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Computation of Eigenfunctions of Nonlinear Boundary-Value -Transmission Problems by Developing Some Approximate Techniques

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abstract: In this study, we investigate a boundary value problem for nonlinear Sturm-Liouville equations with additional transmission conditions at one interior singular point. Known numerical methods are intended for solving initial and boundary value problems without transmission conditions. By modifying the Adomian decomposition method and the differential transform method, we present a new numerical algorithm to compute the eigenvalues and eigenfunctions of the considered boundary-value-transmission problem. Some graphic illustrations of the approximate eigenfunctions are also presented.

Key Words: Nonlinear Sturm-Liouville equations, transmission conditions, approximate eigenfunction, numerical methods.

Contents

1. Introduction

Due to the complexity of the search for analytical solutions of many boundary-value problems for nonlinear differential equations, growing interest has been given to the development of efficient approximate methods for solving of such type problems. Various methods for the approximate solutions of classical boundary value problems have been extended to solve nonlinear problems appearing in physics, engineering and other branches of natural science. These include the differential transformation method(DTM), the Adomian decomposition method (ADM), the finite difference method (FDM), the homotopy perturbation method (HPM), the predictor correctors method(PCM), the shooting method, the weighted residual method and etc. The concept of DTM was first proposed by Zhou [20] in 1986 for solving linear and non-linear initial value problems arising in the analysis of an electric circuit. In 1980 Adomian [1] developed a new semi-analytic method for solving stochastic systems. The main advantage of those methods (DTM and ADM) is that they can be applied to a rather wide class of initial and boundary value problems requiring no linearization, perturbation or discretization techniques which can require massive numerical computation. Moreover, unlike to many numerical methods the DTM and the ADM do not require the calculation of auxiliary parameters and the determination of auxiliary functions or the determination of suitable initial or boundary conditions. Applying the separation of variables method to many problems of mathematical physics usually leads to a boundary value problems of Sturm-Liouville type and not all of them have analytical solutions.

Some special problems of Sturm-Liouville type are solved by different approximate methods. Chen an Ho [6] applied the DTM to solve the linear Sturm-Liouville problem (SLP)

$$(p(x)y'(x))' + (q(x) + \lambda\omega(x))y = 0 \quad x \in (0,1)$$

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$$y(0) + \alpha y'(0) = y(1) + \alpha y'(1) = 0$$

and the obtained solutions were compared with those calculated by other approximate methods. Somali and Gökmen [7] used ADM to calculate eigenvalues and corresponding eigenfunctions of the boundary-value problem for nonlinear Sturm-Liouville equation, given by

$$y''(x) + y^p(x) = \lambda y(x) \quad x \in (0,1)$$

$$y(0) = y(1) = 0.$$

Attili [3] used ADM to calculate eigenvalues of Sturm-Liouville two point boundary-value problem. Hassan [8] provided a comparison for DTM and ADM for linear and nonlinear initial problems. Sheikholeslami and Ganji [14] used DTM to nanofluid flow and heat transfer between parallel plates considering Brownian motion.

Nonlinear Sturm-Liouville problems usually arise from mathematical modeling of many physical systems, such as electrodynamics of complex medium, aerodynamics, polimer rheology, fluid dynamics and etc. In recent years the ADM has been applied to a wide class of initial and boundary value problems. Based on the ADM a new numerical treatment is proposed in [15] to solve linear and non-linear BVP's.

Wazwaz in [16] used ADM to obtain a semi-analytic solutions of nonlinear diffusion equations. Bulut and Evans [5] studied exact and approximate solutions of the Riccati equation. In [2], the ADM has been used to solve homogeneous differential equations with dual variable and dual coefficients. In [17], a new improved ADM is proposed for computing the eigenvalues of the fractional Sturm-Liouville problems. Hassan and Ertürk in [9] used DTM for the approximate solution of the linear and nonlinear higher-order boundary value problems.

In the present article we will develop the DTM and the ADM to compute the spectral characteristics of a new type non-linear Sturm-Liouville problems, the main feature of which is the nature of the imposed boundary conditions. Namely, we will investigate a Sturm-Liouville problem with quadratic nonlinearity together with additional transmission conditions at an interior point of interaction. Such type of problems are called boundary-value-transmission problems (BVTP's) and arise in heat and mass transfer problems, in vibrating string problems when the string loaded with additional point masses, in diffraction problems, in thermal conduction problems for a thin laminated plate (i.e. a plate composed by materials with different characteristics piled in the thickness) and etc. (see, [10], [11], [12], [19] and references, cited therein). Obviously, such type of BVTP's are much more complicated to solve than classical Sturm-Liouville problems. The major difficulty lies in the existence of solutions, since it is not clear how to apply the known numerical and analytical methods to the BVTP's.

Note that, the basic spectral properties of many-interval Sturm-Liouville problems involving transmission conditions have been established by the authors of this study (see, [4], [11], [18], [19]).

The main objective of this study is to adapt the DTM and ADM so that these methods can also be used to solve nonlinear Sturm-Liouville problems associated with additional transmission conditions.

The obtained results demonstrate that the well-known DTM and ADM methods can also be adapted to solve nonlinear differential equations subject to the boundary conditions specified not only at the endpoints of the considered interval, but also at the internal singularity point (such conditions are called transmission conditions).

2. Methodology

Let us briefly describe the basic definitions and operations of the ADM and DTM applied to nonlinear differential equations. Consider a nonlinear differential equation in the operator form

$$Ly + Ry + Ny = g, \tag{2.1}$$

where $L = \frac{d^n}{dx^n}$ is the highest order derivative operator, R is the linear differential operator of less order than L, N is the nonlinear operator and g is the source term. It is obvious that L is the easily invertable operator and L^{-1} is the n-fold integral operator. Applying the inverse operator L^{-1} to the equation (2.1) yields

$$L^{-1}Ly = -L^{-1}Ry - L^{-1}Ny + L^{-1}g.$$

Particularly, if $n=2$ then L^{-1} is the two-fold integral operator, given by $L^{-1}(.)=\int_0^x\int_0^x(.)dxdx$ and therefore $L^{-1}Ly=y-y(0)-xy'(0)$. Thus we obtain that the solution of the equation (2.1) can be written in the form

$$y=y(0)+xy'(0)+L^{-1}g-L^{-1}Ry-L^{-1}Ny. \tag{2.2}$$

Assume that the nonlinear term Ny is an analytic function $f(y)$ and assume that the solution $y=y(x)$ can be decomposed into an infinite series

$$y(x)=y_0(x)+y_1(x)+y_2(x)+... \tag{2.3}$$

Since $f(y)$ is an analytic function we have the following Taylor's expansion about y_0:

$$f(y)=f(y_0)+\frac{f'(y_0)}{1!}(y-y_0)+\frac{f''(y_0)}{2!}(y-y_0)^2+...$$

Since $y-y_0=y_1+y_2+...$, we have

$$f(y)=f(y_0)+f'(y_0)(y_1+y_2+...)+\frac{f''(y_0)}{2!}(y_1+y_2+...)^2+...$$

This expansion can be rearranged by an infinite series of so-called Adomian polynomials [1]

$$f(y)=\sum_{n=0}^{\infty}A_m(y_0,y_1,...,y_m) \tag{2.4}$$

where the Adomian polynomials $A_0,A_1,A_2,...$ are defined as following:
$A_0=f(y_0)$, $A_1=y_1f'(y_0)$, $A_2=y_2f'(y_0)+\frac{1}{2!}y_1^2f''(y_0)$,
$A_3=y_3f'(y_0)+y_1y_2f''(y_0)+\frac{1}{3!}y_1^3f'''(y_0)$,... It is very important that A_0 depends only on y_0, A_1 depends only on y_0 and y_1, A_2 depends on only on y_0, y_1 and y_2 and so on.

Combining (2.2), (2.3) and (2.4) and denoting $y_0=y(0)+xy'(0)+L^{-1}g$ we get

$$\sum_{n=0}^{\infty}y_n=y_0-L^{-1}\left(R\left(\sum_{n=0}^{\infty}y_n\right)\right)-L^{-1}\left(\sum_{n=0}^{\infty}A_n\right).$$

Now by applying the decomposition technique of Adomian we have the following recursive relations:

$$\begin{aligned}
y_1 &= -L^{-1}(Ry_0)-L^{-1}(A_0(y_0))\\
y_2 &= -L^{-1}(Ry_1)-L^{-1}(A_1(y_0,y_1))\\
y_3 &= -L^{-1}(Ry_2)-L^{-1}(A_2(y_0,y_1,y_2))\\
&.....
\end{aligned}$$

Thus we can recursively determine all terms $y_0,y_1,y_2,...$ of the series solution $y=y_0+y_1+y_2,...$

Now we shall describe the basic definitions and properties of the DTM.

Let $y=y(x)$ be any analytic function in some neighborhood of the point $x=x_0$.

Definition 2.1. *The differential transform of the analytic function $y(x)$ is defined as the sequence $(Y(0),Y(1),Y(2),Y(3),...)$ where, $Y(k)=\frac{1}{k!}\left[\frac{d^ky(x)}{dx^k}\right]_{x=x_0}$, $k=0,1,2....$ [20].*

Definition 2.2. *The differential inverse transformation of the sequence $(Y(k))$ is defined by $y=\sum_{k=0}^{\infty}Y(k)(x-x_0)^k$. Here $y(x)$ is said to be the original function and the sequence $(Y(k))$ is said to be the T-transform of y [20].*

Let us denote the T- transform of the original $y(x)$ by $T(y)$. From the definition of the T- transform it follows easily the following properties:

(i) $T(y+z)=T(y)+T(z)$
(ii) $T(\lambda y)=\lambda T(y)$ *for any* $\lambda\in R$
(iii) if $T(y)=(Y(k))$, *then* $T(\frac{dy}{dx})=((k+1)Y(k+1))$ *and* $T(\frac{d^2y}{dx^2})=((k+1)(k+2)Y(k+2))$
(iv) if $T(y)=(Y(k))$, $T(z)=(Z(k))$ *and* $T(yz)=(X(k))$, *then* $X(k)=(Y(k))*(Z(k))=\sum_{k=0}^{n}Y(n)Z(k-n)$

Remark 2.3. *In real applications, the differential inverse transform $T^{-1}(Y(k))$ is defined by a finite sum*

$$T^{-1}(Y(k)) = \sum_{k=0}^{s} Y(k)(x-x_0)^k$$

for sufficiently large s.

3. Development of the Adomian Decomposition Method for Solving Transmission Problems

Note that the ADM is intended for solving initial and boundary value problems without transmission conditions. Our aim here is to develop the ADM to deal also with the similar problems involving additional transmission conditions at some interior singular point. To this let us consider a simple, but illustrative nonlinear Sturm-Liouville equation,

$$y''(x) + y^2(x) = \lambda y(x), \tag{3.1}$$

on the domain $x \in [1,2) \cup (2,3]$ subject to the boundary conditions at the end points $x = 1$ and $x = 3$ given by

$$y(1) = y(3) = 0 \tag{3.2}$$

and additional transmission conditions at the interior singular point $x = 2$, given by

$$y(2-0) = \gamma_1 y(2+0) \tag{3.3}$$

$$y'(2-0) = \gamma_2 y'(2+0) \tag{3.4}$$

where γ_1 and γ_2 are real numbers that will be specified later. By applying an our own approach, at first we will consider some auxiliary initial-value problems on the left side $[1,2)$ and the right side $(2,3]$ of the domain $[1,2) \cup (2,3]$, separately.

Let us consider the following auxiliary initial-value problem on the left interval $[1,2)$, given by

$$y''(x) + y^2(x) = \lambda y(x), \qquad x \in [1,2) \tag{3.5}$$

$$y(1) = 0, \qquad y'(1) = a \tag{3.6}$$

where a is the unknown number that will be calculated later. By virtue of the well-known existence and uniqueness theorem of ordinary differential equation theory, the initial-value problem (3.5)-(3.6) has an unique solution $\widetilde{y}(x)$ (see, [21]). By applying the Adomian decomposition method we have the following recursive relation

$$\widetilde{y}_0(x) = ax - a$$

$$\begin{aligned}
\widetilde{y}_1(x) &= L^{-1}(\lambda \widetilde{y}_0(x) - \widetilde{A}_0) \\
&= \int_1^x \int_1^x \lambda(ax-a) - (ax-a)^2 dx dx \\
&= -\frac{1}{12} a(-2\lambda + a(-1+x))(-1+x)^3
\end{aligned}$$

$$\begin{aligned}
\widetilde{y}_2(x) &= L^{-1}(\lambda \widetilde{y}_1(x) - \widetilde{A}_1) \\
&= \frac{a}{2520}(-1+x)^5(10a^2(-1+x)^2 - 35a(-1+x)\lambda + 21\lambda^2)
\end{aligned}$$

$$\begin{aligned}
\widetilde{y}_3(x) &= L^{-1}(\lambda \widetilde{y}_2(x) - \widetilde{A}_2) \\
&= -\frac{a}{60480}(-1+x)^7(10a^3(-1+x)^3 - 50a^2(-1+x)^2\lambda + 63a(-1+x)\lambda^2 - 12\lambda^3)
\end{aligned}$$

....

Consequently we get the following left-side approximate solution $\widetilde{y}(x)$, given by

$$\begin{aligned}\widetilde{y}(x) &= ax - a - \frac{1}{12}a(-2\lambda + a(-1+x))(-1+x)^3 \\ &+ \frac{a}{2520}(-1+x)^5(10a^2(-1+x)^2 - 35a(-1+x)\lambda + 21\lambda^2) \\ &- \frac{a}{60480}(-1+x)^7(10a^3(-1+x)^3 - 50a^2(-1+x)^2\lambda + 63a(-1+x)\lambda^2 - 12\lambda^3). \end{aligned} \tag{3.7}$$

Now we will consider the following initial-value problem on the right-hand interval $(2,3]$, given by

$$y''(x) + y^2(x) = \lambda y(x), \qquad x \in (2,3] \tag{3.8}$$

$$y(3) = 0, \quad y'(3) = b \tag{3.9}$$

where b is unknown number that will be calculated later.

It is well-known that the problem (3.8)-(3.9) has an unique solution $\widetilde{\widetilde{y}}(x)$ (see, [21]). By using the same technique we can calculate the following first terms of the series solution $\widetilde{\widetilde{y}}(x) = \sum_{n=0}^{\infty} \widetilde{\widetilde{y}}_n(x)$, on the right interval $(2,3]$, given by

$$\widetilde{\widetilde{y}}_0(x) = bx - 3b$$

$$\widetilde{\widetilde{y}}_1(x) = -\frac{1}{12}b(-3+x)^3(b(-3+x) - 2\lambda)$$

$$\widetilde{\widetilde{y}}_2(x) = \frac{b}{2520}(-3+x)^5(10b^2(-3+x)^2 - 35b(-3+x)\lambda + 21\lambda^2)$$

....

Thus, we have the following right-side approximate solution $\widetilde{\widetilde{y}}(x)$ given by,

$$\begin{aligned}\widetilde{\widetilde{y}}(x) &= bx - 3b - \frac{1}{12}b(-3+x)^3(b(-3+x) - 2\lambda) \\ &+ \frac{b}{2520}(-3+x)^5(10b^2(-3+x)^2 - 35b(-3+x)\lambda + 21\lambda^2). \end{aligned} \tag{3.10}$$

Using the solutions (3.7) and (3.10) and to satisfy the transmission conditions, we have the following system of equations, to find unknown numbers a and b

$$\widetilde{y}(2) = \gamma_1 \widetilde{\widetilde{y}}(2)$$

$$\widetilde{y}'(2) = \gamma_2 \widetilde{\widetilde{y}}'(2).$$

By using "Mathematica 8" we can solve this system of equations to find eigenvalues $\lambda_1, \lambda_2, \ldots$ and corresponding eigenfunctions $y_1(x), y_2(x), \ldots$ of the boundary-value-transmission problem (3.1)− (3.4), as follows.

Table 1: The approximate values of the ADM-solutions at the nodal points.

x	y(x) (the case $\lambda = 1, \gamma_1 = 1, \gamma_2 = 1$)	y(x) (the case $\lambda = 2, \gamma_1 = 2, \gamma_2 = -1$)
1.1	0.8374135822	0.4151480442
1.2	1.6750175679	0.8365909708
1.3	2.5001801538	1.2674681270
1.4	3.2868741266	1.7073175342
1.5	3.9979143730	2.1518575892
1.6	4.5900087331	2.5928515921
1.7	5.0227932802	3.0181969260
1.8	5.2730231103	3.4123807106
1.9	5.3550897237	3.7574437536
2.1	5.3550897237	1.7811385128
2.2	5.2730231103	1.5494379309
2.3	5.0227932802	1.3250156536
2.4	4.5900087331	1.1096546336
2.5	3.9979143730	0.9041878745
2.6	3.2868741266	0.7085963698
2.7	2.5001801538	0.5221080419
2.8	1.6750175679	0.3432879321
2.9	0.8374135822	0.1701098883

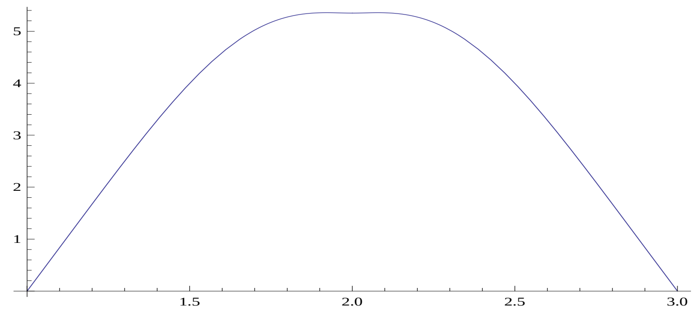

Figure 1: Graph of the approximate ADM-solution for the case $\lambda = 1, \gamma_1 = 1, \gamma_2 = 1$ (the continuous case).

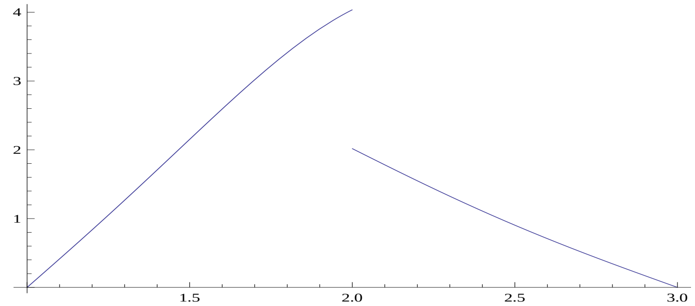

Figure 2: Graph of the approximate ADM-solution for the case $\lambda = 2, \gamma_1 = 2, \gamma_2 = -1$ (the discontinuous case).

4. Development of the Differential Transform Method for Solving Transmission Problems

In this section we shall develop the DTM for solving the same boundary-value-transmission problem (3.1)– (3.4). Devote by $Y^-(k)$ and $Y^+(k)$ the T- transforms of $y(x)$ at the left end-point $x=1$ and the right end-point $x=3$, respectively. If differential transform method is applied to the differential equation, (3.1) in the left interval, i.e. in the neighborhood of the point $x_0=1$, we have

$$Y^-(k+2)=\frac{\lambda Y^-(k)-\sum_{r=0}^{k}Y^-(r)Y^-(k-r)}{(k+2)(k+1)}, \tag{4.1}$$

where $Y^-(k)=\frac{1}{k!}\frac{d^k y(x)}{dx^k}|_{x=1}$. The differential inverse transformation in the left interval has the following form:

$$y^-(x)=Y^-(0)+(x-1)Y^-(1)+...+(x-1)^nY^-(n)+...$$

The first boundary condition $y(1)=0$ becomes $Y^-(0)=0$. Denoting $Y^-(1)=\alpha$ (where α is unknown number that will be calculated later) and then substituting in the recursive relation (4.1), we have $Y^-(2)=0$. Now proceeding the iteration using (4.1) we can calculate the other terms of the T- transform as $Y^-(3)=\frac{\lambda\alpha}{6},\ Y^-(4)=\frac{-\alpha^2}{12},\ Y^-(5)=\frac{\lambda\alpha}{120},\ Y^-(6)=\frac{-\lambda\alpha^2}{72},\ Y^-(7)=\frac{1}{42}\left(\frac{\lambda^2\alpha}{120}+\frac{\alpha^3}{6}\right)$, ...

If we carry out the iteration up to $k=7$, then we have the following approximation of the left solution:

$$\begin{aligned} y^-(x) &= \alpha(x-1)+\frac{\lambda\alpha}{6}(x-1)^3-\frac{\alpha^2}{12}(x-1)^4 \\ &+ \frac{\lambda\alpha}{120}(x-1)^5-\frac{\lambda\alpha^2}{72}(x-1)^6+\frac{1}{42}\left(\frac{\lambda^2\alpha}{120}+\frac{\alpha^3}{6}\right)(x-1)^7 \end{aligned} \tag{4.2}$$

Secondly, let's get the solution for the problem in the right interval $(2,3]$. If differential transform method is applied to the differential equation (3.1) in the neighborhood of the point $x_0=3$, we have

$$Y^+(k+2)=\frac{\lambda Y^+(k)-\sum_{r=0}^{k}Y^+(r)Y^+(k-r)}{(k+2)(k+1)}. \tag{4.3}$$

The differential inverse transformation in the right interval $(2,3]$ has the following form:

$$y^+(x)=Y^+(0)+(x-3)Y^+(1)+...+(x-3)^nY^+(n)+...$$

The second boundary condition $y(3)=0$ becomes $Y^+(0)=0$. Putting $Y^+(1)=\beta$, (where β is unknown number that will be calculated later) and using the recursive relation (4.3) we have $Y^+(2)=0$. By proceeding the iteration and using the relation (4.3) we have $Y^+(3)=\frac{\lambda\beta}{6}$, $Y^+(4)=\frac{-\beta^2}{12}$, $Y^+(5)=\frac{\lambda\beta}{120}$, $Y^+(6)=\frac{-\lambda\beta^2}{72}$, $Y^+(7)=\frac{1}{42}\left(\frac{\lambda^2\beta}{120}+\frac{\beta^3}{6}\right)$, ...

Again, we carry out the iteration up to $k=7$ and applying differential inverse transformation, then we have the following approximation of the right solution

$$\begin{aligned} y^+(x) &= \beta(x-3)+\frac{\lambda\beta}{6}(x-3)^3-\frac{\beta^2}{12}(x-3)^4 \\ &+ \frac{\lambda\beta}{120}(x-3)^5-\frac{\lambda\beta^2}{72}(x-3)^6+\frac{1}{42}\left(\frac{\lambda^2\beta}{120}+\frac{\beta^3}{6}\right)(x-3)^7. \end{aligned} \tag{4.4}$$

Substituting (4.2)-(4.4) in the transmission conditions (3.3)-(3.4) yields

$$\alpha+\frac{\lambda\alpha}{6}-\frac{\alpha^2}{12}+\frac{\lambda\alpha}{120}-\frac{\lambda\alpha^2}{72}+\frac{1}{42}\left(\frac{\lambda^2\alpha}{120}+\frac{\alpha^3}{6}\right)$$
$$=\gamma_1\left(-\beta-\frac{\lambda\beta}{6}-\frac{\beta^2}{12}-\frac{\lambda\beta}{120}-\frac{\lambda\beta^2}{72}-\frac{1}{42}\left(\frac{\lambda^2\beta}{120}+\frac{\beta^3}{6}\right)\right)$$

and

$$\alpha+\frac{\lambda\alpha}{2}-\frac{\alpha^2}{3}+\frac{\lambda\alpha}{24}-\frac{\lambda\alpha^2}{12}+\frac{1}{6}\left(\frac{\lambda^2\alpha}{120}+\frac{\alpha^3}{6}\right)$$
$$=\gamma_2\left(\beta+\frac{\lambda\beta}{2}+\frac{\beta^2}{3}+\frac{\lambda\beta}{24}+\frac{\lambda\beta^2}{12}+\frac{1}{6}\left(\frac{\lambda^2\beta}{120}+\frac{\beta^3}{6}\right)\right).$$

Using "Mathematica 8", we can solve this system of algebraic equations for obtaining approximate values of the unknown numbers α, β and the eigenvalues and corresponding eigenfunctions. Below Table II presents the numerical results for the approximate solution $y(x)$.

Table 2: The approximate values of the DTM-solutions at the nodal points.

x	y(x) (the case $\lambda=1,\gamma_1=1,\gamma_2=1$)	y(x) (the case $\lambda=2,\gamma_1=2,\gamma_2=-1$)
1.1	0.8344762396	0.4186440631
1.2	1.6691708028	0.8436189511
1.3	2.4915688717	1.2780466761
1.4	3.2758492634	1.7213917708
1.5	3.9850957246	2.1692530698
1.6	4.5762921112	2.6132533817
1.7	5.0092610789	3.0411758308
1.8	5.2607059147	3.4374956428
1.9	5.3445151349	3.7844561536
2.1	5.3445151349	1.7934130679
2.2	5.2607059147	1.5598785894
2.3	5.0092610789	1.3338965425
2.4	4.5762921112	1.1171170753
2.5	3.9850957246	0.9103076485
2.6	3.2758492634	0.7134222063
2.7	2.4915688717	0.5256799643
2.8	1.6691708028	0.3456424954
2.9	0.8344762396	0.1712777940

Below, Figure 3 and 4 presented the graphs of the approximate DTM-solutions for some values of the parameters λ,γ_1 and γ_2.

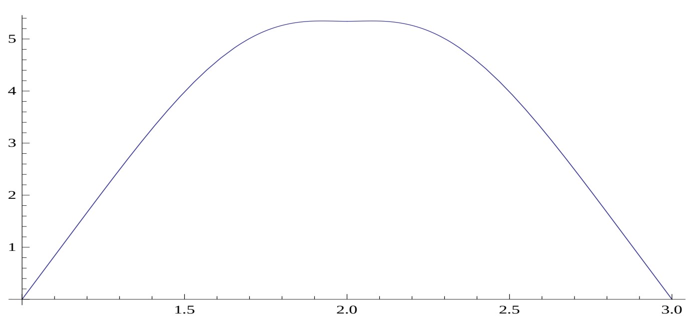

Figure 3: Graph of the approximate DTM-solution for the case $\lambda=1,\gamma_1=1,\gamma_2=1$ (the continuous case).

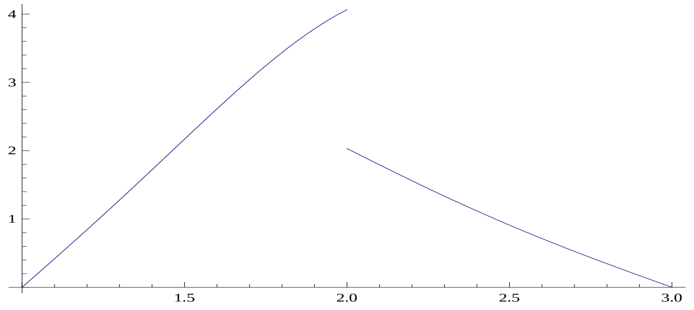

Figure 4: Graph of the approximate DTM-solution for the case $\lambda = 2, \gamma_1 = 2, \gamma_2 = -1$ (the discontinuous case).

Below by $Dy(x)$ and $Ay(x)$ we shall denote the approximation solutions obtained by using DTM and ADM, respectively.

Table 3: The numerical results for DTM and ADM-solutions.

x	Dy(x) (the case $\lambda = 1, \gamma_1 = 1, \gamma_2 = 1$)	Ay(x) (the case $\lambda = 1, \gamma_1 = 1, \gamma_2 = 1$)
1.1	0.8344762396	0.8374135822
1.2	1.6691708028	1.6750175679
1.3	2.4915688717	2.5001801538
1.4	3.2758492634	3.2868741266
1.5	3.9850957246	3.9979143730
1.6	4.5762921112	4.5900087331
1.7	5.0092610789	5.0227932802
1.8	5.2607059147	5.2730231103
1.9	5.3445151349	5.3550897237
2.1	5.3445151349	5.3550897237
2.2	5.2607059147	5.2730231103
2.3	5.0092610789	5.0227932802
2.4	4.5762921112	4.5900087331
2.5	3.9850957246	3.9979143730
2.6	3.2758492634	3.2868741266
2.7	2.4915688717	2.5001801538
2.8	1.6691708028	1.6750175679
2.9	0.8344762396	0.8374135822

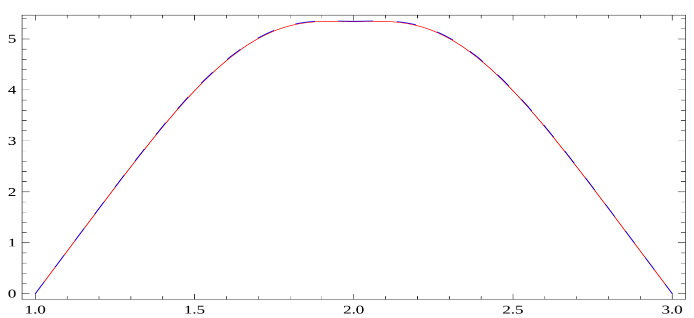

Figure 5: Comparison of the DTM-solution (red line) and the ADM- solution (blue line) for $\lambda = 1, \gamma_1 = 1, \gamma_2 = 1$ (the continuous case).

Table 4: The numerical results for DTM and ADM-solutions.

x	Dy(x) (the case $\lambda = 2, \gamma_1 = 2, \gamma_2 = -1$)	Ay(x) (the case $\lambda = 2, \gamma_1 = 2, \gamma_2 = -1$)
1.1	0.4186440631	0.4151480442
1.2	0.8436189511	0.8365909708
1.3	1.2780466761	1.2674681270
1.4	1.7213917708	1.7073175342
1.5	2.1692530698	2.1518575892
1.6	2.6132533817	2.5928515921
1.7	3.0411758308	3.0181969260
1.8	3.4374956428	3.4123807106
1.9	3.7844561536	3.7574437536
2.1	1.7934130679	1.7811385128
2.2	1.5598785894	1.5494379309
2.3	1.3338965425	1.3250156536
2.4	1.1171170753	1.1096546336
2.5	0.9103076485	0.9041878745
2.6	0.7134222063	0.7085963698
2.7	0.5256799643	0.5221080419
2.8	0.3456424954	0.3432879321
2.9	0.1712777940	0.1701098883

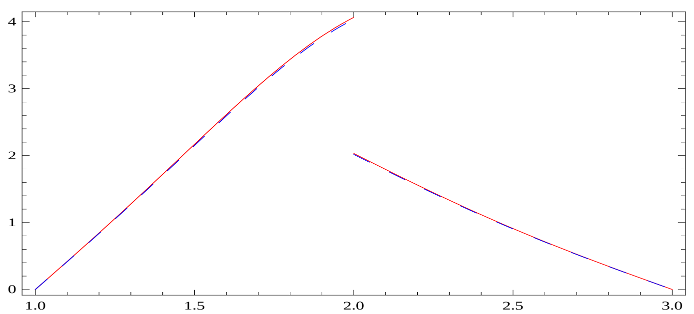

Figure 6: Comparison the DTM-solution (red line) and with the ADM-solution (blue line) for $\lambda = 2, \gamma_1 = 2, \gamma_2 = -1$ (the discontinuous case).

5. Conclusion

Boundary value problems with additional transmission conditions are much more complicated to solve than classical boundary value problems. The major difficulty lies in the existence of eigenvalues and eigenfunctions, since it is not clear how to apply the known numerical and analytical methods to such type of problems. In particular, the well-known DTM and ADM are intended for solving initial and boundary value problems without transmission conditions at some internal singularity points. Such type of problems we call boundary value transmission problems (BVTP's). In this paper we modified these methods to solve also BVTP's for non-linear differential equations. The advantage of our approach is that it can be applied not only to one-interval boundary value problems but also to two-interval boundary value problems involving additional transmission conditions. To justify the proposed modifications we studied a simple but illustrative BVTP's for one non-linear Sturm-Liouville equation. The results are illustrated graphically. The graphs were sketched by using "Mathematica 8".

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