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η-Ricci Soliton in an Indefinite Trans-Sasakian Manifold Admitting Semi-Symmetric Metric Connection

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abstract: In this paper, we intend to study some of the curvature tensor of η-Ricci solitons of indefinite Trans-Sasakian manifold admitting semi-symmetric metric connection.

Key Words: Indefinite Trans-Sasakian manifold, η-Ricci solitons, quasi-conformal curvature tensor, pseudo projective curvature tensor.

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1. Introduction

In the past years Hamilton studied Ricci flow [4] and proved it was in the state of being. This concept was developed to render answer to geometric conjecture of Thurston's theory. Hamilton [5] and Perelman [6] have made a contribution on the study of Ricci flow and the Ricci flow equation is given by

$$\frac{dg}{dt} = -2S \tag{1.1}$$

Generally it has been noted that η-Ricci soliton was introduced by Cho and Kimura [3], further enhanced by Calin and Crasmareanu [2] on Hopf hypersurfaces in complex space forms. In [8], the authors obtain some results on indefinite trans-Sasakian manifold. In this paper, we obtain some results on η-Ricci solitons on some curvature tensors.

Let $(M, \varphi, \xi, \eta, g)$ be an indefinite trans-Sasakian manifold with semi-symmetric metric connection. Considering η-Ricci soliton equation :

$$L_\xi g + 2S + 2\lambda g + 2\mu\eta(X_1)\eta(Y_1) = 0, \tag{1.2}$$

where L_ξ is the Lie derivative operator along the vector field ξ, S is the Ricci tensor field of the metric g. Here λ and μ are real constants.

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In [10], we studied the C-Bochner curvature tensor under D-homothetic deformation in LP-Sasakian manifolds. Further, in [11], we studied some results on indefinite Sasakian manifold admitting quarter-symmetric metric connection and η-Ricci solitons of some curvature tensors.

Somashekhara et al. [9], proved some results on invariant sub-manifolds of LP-Sasakian manifolds endowed with semi-symmetric metric conection and shown that the LP-Sasakian manifold is totally geodesic.

2. Preliminaries

A smooth manifold of dimension (M^n, g) is said to be indefinite almost contact metric manifold [1], if it admits a $(1,1)$ tensor field φ, a structure vector field ξ, a 1-form η and an indefinite metric g such that

$$\varphi\xi = 0,\ \eta(\varphi X_1) = 0\ \eta(\xi) = 1,\ g(\xi.\xi) = \varepsilon, \tag{2.1}$$
$$\eta(X_1) = \varepsilon g(\xi, X_1),\ g(\varphi X_1, \varphi Y_1) = g(X_1, Y_1) - \eta(X_1)\eta(Y_1), \tag{2.2}$$
$$g(\varphi X_1, Y_1) = -g(X_1, \varphi Y_1),\ g(\varphi X_1, X_1) = 0, \varphi^2 X_1 = -X_1 + \eta(X_1)\xi, \tag{2.3}$$

$\forall$ vector fields X_1,Y_1 on M^n, where ε is 1 or -1 accordingly as ξ is space like or time like vector field and rank φ is $n-1$. If

$$d\eta(X_1, Y_1) = g(X_1, \varphi Y_1), \tag{2.4}$$

then $M^n(\varphi, \xi, \eta, g)$ is called an Indefinite contact metric manifold. An Indefinite almost contact metric manifold is said to be Idefinite Trans-Sasakian manifold if

$$(\nabla_{X_1}\varphi)Y_1 = \alpha[g(X_1, Y_1)\xi - \varepsilon\eta(Y_1)X_1] + \beta[g(\varphi X_1, Y_1))\xi - \varepsilon\eta(Y_1)\varphi X_1], \tag{2.5}$$

for any $X_1 Y_1 \ \epsilon\ \Gamma(T\,M^n)$, where ∇ is a metric connection of indefinite metric g, α and β are smooth function on M^n.

Using (2.1), (2.2), (2.3), (2.4) and (2.5), we get some relations:

$$\nabla_{X_1}\xi = \varepsilon\left[-\alpha\varphi X_1 + \beta(X_1 - \eta(X_1)\xi)\right], \tag{2.6}$$
$$\left(\nabla_{X_1}\eta\right)Y_1 = -\alpha g(\varphi X_1, Y_1) + \beta[g(X_1, Y_1) - \varepsilon\eta(X_1)\eta(Y_1)]. \tag{2.7}$$

In reference to [8] the following conditions holds:

$$\begin{aligned}&\tilde{R}(X_1, Y_1)\xi = (\alpha^2 - \beta^2 - \varepsilon\beta)[\eta(Y_1)X_1 - \eta(X_1)Y_1] +\\ &(2\alpha\beta + \varepsilon\alpha)[\eta(Y_1)\varphi X_1 - \eta(X_1)\varphi Y_1] +\\ &\varepsilon[(Y_1\alpha)\varphi X_1 - (X_1\alpha)\varphi Y_1 + (Y_1\beta)\varphi^2 X_1 - (X_1\beta)\varphi^2 Y_1],\end{aligned} \tag{2.8}$$
$$\begin{aligned}&\tilde{R}(\xi, Y_1)Z_1 = (\varepsilon + 2\beta)\left[g(Y_1, Z_1)\xi - \varepsilon\eta(Z_1)Y_1\right] +\\ &(\varepsilon + \beta)[\varepsilon\eta(Y_1)\eta(Z_1)\xi - g(Y_1, Z_1)\xi] + (1 + \varepsilon\beta)\eta(Z_1)[Y_1 - \eta(Y_1)\xi] +\\ &\alpha[g(\varphi Y_1, Z_1)\xi - \varepsilon\eta(Z_1)\varphi Y_1] + (\alpha^2 - \beta^2)[\varepsilon g(Y_1, Z_1)\xi -\\ &\eta(Z_1)Y] + 2\alpha\beta[\varepsilon g(Y_1, \varphi Z_1)\xi + \eta(Z_1)\varphi Y_1] + \varepsilon(Z_1\alpha)\varphi Y_1 +\\ &\varepsilon g(Y_1, \varphi Z_1)grad\alpha - \varepsilon g(\varphi Y_1, \varphi Z_1)grad\beta + \varepsilon(Z_1\beta)[Y_1 - \eta(Z_1)\xi],\end{aligned} \tag{2.9}$$
$$\begin{aligned}&\tilde{S}(Y_1, Z_1) = S(Y_1, Z_1) - [(2\beta + \varepsilon)(n-2) + \beta]g(Y_1, Z_1) +\\ &(1 + \varepsilon\beta)(n-2)\eta(Y_1)\eta(Z_1) + a(n-2)g(\varphi Y_1, Z_1),\end{aligned} \tag{2.10}$$
$$\tilde{Q}Y_1 = QY_1 - [(2\beta + \varepsilon)(n-2) + \beta]Y_1 + (1 + \varepsilon\beta)(n-2)\eta(Y_1)\xi + a(n-2)\varphi Y_1. \tag{2.11}$$

Knowing the relation between Ricci tensor S and Ricci operator Q, we have:

$$S(X_1, Y_1) = g(QX_1, Y_1).$$

From the Lie derivative definition of η-Ricci soliton, we have:

$$S(X_1, Y_1) = A_1 g(X_1, Y_1) + A_2 \eta(X_1)\eta(Y_1), \tag{2.12}$$

where

$$A_1 = -[\varepsilon\beta + \lambda],\ \ A_2 = [\varepsilon\beta - \mu].$$

We now have:

$$QX_1 = A_1 X_1 + A_2 \eta(X_1)\xi \tag{2.13}$$

$$S(X_1, \xi) = A_3 \eta(X_1), \tag{2.14}$$

where

$$A_3 = (\varepsilon A_1 + A_2)$$

. In the view of (2.12), (2.13), equations (2.10) and (2.11) reduces to

$$\tilde{S}(X_1, Y_1) = A_4 g(X_1, Y_1) + A_5 \eta(X_1)\eta(Y_1) + A_6 g(\varphi X_1, Y_1), \tag{2.15}$$

where

$$A_4 = A_1 - ((2\beta + \varepsilon)(n-2) + \beta), A_5 = A_2 + (1 + \varepsilon\beta)(n-2), A_6 = \alpha(n-2).$$

We now have:

$$\tilde{Q}X_1 = A_4 X_1 + A_5 \eta(X_1)\xi + A_6 \varphi X_1, \tag{2.16}$$

$$\tilde{S}(X_1, \xi) = A_7 \eta(X_1), \tag{2.17}$$

where

$$A_7 = \varepsilon A_4 + A_5.$$

$$\tilde{Q}\xi = A_8 \xi, \tag{2.18}$$

where

$$A_8 = A_4 + A_5.$$

3. η-Ricci Solitons in an Indefinite Trans-Sasakian Manifold with Semi-Symmetric Metric Connection Satisfying $\tilde{R}(\xi, X)\tilde{C} = 0$

Let (M^n, g) be a smooth manifold indefinite trans-Sasakian manifold admitting η-Ricci soliton (g, V, λ) with semi-symmetric metric connection. Quasi conformal curvature tensor $\tilde{C}$ on M is defined by

$$\begin{aligned}&\tilde{C}(X_1, Y_1)Z_1 = a\tilde{R}(X_1, Y_1)Z_1 + b[\tilde{S}(Y_1, Z_1)X_1 - \tilde{S}(X_1, Z_1)Y_1 + \\ &g(Y_1, Z_1)\tilde{Q}X_1 - g(X_1, Z_1)\tilde{Q}Y_1] - \left[\frac{r}{n}\right]\left[\frac{a}{n-1} + 2b\right][g(Y_1, Z_1)X_1 - g(X_1, Z_1)Y_1],\end{aligned} \tag{3.1}$$

where r is scalar curvature.
Substitute $Z_1 = \xi$ and equation (3.1) reduces to

$$\begin{aligned}&\tilde{C}(X_1, Y_1)\xi = a\tilde{R}(X_1, Y_1)\xi + b[\tilde{S}(Y_1, \xi)X_1 - \tilde{S}(X_1, \xi)Y_1 + \\ &g(Y_1, \xi)\tilde{Q}X_1 - g(X_1, \xi)\tilde{Q}Y_1] - \left[\frac{r}{n}\right]\left[\frac{a}{n-1} + 2b\right][g(Y_1, \xi)X_1 - g(X_1, \xi)Y_1].\end{aligned} \tag{3.2}$$

In view of the equations (2.1), (2.2), (2.8), (2.15) and (2.16), equation (3.2) reduces to

$$\tilde{C}(X_1,Y_1)\xi = A_9[\eta(Y_1)X_1 - \eta(X_1)Y_1] + A_{10}[\eta(Y_1)\varphi X_1 - \eta(X_1)\varphi Y_1], \tag{3.3}$$

where

$$A_9 = \left(a(\alpha^2 - \beta^2 - \varepsilon\beta) + A_7 b + A_4 b\varepsilon - \frac{r}{n}\left[\frac{a}{n-1} + 2b\right]\xi\right)$$

,

$$A_{10} = (a\alpha\varepsilon + A_6 b\varepsilon)\,.$$

Taking inner product with Z_1, we get

$$\begin{aligned} &-\eta(\tilde{C}(X_1,Y_1)Z_1) = A_9[\eta(Y_1)g(X_1,Z_1) - \eta(X_1)g(Y_1,Z_1)]+\\ &A_{10}[\eta(Y_1)g(\varphi X_1,Z_1) - \eta(X_1)g(\varphi Y_1,Z_1)]. \end{aligned} \tag{3.4}$$

We now assume that the condition $\tilde{R}(\xi,X_1).\tilde{C} = 0$, then we have

$$\begin{aligned} &\tilde{R}(\xi,X_1)(\tilde{C}(Y_1,Z_1)X_2) - \tilde{C}(\tilde{R}(\xi,X_1)Y_1,Z_1)X_2 -\\ &\tilde{C}(Y_1,\tilde{R}(\xi,X_1)Z_1)X_2 - \tilde{C}(Y_1,Z_1)\tilde{R}(\xi,X_1)X_2 = 0, \end{aligned} \tag{3.5}$$

for all vector fields X_1, Y_1, Z_1, X_2 on M.
Substituting $X_2 = \xi$, we get

$$\begin{aligned} &\tilde{R}(\xi,X_1)(\tilde{C}(Y_1,Z_1)\xi) - \tilde{C}(\tilde{R}(\xi,X_1)Y_1,Z_1)\xi -\\ &\tilde{C}(Y_1,\tilde{R}(\xi,X_1)Z_1)\xi - \tilde{C}(Y_1,Z_1)\tilde{R}(\xi,X_1)\xi = 0. \end{aligned} \tag{3.6}$$

Using equation (2.9) and (3.4) and taking inner product with ξ, we get

$$\begin{aligned} &(\beta\varepsilon + \alpha^2 - \beta^2)[g(X_1,\tilde{C}(Y_1,Z_1))\xi + \eta(\tilde{C}(Y_1,Z_1)X_1)]+\\ &\alpha\varepsilon[g(\varphi X_1,\tilde{C}(Y_1,Z_1))\xi + \eta(\tilde{C}(Y_1,Z_1)\varphi X_1)] = 0. \end{aligned} \tag{3.7}$$

Substituting $Z_1 = \xi$, we get

$$\begin{aligned} &(\beta\varepsilon + \alpha^2 - \beta^2)[g(X_1,\tilde{C}(Y_1,\xi))\xi + \eta(\tilde{C}(Y_1,\xi)X_1)]+\\ &\alpha\varepsilon[g(\varphi X_1,\tilde{C}(Y_1,\xi))\xi + \eta(\tilde{C}(Y_1,\xi)\varphi X_1)] = 0. \end{aligned} \tag{3.8}$$

Again from (3.1), we get

$$\begin{aligned} &\tilde{C}(X_1,\xi)Z_1 = a\tilde{R}(X_1,\xi)Z_1 + b[\tilde{S}(\xi,Z_1)X_1 - \tilde{S}(X_1,Z_1)\xi+\\ &g(\xi,Z_1)\tilde{Q}X_1 - g(X_1,Z_1)\tilde{Q}\xi] - \left[\frac{r}{n}\right]\left[\frac{a}{n-1} + 2b\right][g(\xi,Z_1)X_1 - g(X_1,Z_1)\xi]. \end{aligned} \tag{3.9}$$

In view of equations (2.1), (2.2), (2.9), (2.10), (2.11) and taking inner product with ξ in (3.8), we get:

$$\begin{aligned} &A_{11}g(X_1,Y_1) + A_{12}\eta(X_1)\eta(Y_1) + A_{13}g(X_1,\varphi Y_1) -\\ &A_{14}\tilde{S}(X_1,Y_1) + A_{15}g(\varphi X_1,Y_1) + A_{16}g(\varphi X_1,\varphi Y_1) - A_{17}\tilde{S}(\varphi X_1,Y_1) = 0, \end{aligned} \tag{3.10}$$

where

$$
\begin{aligned}
A_{11} &= A_9 + \left[a(\beta^2-\alpha^2-\varepsilon\beta) - A_8 + \frac{r}{n}\left(\frac{a}{n-1}+2b\right)\varepsilon\right](\beta\varepsilon+\alpha^2-\beta^2),\\
A_{12} &= -\left[\varepsilon A_9 - a\left(\beta+\varepsilon(\alpha^2-\beta^2)\right) + (A_7+A_5+A_4)b\varepsilon - \frac{r}{n}\left(\frac{a}{n-1}+2b\right)\right](\beta\varepsilon+\alpha^2-\beta^2),\\
A_{13} &= [A_{10} - a\varepsilon\alpha](\beta\varepsilon+\alpha^2-\beta^2),\\
A_{14} &= b\varepsilon\left(\beta\varepsilon+\alpha^2-\beta^2\right),\\
A_{15} &= \left[A_9 + a(\beta^2-\alpha^2-\varepsilon\beta) - A_8 - \frac{r}{n}\left(\frac{a}{n-1}+2b\right)\varepsilon\right]\varepsilon\alpha,\\
A_{16} &= (A_{10} - a\varepsilon\alpha)\,\varepsilon\alpha,\\
A_{17} &= b\alpha.
\end{aligned}
$$

Again using (2.10), we get

$$A_{18}g(X_1,Y_1) + A_{19}\eta(X_1)\eta(Y_1) + A_{20}g(\varphi X_1,Y_1) - A_{17}S(\varphi X_1,Y_1) = A_{14}S(X_1,Y_1), \tag{3.11}$$

where

$$
\begin{aligned}
A_{18} &= A_{11} + [(2\beta+\varepsilon)(n-2)+\beta]A_{14} + A_{16} + A_{17}\alpha(n-2),\\
A_{19} &= A_{12} - A_{14}(1+\varepsilon\beta)(n-2) - A_{16} - \alpha\varepsilon(n-2)A_{17},\\
A_{20} &= A_{15} - A_{13} - A_{14}(n-2)\alpha + A_{17}[(2\beta+\varepsilon)(n-2)+\beta].
\end{aligned}
$$

Further using (2.1), (2.2), (2.3) and (2.12), equation (3.11) reduces to

$$S(X_1,Y_1) = G_1 g(X_1,Y_1) + G_2\eta(X_1)\eta(Y_1) \tag{3.12}$$

where

$$G_1 = \frac{A_1}{A_{14}},\ G_2 = \frac{A_2}{A_{14}}.$$

Henceforth, we can state the following result:

Theorem 3.1. *An indefinite trans-Sasakian manifold with semi-symmetric metric connection admitting η-Ricci soliton and is quasi-conformally semi-symmetric, that is $\tilde{R}(\xi,X).\tilde{C}=0$, then the manifold is η-Einstein manifold, where $\tilde{C}$ is quasi conformal curvature tensor and $\tilde{R}(\xi,X)$ is derivation of tensor algebra of the tangent space of the manifold.*

4. η-Ricci solitons in an Indefinite Trans-Sasakian Manifold with Semi-Symmetric Metric Connection Satisfying $\tilde{R}(\xi,X)\tilde{P}=0$

Let (M^n,g) be a smooth manifold indefinite trans-Sasakian manifold admitting η-Ricci soliton (g,V,λ) with semi-symmetric metric connection. Pseudo projective curvature tensor $\tilde{P}$ on M is defined by

$$
\begin{aligned}
&\tilde{P}(X_1,Y_1)Z_1 = a\tilde{R}(X_1,Y_1)Z_1 + b[\tilde{S}(Y_1,Z_1)X_1 - \tilde{S}(X_1,Z_1)Y_1] - \\
&\frac{r}{n}\left[\frac{a}{n-1}\right][g(Y_1,Z_1)X_1 - g(X_1,Z_1)Y_1],
\end{aligned} \tag{4.1}
$$

where r is scalar curvature.
Substituting $Z_1=\xi$ and equation (4.1) reduces to

$$
\begin{aligned}
&\tilde{P}(X_1,Y_1)\xi = a\tilde{R}(X_1,Y_1)\xi + b[\tilde{S}(Y_1,\xi)X_1 - \tilde{S}(X_1,\xi)Y_1] - \\
&\frac{r}{n}\left[\frac{a}{n-1}\right][g(Y_1,\xi)X_1 - g(X_1,\xi)Y_1].
\end{aligned} \tag{4.2}
$$

In view of (2.1), (2.2), (2.8), (2.15) and (2.16), equation (4.2) reduces to

$$\tilde{P}(X_1, Y_1)\xi = A_{21}[\eta(Y_1)X_1 - \eta(X_1)Y_1)] + A_{22}[\eta(Y_1)\varphi X_1 - \eta(X_1)\varphi Y_1)], \tag{4.3}$$

where

$$A_{21} = a(\alpha^2 - \beta^2 - \varepsilon\beta) + A_7 b - \frac{r}{n}\left[\frac{a}{n-1}\right]\varepsilon, \;\; A_{22} = a\varepsilon\alpha.$$

Taking inner product with Z_1, we get

$$\begin{aligned}&\eta(\tilde{P}(X_1, Y_1))Z_1 = A_{21}[\eta(Y_1)g(X_1, Z_1) - \eta(X_1)g(Y_1, Z_1)]+\\ &A_{22}[\eta(Y_1)g(\varphi X_1, Z_1) - \eta(X_1)g(\varphi Y_1, Z_1)].\end{aligned} \tag{4.4}$$

Assuming condition $\tilde{R}(\xi, X).\tilde{P} = 0$ holds in M, we have

$$\begin{aligned}&\tilde{R}(\xi, X_1)(\tilde{P}(Y_1, Z_1)X_2) - \tilde{P}(\tilde{R}(\xi, X_1)Y_1, Z_1)X_2 - \tilde{P}(Y_1, \tilde{R}(\xi, X_1)Z_1)X_2-\\ &\tilde{P}(Y_1, Z_1)\tilde{R}(\xi, X_1)X_2 = 0,\end{aligned} \tag{4.5}$$

for all vector fields X_1, Y_1, Z_1 and X_2 on M.
Now substituting $X_2 = \xi$

$$\begin{aligned}&\tilde{R}(\xi, X_1)(\tilde{P}(Y_1, Z_1)\xi) - \tilde{P}(\tilde{R}(\xi, X_1)Y_1, Z_1)\xi - \tilde{P}(Y_1, \tilde{R}(\xi, X_1)Z_1)\xi-\\ &\tilde{P}(Y_1, Z_1)\tilde{R}(\xi, X_1)\xi = 0.\end{aligned} \tag{4.6}$$

Using the equations (2.9), (3.4) and contracting with ξ, we have

$$\begin{aligned}&(\alpha^2 - \beta^2 + \varepsilon\beta)[g(X_1, \tilde{P}(Y_1, Z_1)\xi) + \eta(\tilde{P}(Y_1, Z_1)X_1)]+\\ &\alpha\varepsilon[g(\varphi X_1, \tilde{P}(Y_1, Z_1)\xi) + \eta(\tilde{P}(Y_1, Z_1)\varphi X_1)] = 0.\end{aligned} \tag{4.7}$$

Substitute $Z_1 = \xi$, we get

$$\begin{aligned}&(\alpha^2 - \beta^2 + \varepsilon\beta)[g(X_1, \tilde{P}(Y_1, \xi)\xi) + \eta(\tilde{P}(Y_1, \xi)X_1)]+\\ &\alpha\varepsilon[g(\varphi X_1, \tilde{P}(Y_1, \xi)\xi) + \eta(\tilde{P}(Y_1, \xi)\varphi X_1)] = 0.\end{aligned} \tag{4.8}$$

Again from (4.1), we get

$$\begin{aligned}&\tilde{P}(X_1, \xi)Z_1 = a\tilde{R}(X_1, \xi)Z_1 + b[\tilde{S}(\xi, Z_1)X_1 - \tilde{S}(X_1, Z_1)\xi]-\\ &\frac{r}{n}\left[\frac{a}{n-1}\right][g(\xi, Z_1)X_1 - g(X_1, Z_1)\xi].\end{aligned} \tag{4.9}$$

Using (2.1), (2.2), (2.9), (2.10), (2.11) and contracting with ξ in (4.9), we have

$$\begin{aligned}&A_{23}g(X_1, Y_1) + A_{24}\eta(X_1)\eta(Y_1) + A_{25}g(X_1, \varphi Y_1) - A_{26}\tilde{S}(X_1, Y_1)+\\ &A_{27}g(\varphi X_1, Y_1) + A_{28}g(\varphi X_1, \varphi Y_1) - A_{29}\tilde{S}(\varphi X_1, Y_1) = 0,\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}
A_{23} &= A_{21} + \left[a(\beta^2 - \alpha^2 - \varepsilon\beta) + \frac{r}{n}\left(\frac{a}{n-1}\right)\varepsilon\right](\beta\varepsilon + \alpha^2 - \beta^2),\\
A_{24} &= -\left[\varepsilon A_{21} + a\left(\beta + \varepsilon(\alpha^2 - \beta^2)\right) - A_7 b\varepsilon - \frac{r}{n}\left(\frac{a}{n-1}\right)\right](\beta\varepsilon + \alpha^2 - \beta^2),\\
A_{25} &= [A_{22} - a\varepsilon\alpha](\beta\varepsilon + \alpha^2 - \beta^2),\\
A_{26} &= b\varepsilon(\beta\varepsilon + \alpha^2 - \beta^2),\\
A_{27} &= (A_{21} + a(\beta^2 - \alpha^2 - \varepsilon\beta))\alpha\varepsilon,\\
A_{28} &= (A_{22} - a\alpha\xi)\alpha\varepsilon,\\
A_{29} &= b\alpha.
\end{aligned}$$

Again using (2.10), we get

$$A_{30}g(X_1, Y_1) + A_{31}\eta(X_1)\eta(Y_1) + A_{32}g(\varphi X_1, Y_1) - A_{29}S(\varphi X_1, Y_1) - A_{26}S(X_1, Y_1) = 0, \tag{4.11}$$

where

$$A_{30} = A_{23} + [(2\beta + \varepsilon)(n-2) + \beta]A_{26} + A_{28} + \alpha(n-2)A_{29},$$
$$A_{31} = A_{24} - A_{26}(1 + \varepsilon\beta)(n-2) - A_{28} - A_{29}\varepsilon\alpha(n-2),$$
$$A_{32} = A_{27} - A_{25} - A_{26}\alpha(n-2) + A_{29}[(2\beta + \varepsilon)(n-2) + \beta].$$

Further using (2.1), (2.2), (2.3) and (2.12) equation (4.11) reduces to

$$S(X_1, Y_1) = G_3 g(X_1, Y_1) + G_4\eta(X_1)\eta(Y_1), \tag{4.12}$$

where

$$G_3 = \frac{A_1}{A_{26}}, \; G_4 = \frac{A_2}{A_{26}}.$$

In view of the above, we have the following:

Theorem 4.1. *An indefinite trans-Sasakian manifold with semi-symmetric metric connection admitting η-Ricci soliton and is pseudo projective semi-symmetric, that is $\tilde{R}(\xi, X_1).\tilde{P} = 0$, then the manifold is η-Einstein manifold, where $\tilde{P}$ is pseudo projective curvature tensor and $\tilde{R}(\xi, X_1)$ is derivation of tensor algebra of the tangent space of the manifold.*

5. η-Ricci Solitons in an Indefinite Trans-Sasakian Manifold with Semi-Symmetric Metric Connection Satisfying $\tilde{R}.\tilde{S} = 0$

Let (M^n, g) be a smooth manifold indefinite trans-Sasakian manifold admitting η-Ricci soliton (g, V, λ) with semi-symmetric metric connection. Assuming the condition $\tilde{R}(X_1, Y_1).\tilde{S} = 0$ holds:

$$\tilde{S}(\tilde{R}(X_1, Y_1)Z_1, X_2) + \tilde{S}(Z_1, \tilde{R}(X_1, Y_1), X_2) = 0. \tag{5.1}$$

Substituting $X_1 = \xi$ in (5.1), we have

$$\tilde{S}(\tilde{R}(\xi, Y_1)Z_1, X_2) + \tilde{S}(Z_1, \tilde{R}(\xi, Y_1), X_2) = 0. \tag{5.2}$$

Using (2.1), (2.9), (2.15) and putting $X_2 = \xi$, we get

$$A_{33}g(Y_1, Z_1) + A_{34}\eta(Y_1)\eta(Z_1) + A_{35}g(Y_1, \varphi Z_1) - \alpha\varepsilon S(\varphi Y_1, Z_1) + (\beta^2 - \alpha^2 - \varepsilon\beta)S(Y_1, Z_1) = 0, \tag{5.3}$$

where

$$A_{33} = A_7(\beta + (\alpha^2 - \beta^2)) + \alpha^2\varepsilon(n-2) - (\beta^2 - \alpha^2 - \varepsilon\beta)((2\beta + \varepsilon)(n-2) + \beta),$$
$$A_{34} = A_7\varepsilon\beta - \alpha^2(n-2) + (\beta^2 - \alpha^2 - \varepsilon\beta)(1 + \varepsilon\beta)(n-2),$$
$$A_{35} = A_7\alpha + \alpha\varepsilon((2\beta + \varepsilon)(n-2) + \beta) + \alpha(n-2).$$

Further on simplification, we get

$$S(Y_1, Z_1) = A_1 g(Y_1, Z_1) + A_2\eta(Y_1)\eta(Z_1). \tag{5.4}$$

Theorem can be stated as follows

Theorem 5.1. *An indefinite trans-Sasakian manifold with semi-symmetric metric connection admitting η-Ricci soliton $\tilde{R}(X_1, Y_1).\tilde{S} = 0$ then the manifold is an η-Einstein manifold.*

6. η-Ricci Solitons in an Indefinite Trans-Sasakian Manifold with Semi-Symmetric Metric Connection Satisfying $\tilde{P}.\tilde{S} = 0$

Let M be an n-dimensional indefinite trans-Sasakian manifold with semi-symmetric metric connection admitting η-Ricci soliton (g, V, λ). Since $\tilde{P}$ is the projective curvature tensor and $\tilde{S}$ is the Ricci-tensor in semi-symmetric metric connection are defined by

$$\tilde{P}(X_1, Y_1)Z_1 = \tilde{R}(X_1, Y_1)Z_1 - \frac{1}{n-1}[\tilde{S}(Y_1, Z_1)X_1 - \tilde{S}(X_1, Y_1)Z_1]. \tag{6.1}$$

Assuming the condition $P(X_1, Y_1).\tilde{S}(Z_1, X_2) = 0$ holds:

$$\tilde{S}(\tilde{P}(X_1, Y_1)Z_1, X_2) + \tilde{S}(Z_1, \tilde{P}(X_1, Y_1)X_2) = 0. \tag{6.2}$$

Substituting $X_1 = \xi$ in (6.2), we have

$$\tilde{S}(\tilde{P}(\xi, Y_1)Z_1, X_2) + \tilde{S}(Z_1, \tilde{P}(\xi, Y_1)X_2) = 0. \tag{6.3}$$

Using (2.1), (2.9), (2.15) and (6.1), we get

$$\begin{aligned} &A_7(\beta + \alpha^2 - \beta^2)g(Y_1, Z_1) + A_7 g(\varphi Y_1, Z_1) + \\ &(\beta^2 - \alpha^2 - \varepsilon\beta)\tilde{S}(Y_1, Z_1) - \alpha\varepsilon\tilde{S}(\varphi Y_1, Z_1) = 0. \end{aligned} \tag{6.4}$$

Agian using (2.10), we get

$$\begin{aligned} &A_{36} g(Y_1, Z_1) + A_{37}\eta(Y_1)\eta(Z_1) + A_{38} g(\varphi Y_1, Z_1) - \\ &\alpha\varepsilon S(\varphi Y_1, Z_1) + (\beta^2 - \alpha^2 - \varepsilon\beta)S(Y_1, Z_1) = 0, \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} A_{36} &= A_7(\beta + \alpha^2 - \beta^2) - (\beta^2 - \alpha^2 - \varepsilon\beta)((2\beta + \varepsilon)(n-2) + \beta) + \alpha^2\varepsilon(n-2), \\ A_{37} &= (1 + \varepsilon\beta)(n-2)(\beta^2 - \alpha^2 - \varepsilon\beta) - \alpha^2(n-2), \\ A_{38} &= A_7\alpha + \alpha(n-2)(\beta^2 - \alpha^2 - \varepsilon\beta) + \alpha\varepsilon((2\beta + \varepsilon)(n-2) + \beta). \end{aligned}$$

Further on simplification, we get

$$S(Y_1, Z_1) = A_1 g(Y_1, Z_1) + A_2\eta(Y_1)\eta(Z_1). \tag{6.6}$$

Hence, the result can be stated as:

Theorem 6.1. *An indefinite trans-Sasakian manifold with semi-symmetric metric connection admitting η-Ricci soliton and is projective semi-symmetric, that is $\tilde{P}.\tilde{S} = 0$ then the manifold is an η-Einstein manifold.*

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