

A New Construction For Harmonic Evolute Surfaces Of Quasi Tangent Surfaces With Quasi Frame

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ABSTRACT: In this paper, we study a harmonic evolute surface of quasi tangent surface associated with quasi frame. We construct quasi tangent surface with first and second fundamental forms. Moreover, we determine harmonic evolute surface of quasi tangent surface by using these fundamental forms. Finally, we obtain some new results about these new surfaces.

Key Words: Tangent spherical image, inextensible flows, curvatures.

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1. Introduction

Surfaces are one of the most interesting subject of differential geometry since the application to physics and engineering especially, geophysics and petroluem engineering is countless [9]-[11], [13]. Ruled surfaces which is generated by the motion of a straight line along a curve are studied widely by many researchers after it was found by Gaspard Mongea. Kimm and Yoon in 2004, Sarioglugil and Tutar in 2007, Aydemir and Orbay in 2009, Senturk and Yuce in 2015, Kaymanlı et al, in 2019, Korpınar and Kaymanlı in 2020 studied ruled surfaces in both Euclidean and Minkowski spaces to name a few [1], [5]-[8], [12], [19], [20].

The harmonic evolute of various surfaces has been studied in depth by many researchers throughout the years. Šipuš and Vladimir researched The harmonic evolute of a surface in Minkowski 3-space in [21]. Following Šipuš and Vladimir’s findings in 2014, Protrka in examined harmonic evolutes of timelike ruled surfaces in Minkowski space in 2015 and the harmonic evolute of a helicoidal surfaces in Minkowski 3-space in 2017 in [17], [18]. Most recently in 2019, Lopez et al, in [15] analyzed harmonic evolutes of B-scrolls with constant mean curvature in Lorentz-Minkowski space. Following these existing studies, we have looked into harmonic evolute surfaces of the ruled surfaces generated by quasi tangent vector by using quasi frame.

This paper is formed of three sections. The first section is about imperative knowledge on the differential geometric construction of the frames in the 3 dimensional Euclidean space that is, Serret Frenet frame, quasi frame and the relation between these frames are examined. By calculating the first and the second fundamental forms, some geometric properties such as Gaussian and the mean curvatures of the surfaces are given. In the second section, we define the harmonic evolute surface of quasi tangent surface associated with quasi frame. The necessary and sufficient conditions of how the qausi tangent surface and its harmonic evolute surface can or can not be a Bonnet surface are given. In addition to this, the quasi tangent surface of a helix and a harmonic evolute of the quasi tangent surface are illustrated.

2. Preliminaries

By way of design and style, this model is kind of a moving frame with regards to a particle. In the quick stages of regular differential geometry, the Frenet-Serret frame was applied to create a curve in location. After that, Frenet-Serret frame is established by way of subsequent equations for a presented framework [3],

$$\begin{bmatrix} \nabla_{\mathbf{t}} \mathbf{t} \\ \nabla_{\mathbf{t}} \mathbf{n} \\ \nabla_{\mathbf{t}} \mathbf{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}, \quad (1.1)$$

where $\kappa = \|\mathbf{t}'\|$ and τ are the curvature and torsion of γ , respectively.

After Bishop in 1975 showed that there is more than one way to frame a curve in [2], Yilmaz and Turgut introduced second type of Bishop frame in [22]. Besides them, Dede et. al. in defined quasi frame in [4]. The quasi frame of a regular curve γ is given by

$$\mathbf{t}_q = \mathbf{t}, \mathbf{n}_q = \frac{\mathbf{t} \wedge \mathbf{k}}{\|\mathbf{t} \wedge \mathbf{k}\|}, \mathbf{b}_q = \mathbf{t}_q \wedge \mathbf{n}_q, \quad (1.2)$$

where \mathbf{k} is the projection vector.

For simplicity, we have chosen the projection vector $\mathbf{k} = (0, 0, 1)$ in this paper. However, the q-frame is singular in all cases where \mathbf{t} and \mathbf{k} are parallel. Thus, in those cases where \mathbf{t} and \mathbf{k} are parallel, the projection vector \mathbf{k} can be chosen as $\mathbf{k} = (0, 1, 0)$ or $\mathbf{k} = (1, 0, 0)$.

If the angle between the quasi normal vector \mathbf{n}_q and the normal vector \mathbf{n} is chosen as ψ , then the following relation is obtained between the quasi and FS frame.

$$\begin{aligned} \mathbf{t}_q &= \mathbf{t}, \\ \mathbf{n}_q &= \cos \psi \mathbf{n} + \sin \psi \mathbf{b}, \\ \mathbf{b}_q &= -\sin \psi \mathbf{n} + \cos \psi \mathbf{b}. \end{aligned} \quad (1.3)$$

Therefore, by using the equations (1 – 3) the variation of parallel adapted quasi frame is obtained by

$$\begin{aligned} \nabla_{\mathbf{t}_q} \mathbf{t}_q &= \varkappa_1 \mathbf{n}_q + \varkappa_2 \mathbf{b}_q, \\ \nabla_{\mathbf{t}_q} \mathbf{n}_q &= -\varkappa_1 \mathbf{t}_q + \varkappa_3 \mathbf{b}_q, \\ \nabla_{\mathbf{t}_q} \mathbf{b}_q &= -\varkappa_2 \mathbf{t}_q - \varkappa_3 \mathbf{n}_q, \end{aligned}$$

where

$$\varkappa_1 = \kappa \cos \psi, \quad \varkappa_2 = -\kappa \sin \psi, \quad \varkappa_3 = \psi' + \tau,$$

and the vector products of the quasi vectors are given by

$$\mathbf{t}_q \times \mathbf{n}_q = \mathbf{b}_q, \quad \mathbf{n}_q \times \mathbf{b}_q = \mathbf{t}_q, \quad \mathbf{b}_q \times \mathbf{t}_q = \mathbf{n}_q.$$

Let n be the standard unit normal vector field on a surface ϕ defined by

$$n = \frac{\phi_s \wedge \phi_t}{\|\phi_s \wedge \phi_t\|},$$

where $\phi_s = \partial\phi/\partial s$, $\phi_t = \partial\phi/\partial t$, respectively. Then, the first fundamental form **I** and the second fundamental form **II** of a surface ϕ are defined by

$$\begin{aligned} \mathbf{I} &= E ds^2 + 2F ds dt + G dt^2, \\ \mathbf{II} &= e ds^2 + 2f ds dt + g dt^2, \end{aligned}$$

where

$$\begin{aligned} E &= \langle \phi_s, \phi_s \rangle, \quad F = \langle \phi_s, \phi_t \rangle, \quad G = \langle \phi_t, \phi_t \rangle, \\ e &= \langle \phi_{ss}, n \rangle, \quad f = \langle \phi_{st}, n \rangle, \quad g = \langle \phi_{tt}, n \rangle. \end{aligned}$$

On the other hand, the Gaussian curvature K and the mean curvature H are

$$K = \frac{eg - f^2}{EG - F^2}, \quad H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}$$

and the principal curvatures k_1 and k_2 are

$$k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K},$$

respectively, [14], [16].

Theorem 2.1. *The surface is minimal if and only if it has vanishing mean curvature [3], [14].*

3. Harmonic Evolute Surfaces of Tangent Surface

In this section, we aim to explore harmonic evolute surface of quasi tangent surface associated with quasi frame when the mean curvature does not vanish.

Firstly, we construct quasi tangent surface of a quasi curve as

$$\phi^{\mathbf{tq}}(s, t) = \alpha + t\mathbf{t}_q.$$

Definition 3.1. *If $E = G$, $F = 0$, $f = c \neq 0$ ($c = \text{const.}$) are satisfied then the surface is called A-net on a surface [3].*

Theorem 3.2. *A surface to be a Bonnet surface if and only if surface has an A-net [3].*

Theorem 3.3. *Let $\phi^{\mathbf{tq}}$ be a quasi tangent surface of a quasi curve in space. $\phi^{\mathbf{tq}}$ is minimal iff*

$$(\varkappa_2(\varkappa_1 + t(\varkappa_1)_s - \varkappa_2 t \varkappa_3) - \varkappa_1(\varkappa_2 + (\varkappa_2)_s t + t \varkappa_1 \varkappa_3)) = 0.$$

Proof. From the definition of quasi tangent surface, we have

$$\begin{aligned} \phi_s^{\mathbf{tq}} &= \mathbf{t}_q + t \varkappa_1 \mathbf{n}_q + \varkappa_2 t \mathbf{b}_q, \\ \phi_t^{\mathbf{tq}} &= \mathbf{t}_q. \end{aligned}$$

By using this field, the coefficients of the first fundamental form are given

$$\begin{aligned} E &= 1 + t^2 \varkappa_1^2 + \varkappa_2^2 t^2, \\ F &= G = 1. \end{aligned}$$

The second partial derivatives of $\phi^{\mathbf{tq}}$ are expressed as follows:

$$\begin{aligned} \phi_{ss}^{\mathbf{tq}} &= -(t \varkappa_1^2 + \varkappa_2^2 t) \mathbf{t}_q + (\varkappa_1 + t \varkappa_1' - \varkappa_2 t \varkappa_3) \mathbf{n}_q + (\varkappa_2 + \varkappa_2' t + t \varkappa_1 \varkappa_3) \mathbf{b}_q, \\ \phi_{ts}^{\mathbf{tq}} &= \varkappa_1 \mathbf{n}_q + \varkappa_2 \mathbf{b}_q, \\ \phi_{tt}^{\mathbf{tq}} &= 0. \end{aligned}$$

So, an algebraic calculus shows that

$$\phi_s^{\mathbf{tq}} \times \phi_t^{\mathbf{tq}} = -t \varkappa_1 \mathbf{b}_q + \varkappa_2 t \mathbf{n}_q.$$

Moreover, by the definition of the unit normal vector, we have

$$n^{\mathbf{tq}} = \frac{\varkappa_2}{\sqrt{\varkappa_1^2 + \varkappa_2^2}} \mathbf{n}_q - \frac{\varkappa_1}{\sqrt{\varkappa_1^2 + \varkappa_2^2}} \mathbf{b}_q.$$

Therefore, the coefficients of the second fundamental form are given

$$\begin{aligned} e &= \frac{\varkappa_2(\varkappa_1 + t(\varkappa_1)_s - \varkappa_2 t \varkappa_3)}{\sqrt{\varkappa_1^2 + \varkappa_2^2}} - \frac{\varkappa_1(\varkappa_2 + (\varkappa_2)_s t + t \varkappa_1 \varkappa_3)}{\sqrt{\varkappa_1^2 + \varkappa_2^2}}, \\ f &= 0, \\ g &= 0. \end{aligned}$$

The mean curvature of $\phi^{t\mathbf{a}}$ is presented

$$H^{t\mathbf{a}} = \frac{1}{2t^2(\varkappa_1^2 + \varkappa_2^2)^{\frac{3}{2}}}(\varkappa_2(\varkappa_1 + t(\varkappa_1)_s - \varkappa_2 t \varkappa_3) - \varkappa_1(\varkappa_2 + (\varkappa_2)_s t + t \varkappa_1 \varkappa_3)).$$

This completes the proof. \square

Theorem 3.4. *Let $\phi^{t\mathbf{a}}$ be a quasi tangent surface of a quasi curve in space. $\phi^{t\mathbf{a}}$ is not a Bonnet surface*

A harmonic evolute surface of the quasi tangent surface is given by

$$\phi^h(s, t) = \phi^{t\mathbf{a}}(s, t) + \frac{1}{H^{t\mathbf{a}}} n^{t\mathbf{a}}.$$

Theorem 3.5. *A harmonic evolute surface of $\phi^{t\mathbf{a}}$ is given by*

$$\phi^h(s, t) = \alpha + t t_{\mathbf{a}} + \frac{\varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}} \mathbf{n}_{\mathbf{a}} - \frac{\varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}} \mathbf{b}_{\mathbf{a}}.$$

Theorem 3.6. *A harmonic evolute surface of $\phi^{t\mathbf{a}}$ is a Bonnet surface if and only if*

$$\begin{aligned} & \left(1 + \frac{\varkappa_2 \varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}} - \frac{\varkappa_1 \varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)^2 + \left(\left(\frac{\varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_s + \frac{\varkappa_3 \varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}} + t \varkappa_1\right)^2 \\ & + \left(\frac{\varkappa_3 \varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}} - \left(\frac{\varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_s + t \varkappa_2\right)^2 = 1 + \left(\left(\frac{\varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_t\right)^2 + \left(\left(\frac{\varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_t\right)^2, \\ & \left(\left(\frac{\varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_s + \frac{\varkappa_3 \varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}} + t \varkappa_1\right) \left(\frac{\varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_t - \left(\frac{\varkappa_3 \varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right. \\ & \left. - \left(\frac{\varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_s + t \varkappa_2\right) \left(\frac{\varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_t = 0, \end{aligned}$$

and

$$\begin{aligned} & \left(\varkappa_2 \left(\frac{\varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_t - \varkappa_1 \left(\frac{\varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_t\right) \frac{1}{\pi} \left(\left(\frac{\varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_t\right) \left(\left(\frac{\varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_s\right) \\ & + \frac{\varkappa_3 \varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}} + t \varkappa_1 + \left(\frac{\varkappa_3 \varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}} - \left(\frac{\varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_s + t \varkappa_2\right) \left(\frac{\varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_t \\ & + \left(\varkappa_1 + \left(\frac{\varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_t s + \varkappa_3 \left(\frac{\varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_t\right) \frac{1}{\pi} \left(\left(1 + \frac{\varkappa_2 \varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right.\right. \\ & \left. - \frac{\varkappa_1 \varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right) \left(\frac{\varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_t + \left(\frac{\varkappa_3 \varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}} - \left(\frac{\varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_s + t \varkappa_2\right) \\ & + \left(\varkappa_3 \left(\frac{\varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_t - \left(\frac{\varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_t s + \varkappa_2\right) \frac{1}{\pi} \left(\left(1 + \frac{\varkappa_2 \varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right.\right. \\ & \left. - \frac{\varkappa_1 \varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right) \left(\frac{\varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_t - \left(\left(\frac{\varkappa_2}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}}\right)_s + \frac{\varkappa_3 \varkappa_1}{H^{t\mathbf{a}} \sqrt{\varkappa_1^2 + \varkappa_2^2}} + t \varkappa_1\right) = \varrho, \end{aligned}$$

where ϱ is constant.

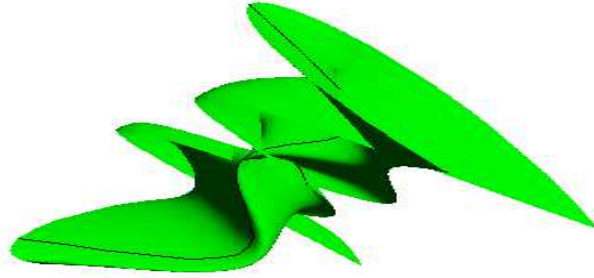
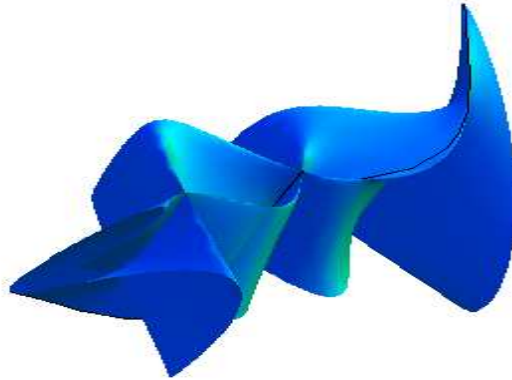


Figure 1: Quasi tangent surface of helix

Figure 2: A harmonic evolute surface of ϕ^{tq}

4. Conclusion

In conclusion, a harmonic evolute surface of quasi tangent surface associated with quasi frame was studied throughout this paper. Initially, quasi tangent surface was constructed. We then established harmonic evolute surface of quasi tangent surface by using quasi frame and mean curvature. Finally, our studies have enabled us to gain new results about these surfaces.

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