\}.$$

is not a Ransford set [14]. Hence the trace pseudospectrum is not a Ransford spectrum. The map

$$T \to \mathrm{Tr}(T)$$

is continuous linear functional and hence Ω is an open set. Important properties of the trace of $T, B \in \mathcal{M}_n(\mathbb{C})$ are

$$\mathrm{Tr}(TB) = \mathrm{Tr}(BT),$$

$$\mathrm{Tr}(\alpha T) = \alpha \mathrm{Tr}(T) \text{ with } \alpha \in \mathbb{C},$$

$$\mathrm{Tr}(T + B) = \mathrm{Tr}(T) + \mathrm{Tr}(B).$$

Now, we introduce the new concept of the trace pseudospectrum in the following definition.

Definition 2.1. *For $\varepsilon > 0$, the trace pseudospectrum of $T \in \mathcal{M}_n(\mathbb{C})$ is denoted by $\mathrm{Tr}_\varepsilon(T)$ and is defined as*

$$\mathrm{Tr}_\varepsilon(T) = \sigma(T) \bigcup \big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda I - T)| \leq \varepsilon\big\}.$$

The trace pseudoresolvent of T is denoted by $\mathrm{Tr}\rho_\varepsilon(T)$ and is defined as

$$\mathrm{Tr}\rho_\varepsilon(T) = \rho(T) \bigcap \big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda I - T)| > \varepsilon\big\}.$$

The following theorem gives some properties of the trace pseudospectrum that follow in a straightforward manner from the definition of the trace pseudospectrum.

Theorem 2.2. *Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then,*

(i) $\mathrm{Tr}_0(T) = \bigcap\limits_{\varepsilon > 0} \mathrm{Tr}_\varepsilon(T)$.

(ii) *If $0 < \varepsilon_1 < \varepsilon_2$, then $\mathrm{Tr}_{\varepsilon_1}(T) \subset \mathrm{Tr}_{\varepsilon_2}(T)$.*

(iii) *$\mathrm{Tr}_\varepsilon(T)$ is a non-empty compact subset of $\mathbb{C}$.*

(iv) *If $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}\backslash\{0\}$. Then, $\mathrm{Tr}_\varepsilon(\beta T + \alpha I) = \beta \mathrm{Tr}_{\frac{\varepsilon}{|\beta|}}(T) + \alpha$.*

(v) *$\mathrm{Tr}_\varepsilon(\alpha I) = \big\{\lambda \in \mathbb{C} : n|\lambda - \alpha| \leq \varepsilon\big\}$ for all $\lambda, \alpha \in \mathbb{C}$.*

Proof. The first two items can be easily checked using the definition of trace pseudospectrum, so we only include the proof of item (iii), (iv) and (v).

(iii) Using the continuity from $\mathbb{C}$ to $[0, \infty[$ of the map

$$\lambda \to |\mathrm{Tr}(\lambda I - T)|,$$

we get that $\mathrm{Tr}_\varepsilon(T)$ is a compact set in the complex plane containing the eigenvalues of T.

(iv) Since

$$\begin{aligned}
\mathrm{Tr}_\varepsilon(\beta T + \alpha I) &= \Big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda I - \beta T - \alpha I)| \leq \varepsilon\Big\} \\
&= \left\{\lambda \in \mathbb{C} : |\beta| \left|\mathrm{Tr}\left(\frac{\lambda - \alpha}{\beta} I - T\right)\right| \leq \varepsilon\right\} \\
&= \left\{\lambda \in \mathbb{C} : \left|\mathrm{Tr}\left(\frac{\lambda - \alpha}{\beta} I - T\right)\right| \leq \frac{\varepsilon}{|\beta|}\right\}.
\end{aligned}$$

Then, $\lambda \in \mathrm{Tr}_\varepsilon(\beta T + \alpha I)$. Hence, $\dfrac{\lambda - \alpha}{\beta} \in \mathrm{Tr}_{\frac{\varepsilon}{|\beta|}}(T)$. Thus, $\lambda \in \beta \mathrm{Tr}_{\frac{\varepsilon}{|\beta|}}(T) + \alpha$.

(v) Let $\lambda \in \mathrm{Tr}_\varepsilon(\alpha I)$, then

$$\begin{aligned} |\mathrm{Tr}(\lambda I - \alpha I)| &= |\lambda - \alpha||\mathrm{Tr}(I)| \\ &= n|\lambda - \alpha| \\ &\leq \varepsilon. \end{aligned}$$

which yields $\mathrm{Tr}_\varepsilon(\alpha I) = \big\{\lambda \in \mathbb{C} : n|\lambda - \alpha| \leq \varepsilon\big\}$ for all $\lambda, \alpha \in \mathbb{C}$. □

We are now ready to present the following.

Theorem 2.3. *Let $T \in \mathcal{M}_n(\mathbb{C})$, $\lambda \in \mathbb{C}$, and $\varepsilon > 0$. If there is $D \in \mathcal{M}_n(\mathbb{C})$ such that $|\mathrm{Tr}(D)| \leq \varepsilon$ and $\mathrm{Tr}(\lambda - T - D) = 0$. Then, $\lambda \in \mathrm{Tr}_\varepsilon(T)$.*

Proof. We assume that there exists $D \in \mathcal{M}_n(\mathbb{C})$ such that $|\mathrm{Tr}(D)| \leq \varepsilon$ and

$$\mathrm{Tr}(\lambda - T - D) = 0.$$

Then,

$$|\mathrm{Tr}(\lambda - T)| = |\mathrm{Tr}(D)| \leq \varepsilon.$$

Thus, $\lambda \in \mathrm{Tr}_\varepsilon(T)$. □

Theorem 2.4. *Let $T \in \mathcal{M}_n(\mathbb{C})$, $\lambda \in \mathbb{C}$, and $\varepsilon > 0$. If $\lambda \in \mathrm{Tr}_\varepsilon(T)$. Then, there is $D \in \mathcal{M}_n(\mathbb{C})$ such that $|\mathrm{Tr}(D)| \leq \varepsilon$ and $\mathrm{Tr}(\lambda - T - D) = 0$.*

Proof. Let $\lambda \in \mathrm{Tr}_\varepsilon(T)$. Then,

$$|\mathrm{Tr}(\lambda - T)| \leq \varepsilon.$$

Now, we consider the matrix D by

$$D = \frac{\mathrm{Tr}(\lambda - T)}{n}\, I.$$

It is clear that, $D \in \mathcal{M}_n(\mathbb{C})$ and

$$|\mathrm{Tr}(D)| = \left|\mathrm{Tr}\left(\frac{\mathrm{Tr}(\lambda - T)}{n}\, I\right)\right| = \frac{|\mathrm{Tr}(\lambda - T)|}{n}\, \mathrm{Tr}(I) \leq \varepsilon.$$

Also, we have

$$\mathrm{Tr}(\lambda - T - D) = \mathrm{Tr}\left(\lambda - T - \frac{\mathrm{Tr}(\lambda - T)}{n}\, I\right) = 0.$$

□

Remark 2.5. *In summary, at the present moment we have shown that from Theorems 2.3 and 2.4, that for $T \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$*

$$\mathcal{J}(T, \varepsilon) = \mathrm{Tr}_\varepsilon(T),$$

where

$$\mathcal{J}(T, \varepsilon) := \big\{\lambda \in \mathbb{C} : \ \mathrm{Tr}(\lambda - T - D) = 0 \text{ for some } D \in \mathcal{M}_n(\mathbb{C}), |\mathrm{Tr}(D)| \leq \varepsilon\big\}.$$

Theorem 2.6. *$T \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then,*

$$\mathrm{Tr}_\delta(T) + \Theta_\varepsilon \subseteq \mathrm{Tr}_{n\varepsilon+\delta}(T), \tag{2.1}$$

holds for $\delta, \varepsilon > 0$ with Θ_ε, denoting the closed disk in the complex plane centered at the origin with radius ε. If we take $\delta = 0$, we obtain an inner bound for $\mathrm{Tr}_\varepsilon(T)$, namely

$$\mathrm{Tr}_0(T) + \Theta_\varepsilon \subseteq \mathrm{Tr}_{n\varepsilon}(T). \tag{2.2}$$

Proof. Let $\lambda \in \text{Tr}_\delta(T)+\Theta_\varepsilon$. Then, there exists $\lambda_1 \in \text{Tr}_\delta(T)$ and $\lambda_2 \in \Theta_\varepsilon$ such that $\lambda = \lambda_1+\lambda_2$. Therefore, $|\text{Tr}(\lambda_1 I - T)| \leq \delta$ and $|\lambda_2| \leq \varepsilon$. Now, we have

$$\begin{aligned}|\text{Tr}(\lambda I - T)| &= |\text{Tr}((\lambda_1+\lambda_2)I - T)| \\ &= |\text{Tr}(\lambda_2 I) + \text{Tr}(\lambda_1 I - T)| \\ &\leq |\lambda_2||\text{Tr}(I)| + |\text{Tr}(\lambda_1 I - T)| \\ &\leq n|\lambda_2| + |\text{Tr}(\lambda_1 I - T)| \leq n\varepsilon + \delta,\end{aligned}$$

so that (2.1) holds. Finally, let $\delta = 0$, then the desired inclusion (2.2) is obtained. □

Theorem 2.7. *Let $T, B \in \mathcal{M}_n(\mathbb{C})$ such that $TB = BT$ and $\varepsilon > 0$. If T is normal (i.e. $T^*T = TT^*$), then*

$$\text{Tr}_\varepsilon(T+B) \subseteq \sigma(T) + \text{Tr}_\varepsilon(B).$$

Proof. We assume that T is normal, so there exists a unitary matrix $U \in \mathcal{M}_n(\mathbb{C})$ such that

$$U^*TU = \lambda_1 I_{n_1} \oplus \lambda_2 I_{n_2} \oplus \ldots \oplus \lambda_k I_{n_k}.$$

Using the condition $TB = BT$ implies that

$$U^*BU = T_1 \oplus T_2 \ldots \oplus T_k$$

where, $T_i \in \mathcal{M}_{n_k}(\mathbb{C})$, $i = 1, \ldots, k$. Using Theorems 2.2, we have

$$\begin{aligned}\text{Tr}_\varepsilon(T+B) &= \text{Tr}_\varepsilon(U^*TU + U^*BU) \\ &= \text{Tr}_\varepsilon((\lambda_1 I_{n_1} + T_1) \oplus \ldots \oplus (\lambda_k I_{n_k} + T_k)) \\ &= \bigcup_{i=1}^{k} \text{Tr}_\varepsilon(\lambda_i I_{n_i} + T_i) \\ &= \bigcup_{i=1}^{k} \lambda_i + \text{Tr}_\varepsilon(T_i) \\ &\subseteq \sigma(T) + \text{Tr}_\varepsilon(B).\end{aligned}$$

The proof is thus complete. □

The next corollary is a consequence of Theorem 2.7.

Remark 2.8. *If $B = 0_{n\times n}$, then*

$$\text{Tr}_\varepsilon(T) \subseteq \sigma(T) + \Big\{\lambda \in \mathbb{C} : |\lambda| \leq \frac{\varepsilon}{n}\Big\}.$$

The following example proves that the condition (T is normal) in Theorem 2.7 is necessary.

Example: Let $\alpha \in \mathbb{C}$ with $\alpha \neq \beta \neq 0$ and let $T = B = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}$. It is clear that T and B are not normal. Using Theorem 2.2 we have

$$\text{Tr}_\varepsilon(T+B) = \text{Tr}_\varepsilon(2T) = \big\{\lambda \in \mathbb{C} : |2\lambda| \leq \varepsilon\big\},$$

$$\sigma(T) + \text{Tr}_\varepsilon(B) = \big\{\pm\sqrt{\alpha\beta}\big\} + \big\{\lambda \in \mathbb{C} : |2\lambda| \leq \varepsilon\big\}.$$

Hence,

$$\text{Tr}_\varepsilon(T+B) \nsubseteq \sigma(T) + \text{Tr}_\varepsilon(B).$$

Remark 2.9. *Let $T, B \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then, from Theorem 2.7, we obtain the following inequality,*

$$\mathrm{Trr}_\varepsilon(T+B) \le r(T) + \mathrm{Trr}_\varepsilon(B).$$

Also, if $B = 0_{n\times n}$ we have

$$\mathrm{Trr}_\varepsilon(T) \le r(T) + \tfrac{\varepsilon}{n}.$$

Each $T \in \mathcal{M}_n(\mathbb{C})$ can be written in exactly one way as

$$T = \mathrm{Re}(T) + i\mathrm{Im}(T) \text{ (Cartesian decomposition)}$$

in which $\mathrm{Re}(T) = \frac{T+T^*}{2}$ denote its real part and $\mathrm{Im}(T) = \frac{T-T^*}{2i}$ denote its imaginary part. Then, we can see the foregoing discussion in the following ramark.

Remark 2.10. *Let $T \in \mathcal{M}_n(\mathbb{C})$ be a normal matrix such that its spectrum is symmetric with respect to the origin and $\varepsilon > 0$. Then*

$$2\mathrm{Tr}_{\frac{\varepsilon}{2}}\big(\mathrm{Re}(T) \oplus i\mathrm{Im}(T)\big) \subseteq \sigma(T) + \mathrm{Tr}_\varepsilon(T^*).$$

Theorem 2.11. *Let $T, B \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then,*

$$\mathrm{Tr}_{\frac{\varepsilon}{2}}(T) + \mathrm{Tr}_{\frac{\varepsilon}{2}}(B) \subseteq \mathrm{Tr}_\varepsilon(T+B).$$

Proof. For the first inclusion, let $\lambda \in \mathrm{Tr}_{\frac{\varepsilon}{2}}(T) + \mathrm{Tr}_{\frac{\varepsilon}{2}}(B)$. Then, there exists $\lambda_1 \in \mathrm{Tr}_{\frac{\varepsilon}{2}}(T)$ and $\lambda_2 \in \mathrm{Tr}_{\frac{\varepsilon}{2}}(B)$ such that $\lambda = \lambda_1 + \lambda_2$. Therefore,

$$|\mathrm{Tr}(\lambda_1 - T)| \le \frac{\varepsilon}{2} \text{ and } |\mathrm{Tr}(\lambda_2 - T)| \le \frac{\varepsilon}{2}.$$

On the other hand,

$$\begin{aligned} |\mathrm{Tr}(\lambda - T - B)| &= |\mathrm{Tr}(\lambda_1 - T + \lambda_2 - B)| \\ &\le |\mathrm{Tr}(\lambda_1 - T)| + |\mathrm{Tr}(\lambda_2 - B)| \\ &\le \varepsilon \end{aligned}$$

Then, $\lambda \in \mathrm{Tr}_\varepsilon(T+B)$. □

Definition 2.12. *Given $T, B \in \mathcal{M}_n(\mathbb{C})$. T is said to be diagonally similar to B if there exists a nonsingular diagonal matrix V such that $T = VBV^{-1}$ if, in addition, V can be chosen to be unitary, then we say T is unitarily diagonally similar to B.*

Theorem 2.13. *Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then,*

(i) If $T \in \mathcal{M}_n(\mathbb{C})$ is diagonally similar to B or T is unitarily diagonally similar to $B \in \mathcal{M}_n(\mathbb{C})$, then $\mathrm{Tr}_\varepsilon(T) = \mathrm{Tr}_\varepsilon(B)$.

(ii) The map $T \to \mathrm{Tr}_\varepsilon(T)$ is an upper semi continuous function from $\mathcal{M}_n(\mathbb{C})$ to compact subsets of $\mathbb{C}$.

Proof. (i) Let $\lambda \in \mathrm{Tr}_\varepsilon(T)$, if and only if

$$\begin{aligned} |\mathrm{Tr}(\lambda I - B)| &= |\mathrm{Tr}(V^{-1}(\lambda I - T)V)|, \ \big(\text{using } \mathrm{Tr}(TB) = \mathrm{Tr}(BT)\big) \\ &= |\mathrm{Tr}(\lambda I - T)| \le \varepsilon. \end{aligned}$$

if and only if, $\lambda \in \mathrm{Tr}_\varepsilon(B)$.

(ii) Obvious, from Definition 2.1. □

The following example shows that the converse of the assertion (i) is not true.

Example: Let $T = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then, T and B are not similar and for $\varepsilon > 0$, we have

$$\mathrm{Tr}_\varepsilon(T) = \mathrm{Tr}_\varepsilon(B) = \big\{\lambda \in \mathbb{C} : 3|\lambda - 1| \leq \varepsilon\big\}.$$

The following example shows that for matrices, the eigenvalues coincide does not imply that the trace pseudospectrum also coincides.

Example: Let $\alpha, \delta \in \mathbb{C}$ with $\alpha \neq \delta \neq 0$ and let

$$T = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \delta \end{pmatrix} \text{ and } B = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{pmatrix}.$$

Then,

$$\sigma(T) = \sigma(B) = \{\alpha, \delta\}.$$

But, a simple computation shows that

$$\begin{aligned} \mathrm{Tr}_\varepsilon(T) &= \big\{\lambda \in \mathbb{C} : |\lambda - 3^{-1}(2\alpha - \delta)| \leq 3^{-1}\varepsilon\big\} \\ &\neq \big\{\lambda \in \mathbb{C} : |\lambda - 3^{-1}(\alpha + 2\delta)| \leq 3^{-1}\varepsilon\big\} = \mathrm{Tr}_\varepsilon(B). \end{aligned}$$

Theorem 2.14. *Let $T, B \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$. Then,*

$$\mathrm{Tr}_\varepsilon(TB) = \mathrm{Tr}_\varepsilon(BT).$$

Proof. Let $\lambda \in \mathrm{Tr}_\varepsilon(TB)$, then

$$\begin{aligned} \varepsilon \geq |\mathrm{Tr}(\lambda I - TB)| &= |\mathrm{Tr}(\lambda I) + \mathrm{Tr}(-TB)| \\ &= |\mathrm{Tr}(\lambda I) + \mathrm{Tr}(-BT)| \\ &= |\mathrm{Tr}(\lambda I - BT)|. \end{aligned}$$

Hence, $\lambda \in \mathrm{Tr}_\varepsilon(BT)$. Thus,

$$\mathrm{Tr}_\varepsilon(TB) \subseteq \mathrm{Tr}_\varepsilon(BT).$$

Using a similar reasoning to the first inclusion, we deduce that

$$\mathrm{Tr}_\varepsilon(BT) \subseteq \mathrm{Tr}_\varepsilon(TB).$$

□

Theorem 2.15. *Let $T \in \mathcal{M}_n(\mathbb{C})$, $\varepsilon > 0$ and $\lambda_1, \dots \lambda_k$ be the distinct eigenvalues of T with algebraic multiplicity $m_1, \dots, m_k$ respectively. Then,*

$$\mathrm{Tr}_\varepsilon(T) = \Big\{\lambda \in \mathbb{C} : \textstyle\sum_{i=1}^{k} m_i|\lambda - \lambda_i| \leq \varepsilon\Big\}.$$

Proof. From the Schur decomposition there exist an upper triangular matrix $U \in \mathcal{M}_n(\mathbb{C})$ with diagonal entries as eigenvalues of T and a unitary matrix Q such that $T = QUQ^{-1}$. Then,

$$\begin{aligned} \mathrm{Tr}_\varepsilon(T) = \mathrm{Tr}_\varepsilon(U) &= \big\{\lambda \in \mathbb{C} : |\mathrm{Tr}(\lambda I - U)| \leq \varepsilon\big\} \\ &= \Big\{\lambda \in \mathbb{C} : \sum_{i=1}^{k} m_i|\lambda - \lambda_i| \leq \varepsilon\Big\}. \end{aligned}$$

□

Theorem 2.16. *Let $T \in \mathcal{M}_n(\mathbb{C})$ and $N \in \mathcal{M}_n(\mathbb{C})$ is a nilpotent matrix and $\varepsilon > 0$. Then,*

$$\mathrm{Tr}_\varepsilon(T+N) = \mathrm{Tr}_\varepsilon(T).$$

Proof. " $\subseteq$ " Let $\lambda \in \mathrm{Tr}_\varepsilon(T+N)$, then $|\mathrm{Tr}(\lambda - T - N)| \leq \varepsilon$. Since

$$|\mathrm{Tr}(\lambda - T) - \mathrm{Tr}(N)| \leq \varepsilon.$$

Using the fact that the matrix trace vanishes on nilpotent matrices, therefore

$$\lambda \in \mathrm{Tr}_\varepsilon(T).$$

Hence,

$$\mathrm{Tr}_\varepsilon(T+N) \subseteq \mathrm{Tr}_\varepsilon(T).$$

" $\supseteq$ " Let $\lambda \in \mathrm{Tr}_\varepsilon(T)$, then $|\mathrm{Tr}(\lambda - T)| \leq \varepsilon$. Now, we can write for any $\lambda \in \mathbb{C}$

$$|\mathrm{Tr}(\lambda - T)| = |\mathrm{Tr}(\lambda - T - N + N)| = |\mathrm{Tr}(\lambda - T - N) + \mathrm{Tr}(N)|.$$

Because, $\mathrm{Tr}(N) = 0$, it follows that $|\mathrm{Tr}(\lambda - T - N)| \leq \varepsilon$. Consequently,

$$\mathrm{Tr}_\varepsilon(T) \subseteq \mathrm{Tr}_\varepsilon(T+N).$$

□

3. Trace pseudospectral mapping Theorem.

The following is a Trace pseudospectral mapping theorem for complex analytic functions. Let $T \in \mathcal{M}_n(\mathbb{C})$ and let f be an analytic function defined on $\mathcal{O}$, an open set containing $\mathrm{Tr}_0(T)$. To state our results, we introduce the functions for each $\varepsilon > 0$,

$$\varphi(\varepsilon) = \sup_{\lambda \in \mathrm{Tr}_\varepsilon(T)} |\mathrm{Tr}\big(f(\lambda)I - f(T)\big)|.$$

and suppose there exists $\varepsilon_0 > 0$ such that $\mathrm{Tr}_{\varepsilon_0}(f(T)) \subseteq f(\mathcal{O})$. Then, for $0 < \varepsilon < \varepsilon_0$ we define

$$\phi(\varepsilon) = \sup_{\mu \in f^{-1}(\mathrm{Tr}_\varepsilon(T)) \cap \mathcal{O}} |\mathrm{Tr}(\mu - T)|.$$

It is sharp in the sense that the functions φ and ϕ measure the sizes of the trace pseudospectra are optimal. Actually, the theorem is an easy consequence of the definitions of these functions.

Lemma 3.1. *Let $T \in \mathcal{M}_n(\mathbb{C})$ and $\varepsilon > 0$, then $\varphi(\varepsilon)$ and $\phi(\varepsilon)$ are well defined,*

$$\lim_{\varepsilon \to 0} \varphi(\varepsilon) = 0 \text{ and } \lim_{\varepsilon \to 0} \phi(\varepsilon) = 0.$$

Proof. In the order to prove that $\varphi(\varepsilon)$ is well defined, we define $h : \mathbb{C} \to \mathbb{R}_+$

$$h(\lambda) = |\mathrm{Tr}\big(f(\lambda)I - f(T)\big)|$$

Since $h(\lambda)$ is continuous and $\mathrm{Tr}_\varepsilon(T)$ is a compact subset of $\mathbb{C}$, then it is clear that

$$\varphi(\varepsilon) = \sup\big\{h(\lambda) : \lambda \in \mathrm{Tr}_\varepsilon(T)\big\}.$$

Thus, $\varphi(\varepsilon)$ is well defined. Now, let assume that there exists $\varepsilon_0 > 0$ such that

$$\mathrm{Tr}_{\varepsilon_0}(f(T)) \subseteq f(\mathcal{O}).$$

We show that for $0 < \varepsilon < \varepsilon_0$, $\phi(\varepsilon)$ is well defined. Define $g : \mathbb{C} \to \mathbb{R}_+$,

$$g(\mu) = |\mathrm{Tr}(\mu - T)|.$$

Since g is continuous for all $\mu \in \mathbb{C}$, then $\phi(\varepsilon)$ is well defined. It is also clear that $\varphi(\varepsilon)$ and $\phi(\varepsilon)$ are a monotonically non-decreasing function and $\varphi(\varepsilon)$ and $\phi(\varepsilon)$ goes to zero as ε goes to zero. □

Theorem 3.2. *Let $T \in \mathcal{M}_n(\mathbb{C})$ and let f be an analytic function defined on $\mathcal{O}$, an open set containing $\mathrm{Tr}_0(T)$. Then, for each $\varepsilon > 0$, we have*

$$f(\mathrm{Tr}_\varepsilon(T)) \subseteq \mathrm{Tr}_{\varphi(\varepsilon)}(f(T)),$$

where, $\varphi(\varepsilon)$ defined above.

Proof. Let $\lambda \in \mathrm{Tr}_\varepsilon(T)$. Then, using Lemma 3.1 we obtain that $\varphi(\varepsilon)$ is well defined and $\lim\limits_{\varepsilon \to 0} \varphi(\varepsilon) = 0$. Therefore, $h(\lambda) \leq \varphi(\varepsilon)$. Hence

$$|\mathrm{Tr}\big(f(\lambda)I - f(T)\big)| := h(\lambda) \leq \varphi(\varepsilon).$$

Thus, $f(\lambda) \in \mathrm{Tr}_{\varphi(\varepsilon)}(f(T))$. This means that

$$f(\mathrm{Tr}_\varepsilon(T)) \subseteq \mathrm{Tr}_{\varphi(\varepsilon)}(f(T)).$$

□

Theorem 3.3. *Let $T \in \mathcal{M}_n(\mathbb{C})$ and let f be an analytic function defined on $\mathcal{O}$, an open set containing $\mathrm{Tr}_0(T)$. Then, for each $\varepsilon > 0$, we have*

$$\mathrm{Tr}_\varepsilon(f(T)) \subseteq f(\mathrm{Tr}_{\phi(\varepsilon)}(T)).$$

where, $\phi(\varepsilon)$ defined above.

Proof. Let $\lambda \in \mathrm{Tr}_\varepsilon(f(T))$. Then, using Lemma 3.1 we obtain the existence of $\varepsilon_0 > 0$ such that

$$\mathrm{Tr}_\varepsilon(f(T)) \subseteq \mathrm{Tr}_{\varepsilon_0}(f(T)) \subseteq f(\mathcal{O}).$$

Consider $\mu \in \mathcal{O}$ such that $\lambda = f(\mu)$. Then $\mu \in f^{-1}(\mathrm{Tr}_\varepsilon(T))$, hence

$$g(\mu) \leq \phi(\varepsilon).$$

Therefore,

$$|\mathrm{Tr}\big(\mu I - T\big)| := g(\mu) \leq \phi(\varepsilon)$$

Thus, $\mu \in \mathrm{Tr}_{\phi(\varepsilon)}(T)$. Then, $\lambda = f(\mu) \in f(\mathrm{Tr}_{\phi(\varepsilon)}(T))$. This means that

$$\mathrm{Tr}_\varepsilon(f(T)) \subseteq f(\mathrm{Tr}_{\phi(\varepsilon)}(T)).$$

□

Corollary 3.4. *Combining the two inclusions in Theorems 3.2 and 3.3, we get*

$$f(\mathrm{Tr}_\varepsilon(T)) \subseteq \mathrm{Tr}_{\varphi(\varepsilon)}(f(T)) \subseteq f(\mathrm{Tr}_{\phi(\varphi(\varepsilon))}(T) \text{ and}$$
$$\mathrm{Tr}_\varepsilon(f(T)) \subseteq f(\mathrm{Tr}_{\phi(\varepsilon)}(T)) \subseteq \mathrm{Tr}_{\varphi(\phi(\varepsilon))}(f(T)).$$

Here are some remarks.

Remark 3.5. (*i*) *If $\varphi(0) = 0 = \phi(0)$, and then the theorem reduces to*

$$f(\mathrm{Tr}_0(T)) \subseteq \mathrm{Tr}_0(f(T)) \subseteq f(\mathrm{Tr}_0(T)$$

and

$$\mathrm{Tr}_0(f(T)) \subseteq f(\mathrm{Tr}_0(T)) \subseteq \mathrm{Tr}_0(f(T)).$$

(*ii*) *We observe from the definitions of φ and ϕ, that the set inclusions are sharp in the sense that the functions cannot be replaced by smaller functions.*

(iii) If $f(z)=\alpha z+\beta$ where α,β are complex numbers, then

$$\varphi(\varepsilon)=\sup_{\lambda\in \mathrm{Tr}_\varepsilon(T)}\left|\mathrm{Tr}\big((\alpha\lambda+\beta)I-\alpha T-\beta\big)\right|=\sup_{\lambda\in \mathrm{Tr}_\varepsilon(T)}\left|\mathrm{Tr}\big(\alpha(\lambda-T)\big)\right|=|\alpha|\varepsilon$$

and

$$\phi(\varepsilon)=\sup_{\mu\,\in\, f^{-1}(\mathrm{Tr}_\varepsilon(T))\cap\mathcal{O}}|\mathrm{Tr}(\mu-T)|=\sup_{\mu\,\in\,\mathrm{Tr}_{\frac{\varepsilon}{|\alpha|}}(T)}|\mathrm{Tr}(\mu-T)|.$$

Then, $\varphi(\phi(\varepsilon))=\varepsilon$ and $\phi(\varphi(\varepsilon))=\varepsilon$. Hence,

$$f(\mathrm{Tr}_\varepsilon(T))=\mathrm{Tr}_\varepsilon(f(T)).$$

In general, the spectral mapping Theorem is not true for trace pseudospectrum.

Example: Let $\alpha,\beta\in\mathbb{C}$ with $\alpha\neq\beta\neq0$ and let $T=\begin{pmatrix}\alpha & 1\\ 0 & \beta\end{pmatrix}$ and $f(\lambda)=\lambda^2$. Then, $f(T)=\begin{pmatrix}\alpha^2 & \alpha+\beta\\ 0 & \beta^2\end{pmatrix}$. A direct computation shows that

$$\mathrm{Tr}_\varepsilon(f(T))=\big\{\lambda\in\mathbb{C}:|2\lambda-\alpha^2-\beta^2|\le\varepsilon\big\},$$

$$f(\mathrm{Tr}_\varepsilon(T))=\big\{\lambda^2\in\mathbb{C}:|2\lambda-\alpha^2-\beta^2|\le\varepsilon\big\}.$$

We can see for all $\varepsilon>0$ that $\mathrm{Tr}_\varepsilon(f(T))\neq f(\mathrm{Tr}_\varepsilon(T))$.

We close this paper with the following example.

Example: Let us consider the following 3×3 circulant matrix (special type of Toeplitz matrix)

$$T=\begin{pmatrix} a & 2a & 3a\\ 3a & a & 2a\\ 2a & 3a & a\end{pmatrix}$$

Let $f(\lambda)=\lambda^2$. Then, $f(T)=T^2$ is also a circulant matrix and

$$T^2=\begin{pmatrix} 13a^2 & 13a^2 & 10a^2\\ 10a^2 & 13a^2 & 13a^2\\ 13a^2 & 10a^2 & 13a^2\end{pmatrix}.$$

Therefore, for $\varepsilon>0$

$$\mathrm{Tr}_\varepsilon(T)=\big\{\lambda\in\mathbb{C}:3|\lambda-a|\le\varepsilon\big\},$$

$$\varphi(\varepsilon)=\sup_{\lambda\in\mathrm{Tr}_\varepsilon(T)}\left|\mathrm{Tr}\big(\lambda^2I-T^2\big)\right|=\sup_{\lambda\in D(a,\frac{\varepsilon}{3})}3|\lambda^2-13a^2|=3\left|(\frac{\varepsilon}{3}+a)^2-13a^2\right|$$

and

$$\phi(\varepsilon)=\sup_{\mu^2\in D(a,\frac{\varepsilon}{3})}3|\mu-a|=|\sqrt{3\varepsilon+9a}-3a|.$$

Consequently,

$$\big(\mathrm{Tr}_\varepsilon(T)\big)^2\subseteq\mathrm{Tr}_{3|(\frac{\varepsilon}{3}+a)^2-13a^2|}(T^2)\text{ and}$$

$$\mathrm{Tr}_\varepsilon(T^2)\subseteq\big(\mathrm{Tr}_{|\sqrt{3\varepsilon+9a}-3a|}(T)\big)^2.$$

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References

1. A. Ammar, A. Jeribi and K. Mahfoudhi, *A characterization of the essential approximation pseudospectrum on a Banach space*, *Filomat* **31**, (11), 3599-3610 (2017).

2. A. Ammar, A. Jeribi and K. Mahfoudhi, *A characterization of the condition pseudospectrum on Banach space*, Funct. Anal. Approx. Comput. **10** (2), 13-21 (2018)

3. A. Ammar, A. Jeribi and K. Mahfoudhi, *The condition pseudospectrum subset and related results.* J. Pseudo-Differ. Oper. Appl.(2018) $https://doi.org/10.1007/s11868-018-0265-9$

4. A. Ammar, A. Jeribi, K. Mahfoudhi, *The essential approximate pseudospectrum and related results,* Filomat, **32**, 6, (2018), 2139-2151.

5. A. Ammar, A. Jeribi, K. Mahfoudhi, *Global bifurcation from the real leading eigenvalue of the transport operator, J. Comput. Theor. Transp.*, **46**(4), 229-241.

6. A. Ammar, A. Jeribi, K. Mahfoudhi, *On the Condition Pseudospectrum of a Operator Pencil*, Asian-European Journal of Mathematics (2020)

7. A. Ammar, A. Jeribi, K. Mahfoudhi, *A characterization of Browder's essential approximation and his essential defect pseudospectrum on a Banach space,* Extracta Math., **34**, (1), (2019), 29-40.

8. A. Ammar, A. Jeribi, K. Mahfoudhi, *Generalized trace pseudo-spectrum of matrix pencils*, Cubo Journal of Mathematics, **21**, (02), (2019) 65–76.

9. A. Jeribi. *Spectral theory and applications of linear operators and block operator matrices,* Springer-Verlag, New-York, (2015).

10. R.A. Horn, C.R. Johnson, *Topics in Matrix Analysis,* Cambridge University Press, (1991).

11. K. Mahfoudhi, *Generalized determinant pseudo-spectrum of matrix pencils*, The Journal of Analysis **29** (1), 163-176 (2020).

12. Krishna Kumar. G, *Determinant spectrum: A generalization of eigenvalues*, Funct. Anal. Approx. Comput. **10** (2) (2018), 1–12.

13. S. H. Kulkarni and D. Sukumar, *The condition spectrum,* Acta Sci. Math. (Szeged) **74**, (2008) 625-641 .

14. T. J. Ransford, *Generalised spectra and analytic multivalued functions,* J. London Math. Soc. **29**,(1984) 306- 322 .

15. L. N. Trefethen and M. Embree, *Spectra and pseudospectra: The behavior of nonnormal matrices and operators.* Prin. Univ. Press, Princeton and Oxford, (2005).

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