Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–11.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.51981

Stochastic Stability and Impulsive Vaccination of Multicompartment Nonlinear Epidemic Model with Incidence Rate

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ABSTRACT: In this work, we consider a multicompartment nonlinear epidemic model with temporary immunity and a saturated incidence rate. $N(t)$ at time t, this population is divide into seven sub-classes.

$N(t) = S(t)+E(t)+I(t)+I_1(t)+I_2(t)+I_3(t)+Q(t)$. where $S(t)$,$E(t)$, $I(t)$, $I_1(t)$,$I_2(t)$, $I_3(t)$ and $Q(t)$ denote the sizes of the population susceptible to disease, exposed, infectious members and quarantine members with the possibility of infection through temporary immunity, respectively. The stability of a disease-free status equilibrium and the existence of endemic equilibrium determined by the ratio called the basic reproductive number. The multicompartment non linear epidemic model with saturated rate has been studied the stochastic stability of the free disease equilibrium under certain conditions, and obtain the conditions of global attractivity of the infection.

Key Words: Basic reproduction number, endemic equilibrium, epidemic, model stability, saturated incidence, stochastic stability, vaccination.

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1. Introduction

This paper considers the following nonlinear epidemic model with temporary immunity:

$$\begin{cases} \dot{S}(t) = \lambda + \nu - (\mu + d) S(t) - \beta \frac{S(t)I(t)}{1+aI^2(t)} + \gamma Q(t), \\ \dot{E}(t) = \beta \frac{S(t)I(t)}{1+aI^2(t)} - (\mu_1 + d) E(t) - \beta e^{-\mu_1 \tau} S(t-\tau) I(t-\tau), \\ \dot{I}(t) = \beta e^{-\mu_1 \tau} S(t-\tau) I(t-\tau) - (\mu_2 + d) I(t) - \sum_{i=1}^{3} \alpha_i I(t), \\ \dot{I}_1(t) = \alpha_1 I(t) - \delta_1 I_1(t) - (\mu_3 + d) I_1(t), \\ \dot{I}_2(t) = \alpha_2 I(t) - \delta_2 I_2(t) - (\mu_4 + d) I_2(t), \\ \dot{I}_3(t) = \alpha_3 I(t) - \delta_3 I_3(t) - (\mu_5 + d) I_3(t), \\ \dot{Q}(t) = \sum_{i=1}^{3} \delta_i I_i(t) - (\mu_6 + d) Q(t) - \gamma Q(t). \end{cases} \tag{1.1}$$

2010 *Mathematics Subject Classification:* 34D23,92D30.
Submitted January 28, 2020. Published April 27, 2021

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Consider a population of size $N(t)$ at time t, this population is divide into seven sub-classes, with $N(t) = S(t) + E(t) + I(t) + I_1(t) + I_2(t) + I_3(t) + Q(t)$.

Where $S(t)$,$E(t), I(t), I_1(t)$,$I_2(t), I_3(t)$ and $Q(t)$ denote the sizes of the population susceptible to disease, exposed, infectious members and quarantine members with the possibility of infection through temporary immunity, respectively. The positive constants $\mu, \mu_1, \mu_2, \mu_3, \mu_4, \mu_5$, and μ_6 represent the death rates of susceptible, exposed, infectious and quarantine. Biologically, it is natural to assume that $\mu \leq min\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6\}$. The positive constant d is natural mortality rate. The positive constants λ represent rate of insidence. The positive constant γ represent the recovery rate of infection. The positive constant β is the average numbers of contacts infective for S and I. ν the positive constant is the parameter of emigration. The positive constant $\alpha_1, \alpha_2, \alpha_3$ are the average numbers of contacts. The positive constants $\delta_1, \delta_2, \delta_3$, are the numbers of transfer to conversion of infected people to quarantine.

a is saturation constant and and τ is the length of immunity period.

The form of incidence is $\beta \frac{S(t)I(t)}{1+aI^2(t)}$; which is saturated with the susceptible.

The initial condition of (1) given as:

$$\begin{cases} S(\eta) = \Phi_1(\eta), E(\eta) = \Phi_2(\eta), I(\eta) = \Phi_3(\eta), I_1(\eta) = \Phi_4(\eta), \\ I_2(\eta) = \Phi_5(\eta), I_3(\eta) = \Phi_6(\eta), Q(\eta) = \Phi_7(\eta), -\tau \leq \eta \leq 0. \end{cases} \tag{1.2}$$

Where $\Phi = (\Phi_1, \Phi_2, \Phi_3, \Phi_4, \Phi_5, \Phi_6, \Phi_7)^T \in \mathbb{C}$ such that

$$\begin{cases} S(\eta) = \Phi_1(0) \geq 0, E(\eta) = \Phi_2(0) \geq 0, I(\eta) = \Phi_3(0) \geq 0, I_1(\eta) = \Phi_4(0) \geq 0, \\ I_2(\eta) = \Phi_5(0) \geq 0, I_3(\eta) = \Phi_6(0) \geq 0, Q(\eta) = \Phi_7(0) \geq 0, -\tau \leq \eta \leq 0. \end{cases} \tag{1.3}$$

Let $\mathbb{C}$ denote the Banach space $C\left([-\tau, 0], \mathbb{R}^7\right)$ of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^7$. With a biological meaning, we further assume that $\Phi_i(\eta) = \Phi_i(0) \geq 0$, for $i = 1, 2, 3, 4, 5, 6, 7$.

We have,

$$\dot{N}(t) = \lambda + \nu - \mu S(t) - \mu_1 E(t) - \mu_2 I(t) - \mu_3 I_1(t) - \mu_4 I_2(t) - \mu_5 I_3(t) - \mu_6 Q(t) - dN(t) \tag{1.4}$$

The region Ω is positively invariant set of (1.1),

$$\left\{ \begin{array}{c} \Omega = (S(t), E(t), I(t), I_1(t), I_2(t), I_3(t), Q(t)) \in \mathbb{R}^7, \\ S(t) + E(t) + I(t) + I_1(t) + I_2(t) + I_3(t) + Q(t) \leq N < \frac{\lambda+\nu}{\mu+d} \end{array} \right\} \tag{1.5}$$

Hence, system (1.1), can be rewrite as:

$$\begin{cases} \dot{S}(t) = \lambda + \nu - (\mu + d) S(t) - \beta \frac{S(t)I(t)}{1+aI^2(t)} + \gamma Q(t), \\ \dot{E}(t) = \beta \frac{S(t)I(t)}{1+aI^2(t)} - (\mu_1 + d) E(t) - \beta e^{-\mu_1 \tau} S(t-\tau) I(t-\tau), \\ \dot{I}(t) = \beta e^{-\mu_1 \tau} S(t-\tau) I(t-\tau) - \left(\mu_2 + d + \sum_{i=1}^{3} \alpha_i\right) I(t), \\ \dot{I}_1(t) = \alpha_1 I(t) - (\mu_3 + d + \delta_1) I_1(t), \\ \dot{I}_2(t) = \alpha_2 I(t) - (\mu_4 + d + \delta_2) I_2(t), \\ \dot{I}_3(t) = \alpha_3 I(t) - (\mu_5 + d + \delta_3) I_3(t), \\ \dot{Q}(t) = \sum_{i=1}^{3} \delta_i I_i(t) - (\mu_6 + d + \gamma) Q(t). \end{cases} \tag{1.6}$$

2. Equilibrium points

We calculate the points of equilibrium in the absence and presence of infection.

In the absence of infection, the system (1.6) has a disease-free equilibrium E_0.

$$E_0 = \left(\hat{S}, \hat{E}, \hat{I}, \hat{I}_1, \hat{I}_2, \hat{I}_3, \hat{Q}\right)^T = \left(\frac{\lambda + \nu}{\mu + d}, 0, 0, 0, 0, 0, 0\right)^T. \tag{2.1}$$

Define the quarantine reproduction number as:

$$R_0 = \frac{\beta e^{-\mu_1 \tau}}{\mu + d} \times \frac{(\lambda + \nu)}{\mu_2 + d + \sum_{i=1}^{3} \alpha_i} \tag{2.2}$$

In the presence of infection, substituting in the system, Ω also contains a unique positive, endemic equilibrium

$$E_\tau^* = (S^*, E^*, I^*, I_1^*, I_2^*, I_3^*, Q^*)^T \tag{2.3}$$

$$\begin{cases} S^* = \frac{\mu_2 + d + \sum_{i=1}^{3} \alpha_i}{\beta e^{-\mu_1 \tau}}, \\ E^* = \frac{1}{\mu_1 + d} \left[\beta S^* I^* \left(\frac{1}{1 + a(I^*)^2} - e^{-\mu_1 \tau} \right) \right] \\ I^* = \frac{\beta S^* + \sqrt{\triangle}}{2m}, m = \lambda + \nu - (\mu + d) S^* + \gamma Q^*, \triangle = (\beta S^*)^2 + 4\alpha m \\ I_1^* = \frac{\alpha_1}{\mu_3 + d + \delta_1} I^*, \\ I_2^* = \frac{\alpha_2}{\mu_4 + d + \delta_2} I^*, \\ I_3^* = \frac{\alpha_3}{\mu_5 + d + \delta_3} I^*, \\ O^* = \frac{\sigma}{c_3} I^*, \\ Q^* = \frac{1}{\mu_6 + d + \gamma} \left[\frac{\alpha_1 \delta_1}{\mu_3 + d + \delta_1} + \frac{\alpha_2 \delta_2}{\mu_4 + d + \delta_2} + \frac{\alpha_3 \delta_3}{\mu_5 + d + \delta_3} \right] I^*, \end{cases}$$

2.1. Theorem 1.

Let $(S, E, I, I_1, I_2, I_3, Q)$ the solution of (1.6) wich is defined in $[0, \infty)$.

$$limsup_{t \to \infty} N(t) \leq \frac{\lambda + \nu}{\mu + d}.$$

PROOF.
We have (1.4), then for

$$\dot{N} \leq \lambda + \nu - (\mu + d) N. \tag{2.4}$$

With integrating, we have for

$$t \in (0, t_1), N \leq \frac{\lambda + \nu}{\mu + d} - \left(1 - e^{-(\mu + d)t} \right), \tag{2.5}$$

Then $N \leq 2\frac{\lambda + \nu - \upsilon}{\mu + d}$. The solution $(S, E, I, I_1, I_2, I_3, Q)$ bounded in $[0, T)$. For

$$t \in [0, \infty), N \leq \frac{\lambda + \nu}{\mu + d}. - \left(1 - e^{-(\mu + d)t} \right). \tag{2.6}$$

Finally

$$limsup_{t \to \infty} N(t) \leq \frac{\lambda + \nu}{\mu + d}.$$

2.2. Theorem 2.

The disease-free equilibrium E_0 of the system (1.6) is locally asymptotically stable if $R_0 < 1$.

2.3. Theorem 3.

If $R_0 > 1$, the system (1.6) has a unique, non-trivial equilibrium E_τ^* which is locally asymptotically stable.

3. Stochastic stability of the free disease equilibrium

We limit ourselves here to perturbing only the contact rate so we replace β by $\beta + ab(t)$, where $b(t)$ is white noise (Brownian motion). The system (1.6) transformed to the following Itô stochastic differential equations:

$$\begin{cases} dS = \left[\lambda + \nu - (\mu + d) S(t) - \beta \frac{SI}{1+aI^2} + \gamma Q(t)\right] dt - a\frac{SI}{1+aI^2} db, \\ dE = \left[\beta \frac{SI}{1+aI^2} - (\mu_1 + d) E(t) - \beta e^{-\mu_1 \tau} S(t-\tau) I(t-\tau)\right] dt + a\frac{SI}{1+aI^2} db, \\ dI = \left[\beta e^{-\mu_1 \tau} S(t-\tau) I(t-\tau) - \left(\mu_2 + d + \sum_{i=1}^{3} \alpha_i\right) I(t)\right] dt, \\ dI_1 = [\alpha_1 I - (\mu_3 + d + \delta_1) I_1] dt, \\ dI_2 = [\alpha_2 I - (\mu_4 + d + \delta_2) I_2] dt, \\ dI_3 = [\alpha_3 I - (\mu_5 + d + \delta_3) I_3] dt, \\ dQ = \left[\sum_{i=1}^{3} \delta_i I_i - (\mu_6 + d + \gamma) Q\right] dt. \end{cases} \tag{3.1}$$

3.1. Theorem 4.

If $R_0 < 1$, $E(t)$ and $Q(t)$ are exponentially almost surely stable.
PROOF: Let w ,with Itô's formula, we obtain

$$L = dlog(E + wQ) = \frac{\dot{E} + w\dot{Q}}{E + wQ} \tag{3.2}$$

$$L = \frac{1}{E + wQ} \left[\begin{array}{c} \beta \frac{SI}{1+aI^2} - (\mu_1 + d) E - \beta e^{-\mu_1 \tau} SI \\ + w\left[\sum_{i=1}^{3} \delta_i I_i - (\mu_6 + d + \gamma) Q\right] - \frac{1}{2} \frac{a^2}{E+wQ} \left(\frac{SI}{1+aI^2}\right)^2 \end{array} \right] dt + \frac{a}{E + wQ} \times \frac{SI}{1 + aI^2} db, \tag{3.3}$$

We have

$$L \leq \frac{1}{E + wQ} \left[\begin{array}{c} -\left[\beta \frac{\lambda+\nu}{\mu+d} \left(\frac{1}{1+aI^2} - e^{-\mu_1 \tau}\right)\right] I - [\mu_1 + d] E \\ + (w\delta_1) I_1 + (w\delta_2) I_2 + + (w\delta_3) I_3 - w(\mu_6 + d + \gamma) Q \end{array} \right] dt + \frac{a}{E + wQ} \times \frac{SI}{1 + aI^2} db, \tag{3.4}$$

Then

$$L \leq \frac{-1}{E + wQ} \left[\; [\mu_1 + d] E + w(\mu_6 + d + \gamma) Q \; \right] dt + \frac{a}{E + wQ} \times \frac{SI}{1 + aI^2} db, . \tag{3.5}$$

We suppose that

$$L_1 = min\{(\mu_1 + d), (\mu_6 + d + \gamma)\}. \tag{3.6}$$

Then

$$L \leq -L_1 dt + \frac{aSI}{E + wQ} db. \tag{3.7}$$

With integration, we obtain

$$log(E + wQ) \leq -L_1 dt + a \int_0^t \frac{S(v) I(v)}{E(v) + wQ(v)} db(v). \tag{3.8}$$

We have $\left(\frac{S(v)I(v)}{E(v)+wQ(v)}\right)^2$, is bounded.
So, $lim_{t\to\infty} \frac{S(v)I(v)}{E(v)+wQ(v)} db(v) = 0$,almost surely.
The following form from Doob's martingale inequality combined with Itô isometry see [18].
$limsup_{t\to\infty} \frac{1}{t} log(E + wQ) \leq -L_1$,almost surely.
Finally

- $limsup_{t\to\infty}\frac{1}{t}logE \leq -L_1$,almost surely,
- $limsup_{t\to\infty}\frac{1}{t}logQ \leq -L_1$,almost surely.

4. Impulsive vaccination

The following mathematical model with saturation incidence:

$$\begin{cases} \left\{ \begin{array}{c} \dot{S}(t) = \lambda + \nu - (\mu + d) S(t) - \beta \frac{S(t)I(t)}{1+aI^2(t)} + \gamma Q(t), \\ \dot{E}(t) = \beta \frac{S(t)I(t)}{1+aI^2(t)} - (\mu_1 + d) E(t) - \beta e^{-\mu_1 \tau} S(t-\tau) I(t-\tau), \\ \dot{I}(t) = \beta e^{-\mu_1 \tau} S(t-\tau) I(t-\tau) - \left(\mu_2 + d + \sum_{i=1}^{3} \alpha_i\right) I(t), \\ \dot{I}_1(t) = \alpha_1 I(t) - (\mu_3 + d + \delta_1) I_1(t), \\ \dot{I}_2(t) = \alpha_2 I(t) - (\mu_4 + d + \delta_2) I_2(t), \\ \dot{I}_3(t) = \alpha_3 I(t) - (\mu_5 + d + \delta_3) I_3(t), \\ \dot{Q}(t) = \sum_{i=1}^{3} \delta_i I_i(t) - (\mu_6 + d + \gamma) Q(t), \end{array} \right\}, t \neq kT, k \in \mathbb{N}, \\ \left\{ \begin{array}{c} S(t^+) = (1-\theta) S(t), \\ E(t^+) = E(t), \\ I(t^+) = I(t^+), \\ I_1(t^+) = I_1(t^+), \\ I_2(t^+) = I_2(t^+), \\ I_3(t^+) = I_3(t^+), \\ Q(t^+) = Q(t) + \theta S(t), \end{array} \right\}, t = kT, k \in \mathbb{N}, \end{cases} \tag{4.1}$$

With θ, is pulse vaccination rate, $(0 < \theta < 1)$.
We simplify (4.1) ,and with (1.4) the following system is:

$$\begin{cases} \left\{ \begin{array}{c} \dot{S}(t) = \lambda + \nu - (\mu + d) S(t) - \beta \frac{S(t)I(t)}{1+aI^2(t)} + \gamma Q(t), \\ \dot{I}(t) = \beta e^{-\mu_1 \tau} S(t-\tau) I(t-\tau) - \left(\mu_2 + d + \sum_{i=1}^{3} \alpha_i\right) I(t), \\ \dot{I}_1(t) = \alpha_1 I(t) - (\mu_3 + d + \delta_1) I_1(t), \\ \dot{I}_2(t) = \alpha_2 I(t) - (\mu_4 + d + \delta_2) I_2(t), \\ \dot{I}_3(t) = \alpha_3 I(t) - (\mu_5 + d + \delta_3) I_3(t), \\ \dot{Q}(t) = \sum_{i=1}^{3} \delta_i I_i(t) - (\mu_6 + d + \gamma) Q(t), \\ \dot{N} \approx \lambda + \nu - (\mu + d) N(t) \end{array} \right\}, t \neq kT, k \in \mathbb{N}, \\ \left\{ \begin{array}{c} S(t^+) = (1-\theta) S(t), \\ I(t^+) = I(t^+), \\ I_1(t^+) = I_1(t^+), \\ I_2(t^+) = I_2(t^+), \\ I_3(t^+) = I_3(t^+), \\ Q(t^+) = Q(t) + \theta S(t), \\ N(t^+) = N(t), \end{array} \right\}, t = kT, k \in \mathbb{N}, \end{cases} \tag{4.2}$$

The initial condition of (4.2) given as:

$$\begin{cases} S(\eta) = \Phi_1(\eta), I(\eta) = \Phi_2(\eta), I_1(\eta) = \Phi_3(\eta), \\ I_2(\eta) = \Phi_4(\eta), I_3(\eta) = \Phi_5(\eta), Q(\eta) = \Phi_6(\eta), N(\eta) = \Phi_7(\eta), -\tau \leq \eta \leq 0. \end{cases} \tag{4.3}$$

With a biological meaning, we further assume that: $\Phi_i(\eta) = \Phi_7(0) \geq 0$, for $i = 1, 2, 3, 4, 5, 6, 7$.

The region Ω_1 is positively invariant set of (4.2),

$$\left\{ \begin{array}{c} \Omega_1 = (S(t), I(t), I_1(t), I_2(t), I_3(t), Q(t), N(t)) \in \mathbb{R}^7, \\ S(t) + I(t) + I_1(t) + I_2(t) + I_3(t) + Q(t) \leq N < \frac{\lambda+\nu}{\mu+d}, N(t) \leq \frac{\lambda+\nu}{\mu+d} \end{array} \right\} \tag{4.4}$$

4.1. Lemma 1.

Consider the following impulsive differential system

$$\left\{ \begin{array}{c} \dot{u}(t) = a - bu(t), t \neq kT \\ u\left(t^+\right) = (1-\theta)\, u(t), t = kT. \end{array} \right. \tag{4.5}$$

Where $a > 0, b > 0$, and $(0 < \theta < 1)$. Then there exists a unique positive periodic solution of system (4.2).

$$\tilde{u_e}\,(t) = \frac{a}{b} + \left(u^* - \frac{a}{b}\right) e^{-b(t-kT)}, kT < t \leq (k+1)\,T,$$

Which is globally asymptotically stable, where

$$u^* = \frac{a}{b} \frac{(1-\theta)\left(1-e^{-bT}\right)}{1-(1-\theta)\, e^{-bT}}.$$

4.2. Lemma 2.

Consider the following linear neutral delay equation:

$$\dot{x}(t) = a_1 x(t-\tau) - a_2 x\,(t) \tag{4.6}$$

Where a_1,a_2, and τ ara all positive constants.
For $-\tau \leq t \leq 0, x\,(t) > 0$. We have:

$$If a_1 < a_2, lim_{t\to\infty} x\,(t) = 0 \tag{4.7}$$

4.3. Global attractivity

In this section, we discuss the global attractivity of infection free periodic solution, in which infectious individuals are entirely absent from the population permanently

$$Q(\eta) = \Phi_7(0) \geq 0, -\tau \leq \eta \leq 0.$$

Forall $t \geq 0$,

$$I(t) = 0, I_i(t) = 0.$$

Under this condition, the growth of susceptible individuals , quarataine individuals and total population must satisfy the following impulsive system:

$$\left\{ \begin{array}{c} \left\{ \begin{array}{c} \dot{S}\,(t) = \lambda + \nu - (\mu+d)\, S(t) + \gamma \frac{\lambda+\nu}{\mu+d} \frac{\left(1-e^{-(\mu+d+\gamma)T}\right)}{1-(1-\theta)e^{-(\mu+d+\gamma)T}} + \xi Q\,(t)\,, \\ \dot{Q}\,(t) = -\left(\mu_6 + d + \gamma\right) Q\,(t)\,, \\ \dot{N}(t) \approx \lambda + \nu - (\mu+d)\, N(t), \end{array} \right\}, t \neq kT, k \in \mathbb{N}, \\ \left\{ \begin{array}{c} S\left(t^+\right) = (1-\theta)\, S\,(t)\,, \\ Q\left(t^+\right) = Q\,(t) + \theta S\,(t)\,, \\ N\left(t^+\right) = N\,(t)\,, \end{array} \right\}, t = kT, k \in \mathbb{N}, \end{array} \right. \tag{4.8}$$

We have

$$\left\{ \begin{array}{c} \dot{N}(t) = \lambda + \nu - (\mu+d)\, N(t), \\ N\left(t^+\right) = N\,(t)\,, \end{array} \right. \tag{4.9}$$

From (4.9),

$$\lim_{t\to\infty} N(t) = \frac{\lambda+\nu}{\mu+d} \tag{4.10}$$

We have

$$\begin{cases} \dot{E}(t) = \beta\frac{S(t)I(t)}{1+aI^2(t)} - (\mu_1+d)E(t) - \beta e^{-\mu_1\tau}S(t-\tau)I(t-\tau), \\ E(t^+) = E(t), \end{cases} \tag{4.11}$$

From(4.11) and if $I(t)=0$,

$$lim_{t\to\infty}E(t) = 0 \tag{4.12}$$

We have S,Q oscillate with T in synchronization with the periodic pulse vaccination.
From (4.10) and (4.11), we have

$$Q(t) = \frac{\lambda+\nu}{\mu+d} - S(t), \tag{4.13}$$

and

$$\begin{cases} \dot{S}(t) = (\mu+d+\gamma)\left(\frac{\lambda+\nu}{\mu+d} - S(t)\right), t \neq kT \\ S(t^+) = (1-\theta)S(t), t = kT \end{cases} \tag{4.14}$$

.

According to lemma 1, the periodic solution of system (4.14) is:

$$\tilde{S}_e(t) = \frac{\lambda+\nu}{\mu+d} + \left(S^* - \frac{\lambda+\nu}{\mu+d}\right)e^{-(\mu+d+\gamma)(t-kT)}, kT < t \leq (k+1)T \tag{4.15}$$

It is globally asymptotically stable, wih

$$S^* = \frac{\lambda+\nu}{\mu+d}\frac{(1-\theta)\left(1-e^{-(\mu+d+\gamma)T}\right)}{1-(1-\theta)e^{-(\mu+d+\gamma)T}}. \tag{4.16}$$

4.4. Theorem 5.

The infection-free periodic solution $\left(\tilde{S}_e(t), 0, 0, 0, 0, \frac{\lambda+\nu}{\mu+d} - \tilde{S}_e(t), \frac{\lambda+\nu}{\mu+d}\right)$ of (4.2) is globally attractive provided that $Q^* < 1$.

$$Q^* = \frac{(\lambda+\nu)\beta e^{-\mu_1\tau}\left(1-e^{-(\mu+d+\gamma)T}\right)}{(\mu+d)1-(1-\theta)e^{-(\mu+d+\gamma)T}}.$$

PROOF.
With $Q^* < 1$, we choose ξ sufficiently small,

$$\beta e^{-\mu_1\tau}\left(\frac{\lambda+\nu}{\mu+d}\frac{\left(1-e^{-(\mu+d+\gamma)T}\right)}{1-(1-\theta)e^{-(\mu+d+\gamma)T}} + \xi\right) < \mu_2 + d + \sum_{i=1}^{3}\alpha_i \tag{4.17}$$

We have

$$\dot{S}(t) \leq (\lambda+\nu+\gamma)\left(\frac{\lambda+\nu}{\mu+d} - S(t)\right)$$

Then the following comparison impulsive differential system is:

$$\begin{cases} \dot{x}(t) = (\mu+d+\gamma)\left(\frac{\lambda+\nu}{\mu+d} - x(t)\right), t \neq kT \\ x(t^+) = (1-\theta)x(t), t = kT \end{cases} \tag{4.18}$$

From (4.14), we have the periodic solution of (4.18) is (4.14), with $\tilde{x_e}(t) = \tilde{S}_e(t)$ which is globally asymptotically stable, if we have (4.15).

$(S(t), I(t), I_1(t), I_2(t), I_3(t), Q(t), N(t))$ the solution of (4.2), with (4.3) and $S(0^+) = S_0 > 0$ and $x(t)$ slution of (4.18), with $x(0^+) = S_0 > 0$.

By comparaison theorem for impulsive differential system [12].
There existe an integer $k_1 > 0$, such that

$$S(t) < x(t) < \tilde{x_e}(t) + \xi, kT < t \leq (k+1)T, k > k_1, \tag{4.19}$$

From (4.19)

$$S(t) < \tilde{S_e}(t) + \xi \leq \left(\frac{\lambda + \nu}{\mu + d} \frac{\left(1 - e^{-(\mu+d+\gamma)T}\right)}{1 - (1-\theta) e^{-(\mu+d+\gamma)T}} + \xi \right), kT < t \leq (k+1)T, k > k_1, \tag{4.20}$$

Then we have

$$S(t) \leq \sigma \tag{4.21}$$

With

$$\sigma = \frac{\lambda + \nu}{\mu + d} \frac{\left(1 - e^{-(\mu+d+\gamma)T}\right)}{1 - (1-\theta) e^{-(\mu+d+\gamma)T}} + \xi.$$

From (4.21) and the second equation of (4.2), we have

$$\dot{I}(t) \leq \beta e^{-\mu_1 \tau} \sigma I(t-\tau) - \left(\mu_2 + d + \sum_{i=1}^{3} \alpha_i \right) I(t), t > kT + \tau, k > k_1. \tag{4.22}$$

Consider the following comparaison

$$\dot{y}(t) = \beta e^{-\mu_1 \tau} \sigma I(t-\tau) - \left(\mu_2 + d + \sum_{i=1}^{3} \alpha_i \right) y(t), t > kT + \tau, k > k_1. \tag{4.23}$$

From (4.17),

$$\beta e^{-\mu_1 \tau} \sigma < \mu_2 + d + \sum_{i=1}^{3} \alpha_i.$$

According to lemma 2,

$$lim_{t \to \infty} y(t) = 0.$$

$y(t)$ is the solution of (4.23), with initial condition $y(\eta) = \Phi(\eta) \geq 0, -\tau \leq \eta \leq 0$.
We have $lim_{t \to \infty} I(t) \leq lim_{t \to \infty} y(t) = 0$,then

$$lim_{t \to \infty} I(t) = 0. \tag{4.24}$$

For $\xi_1 > 0$ sufficiently small, there exist an integer $k_2 > 0$, with $k_2 \mathrm{T} > kT + \tau$, with $I(t) < \xi_1$,for all $t > k_2 T$.
We have

$$\begin{cases} \dot{I}_1(t) = \alpha_1 I(t) - (\mu_3 + d + \delta_1) I_1(t), \\ I(t) < \xi_1, for all t > k_2 T. \end{cases} \tag{4.25}$$

Then for $t > k_3 T$, we have

$$\dot{I}_1(t) < \alpha_1 \xi_1 - (\mu_3 + d + \delta_1) I_1(t) \tag{4.26}$$

Consider comparaison equation for $t > k_2 T$

$$\dot{L_1}(t) = \alpha_1 \xi_1 - (\mu_3 + d + \delta_1) L_1(t) \tag{4.27}$$

We see $lim_{t \to \infty} L_1(t) = 0$, by comparaison theorem [12] there exists an integer$k_4 > k_3$, such that for all $t > k_3 T$, and ξ_1 arbitrarily small, we have

$$lim_{t \to \infty} I_1(t) = 0 \tag{4.28}$$

We have

$$\begin{cases} \dot{I}_2(t) = \alpha_2 I(t) - (\mu_4 + d + \delta_2) I_2(t), \\ I(t) < \xi_1, forallt > k_2 T. \end{cases} \tag{4.29}$$

Then for $t > k_4 T$, we have

$$\dot{I}_2(t) < \alpha_2 \xi_1 - (\mu_4 + d + \delta_2) I_2(t) \tag{4.30}$$

Consider comparaison equation for $t > k_2 T$

$$\dot{L}_2(t) = \alpha_2 \xi_1 - (\mu_4 + d + \delta_2) L_2(t) \tag{4.31}$$

We see $lim_{t\to\infty} L_2(t)$=0, by comparaison theorem [12] there exists an integer$k_5 > k_4$, such that for all $t > k_4 T$, and ξ_1 arbitrarily small, we have

$$lim_{t\to\infty} I_2(t) = 0 \tag{4.32}$$

We have

$$\begin{cases} \dot{I}_3(t) = \alpha_3 I(t) - (\mu_5 + d + \delta_3) I_3(t), \\ I(t) < \xi_1, forallt > k_2 T. \end{cases} \tag{4.33}$$

Then for $t > k_5 T$, we have

$$\dot{I}_3(t) < \alpha_3 \xi_1 - (\mu_5 + d + \delta_3) I_3(t) \tag{4.34}$$

Consider comparaison equation for $t > k_2 T$

$$\dot{L}_3(t) = \alpha_3 \xi_1 - (\mu_5 + d + \delta_3) L_3(t) \tag{4.35}$$

We see $lim_{t\to\infty} L_3(t)$=0, by comparaison theorem [12] there exists an integer$k_6 > k_5$, such that for all $t > k_5 T$, and ξ_1 arbitrarily small, we have

$$lim_{t\to\infty} I_3(t) = 0 \tag{4.36}$$

From (4.10), and (4.24) ,there exists $k_7 > k_6$ such that

$$I(t) < \xi_1, andN(t) > \frac{\lambda + \nu}{\mu + d} - \xi_1, fort > k_7 T \tag{4.37}$$

From second equation of (4.1),Then

$$\dot{E}(t) \leq \frac{\lambda + \nu}{\mu + d} \frac{\beta \xi_1}{1 + a\xi_1^2} - (\mu_1 + d) E(t), fort > k_7 T \tag{4.38}$$

There exists $k_8 > k_7$ such that

$$E(t) < \frac{\lambda + \nu}{(\mu + d)^2} \frac{\beta \xi_1}{1 + a\xi_1^2} + \xi_1, fort > k_8 T \tag{4.39}$$

We have the first equation to (4.2), (4.37), and (4.39), we have

$$\dot{S}(t) \geq \lambda + \nu - (\mu + d) S(t) - \beta \frac{S(t)\xi_1}{1 + a\xi_1^2(t)} + \gamma \left(\frac{\lambda + \nu}{\mu + d} - \frac{\beta \xi_1}{1 + a\xi_1^2} \frac{\lambda + \nu}{(\mu + d)^2} - S(t) - 3\xi_1 \right)$$

$$\dot{S}(t) \geq \lambda + \nu + \gamma \frac{\lambda + \nu}{\mu + d} - \frac{\gamma \beta \xi_1}{1 + a\xi_1^2} \frac{\lambda + \nu}{(\mu + d)^2} - 3\gamma \xi_1 - \left(\mu + d + \frac{\beta \xi_1}{1 + a\xi_1^2(t)} + \gamma \right) S(t), \tag{4.40}$$

Consider the comparaison impulsive differential equation for$t > k_8 T$ and $k > k_8$,

$$\begin{cases} \dot{u}(t) = \left(\lambda + \nu + \gamma \frac{\lambda + \nu}{\mu + d} - \frac{\gamma \beta \xi_1}{1 + a\xi_1^2} \frac{\lambda + \nu}{(\mu + d)^2} - 3\gamma \xi_1 \right) - \left(\mu + d + \frac{\beta \xi_1}{1 + a\xi_1^2(t)} + \gamma \right) u(t), t \neq kT \\ u(t^+) = (1 - \theta) u(t), t = kT \end{cases} \tag{4.41}$$

With lemma 1, we have the solution

$$\tilde{u_e}(t) = \Psi + (u^* - \Psi)\, e^{-\left(\mu+d+\frac{\beta\xi_1}{1+a\xi_1^2(t)}+\gamma\right)(t-kT)}, kT < t \leq (k+1)\,T,$$

Which is globally asymptotically stable, where

$$\Psi = \frac{\lambda+\nu+\gamma\frac{\lambda+\nu}{\mu+d} - \frac{\gamma\beta\xi_1}{1+a\xi_1^2}\frac{\lambda+\nu}{(\mu+d)^2} - 3\gamma\xi_1}{\mu+d+\frac{\beta\xi_1}{1+a\xi_1^2(t)}+\gamma}, u^* = \Psi\frac{(1-\theta)\left(1-e^{-\left(\mu+d+\frac{\beta\xi_1}{1+a\xi_1^2(t)}+\gamma\right)T}\right)}{1-(1-\theta)\,e^{-\left(\mu+d+\frac{\beta\xi_1}{1+a\xi_1^2(t)}+\gamma\right)T}}.$$

Consider the comparaison impulsive differential equation, there exists an integer $k_9 > k_8$,

$$S(t) > \tilde{u_e}(t) - \xi_1, kT < t \leq (k+1)\,T, k > k_9 \tag{4.42}$$

From (4.20), (4.42), we have

$$\tilde{S_e}(t) = \frac{\lambda+\nu}{\mu+d}\left(1 - \frac{\theta}{1-(1-\theta)\,e^{-(\mu+d+\gamma)T}}e^{-(\mu+d+\gamma)(t-kT)}\right), kT < t \leq (k+1)\,T, \tag{4.43}$$

(4.43) is globally attractive, then

$$lim_{t\to\infty}S(t) = \tilde{S}(t). \tag{4.44}$$

From (4.43), we have

$$lim_{t\to\infty}E(t) = 0. \tag{4.45}$$

We have

$$lim_{t\to\infty}Q(t) = \frac{\lambda+\nu}{\mu+d} - \tilde{S_e}(t), \tag{4.46}$$

From $(4.10), (4.24), (4.28), (4.32), (4.36), (4.44), (4.45), and\,(4.46)$, the infection free periodic solution $\left(\tilde{S_e}(t), 0, 0, 0, 0, \frac{\lambda+\nu}{\mu+d} - \tilde{S_e}(t), \frac{\lambda+\nu}{\mu+d}\right)$ is globally attractive.

5. Conclusion

This paper addresses the equilibrium and stability of the epidemic model with temporary immunity and saturated incidence rate. Both trivial and endemic equilibrium are founds.

The disease-free equilibrium E_0 is globally asymptotically stable, if $R_0 < 1$, and the system has a unique non-trivial equilibrium E_τ^* which is globally asymptotically stable if $R_0 > 1$. We study stochastic stability of the free disease equilibrium under some conditions, and finally we analyse impulsive vaccination.using theorem 5 imply that the disease dynamics of (22) is determined with $Q^* < 1$.

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