Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–12.
ISSN-0037-8712 in press
doi:10.5269/bspm.51960

A Note on Hermite-based Truncated Euler Polynomials

Waseem A. Khan, Divesh Srivastava and Rifaqat Ali

ABSTRACT: In this paper, we introduce a new class of generating function of Hermite-based truncated Euler polynomials which is a generalization of Hermite-Euler polynomials. By making use of this generating function, we obtain some new interesting properties and relations. Furthermore, we derive some new implicit summation formulae.

Key Words: Hermite polynomials, Euler polynomials, truncated Euler polynomials, truncated Hermite-Euler polynomials.

Contents

1 Introduction **1**

2 Hermite-based truncated Euler polynomials **3**

3 Summation formulae for Hermite-based truncated Euler polynomials **9**

4 Conclusion **11**

1. Introduction

Generalized and multivariable forms of the special functions of mathematical physics have witnessed a significant evolution during the recent years. In particular, the special polynomials of two variables provided new means of analysis for the solution of large classes of partial differential equations often encountered in physical problems. Most of the special functions of mathematical physics and their generalizations have been suggested by physical problems. For example (see [4-7], [13-25])

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ are defined by the generating function (see [2], [4]):

$$e^{xt+yt^2}=\sum_{n=0}^{\infty}H_n(x,y)\frac{t^n}{n!}, \tag{1.1}$$

are solutions of the heat equation

$$\frac{\partial}{\partial y}H_n(x,y)=\frac{\partial^2}{\partial x^2}H_n(x,y),$$

$$H_n(x,0)=x^n.$$

For non-negative integer m, the truncated Euler polynomials are defined by means of the following generating function (see [13]):

$$\frac{\frac{2t^m}{m!}}{e^t+1-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}e^{xt}=\sum_{n=0}^{\infty}E_{m,n}(x)\frac{t^n}{n!}. \tag{1.2}$$

Taking $m=0$ in (1.2) reduces to

2010 *Mathematics Subject Classification:* Primary: 11B68; Secondary: 11B83, 33C45, 05A15, 05A19.
Submitted January 26, 2020. Published April 19, 2020

$$\frac{2}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}E_{0,n}(x)\frac{t^n}{n!},(|\ t\ |<\pi), \tag{1.3}$$

where, $E_n(x)$ are called the Euler polynomials (see [23]).

When $x=0$ in (1.2), $E_{m,n}=E_{m,n}(0)$ are called the truncated Euler numbers given by

$$\frac{\frac{2t^m}{m!}}{e^t+1-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}=\sum_{n=0}^{\infty}E_{m,n}\frac{t^n}{n!}. \tag{1.4}$$

Recently, Hassen et al. [9] introduced hypergeometric Bernoulli polynomials by means of the following generating function:
For $N\geq 1$, the hypergeometric Bernoulli numbers $B_{N,n}$ are defined by means of the following generating function

$$\frac{1}{{}_1F_1(1;N+1;t)}=\frac{\frac{t^N}{N!}}{e^t-\sum\limits_{n=0}^{N-1}\frac{t^n}{n!}}=\sum_{n=0}^{\infty}B_{N,n}\frac{t^n}{n!},$$

where

$${}_1F_1(a;b;z)=\sum_{n=0}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!}$$

is the confluent hypergeometric function with $(x)_n=x(x+1)\cdots(x+n-1),(n\geq 1)$ and $(x)_0=1$. When $N=1$, $B_n=B_{1,n}$ are called the classical Bernoulli numbers (with $B_1=-\frac{1}{2}$) defined by

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}.$$

In particular, the hypergeometric Bernoulli polynomials are defined by means of the following generating function (see [8], [9], [12]):

$$\frac{\frac{t^m}{m!}}{e^t-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}e^{xt}=\sum_{n=0}^{\infty}B_{m,n}(x)\frac{t^n}{n!}, \tag{1.5}$$

or

$$\frac{e^{xt}}{{}_1F_1(1;m+1;t)}=\sum_{n=0}^{\infty}B_{m,n}(x)\frac{t^n}{n!}. \tag{1.6}$$

If we take $m=0$, Eq. (1.5) becomes (see [13]):

$$e^{(x-1)t}=\sum_{n=0}^{\infty}B_{0,n}(x)\frac{t^n}{n!},$$

from where we see that $B_{0,n}(x)=(x-1)^n$.

On setting $m=1$ in (1.5), $B_n(x)=B_{1,n}(x)$ are called the classical Bernoulli polynomials given by

$$\frac{t}{e^t-1}e^{xt}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}.$$

For $\lambda \in \mathbb{C}$ with $\lambda \neq 1$ and a non negative integer r, Frobenius-Euler polynomials $H_n^{(r)}(x|\lambda)$ are defined by means of the following generating function (see [1])

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!}. \tag{1.7}$$

Observe that $H_n^{(1)}(x,\lambda) = H_n(x,\lambda)$, which denotes the Frobenius-Euler polynomials and $H_n^{(r)}(0;\lambda) = H_n^{(r)}(\lambda)$, which denotes the Frobenius-Euler numbers of order r. $H_n(x;-1) = E_n(x)$, which denotes the Euler polynomials (see [3], [8-11]).

In the year 2014, Pathan and Khan [23] introduced generalized Hermite-Euler polynomials of two variable by means of the following generating function

$$\left(\frac{2}{e^t+1}\right)^\alpha e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HE_n^{(\alpha)}(x,y)\frac{t^n}{n!}. \tag{1.8}$$

For $\alpha = 1$, Eq. (1.8) reduces to the known result of Dattoli et al. [4].

The generating function of the Stirling numbers of the second kind denoted by $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ is given by ([27])

$$\frac{(e^t-1)^k}{k!} = \sum_{n=k}^{\infty} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{t^n}{n!}. \tag{1.9}$$

The falling factorial $(x)_n$ and the rising factorial $(x)^{(n)}$ are defined by $(x)_n = x(x-1)\cdots(x-n+1)$ and $(x)^{(n)} = x(x+1)\cdots(x+n-1), (n \geq 1)$ with $(x)_0 = (x)^{(0)} = 1$, respectively.

The object of this paper is to present a systematic account of these families in a unified and generalized form. We develop some elementary properties and derive the implicit summation formulae for the Hermite-based truncated Euler polynomials by using different analytical means on their respective generating functions. The approach given in recent papers of Khan [16] and Pathan and Khan [23] has indeed allowed the derivation of implicit summation formulae in the two-variable Hermite-Euler polynomials. In addition to this, we give some relevant connections between Hermite and truncated Euler polynomials.

2. Hermite-based truncated Euler polynomials

In this section, we define Hermite-based truncated Euler polynomials ${}_HE_{m,n}(x,y)$ and investigate its properties. Now we start at the following definition.

.

Definition 2.1. *For nonnegative integer m, we define the Hermite-based truncated Euler polynomials ${}_HE_{n,m}(x,y)$ by means of the following generating function in a suitable neighborhood of $t = 0$:*

$$\frac{\frac{2t^m}{m!}}{e^t+1-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HE_{n,m}(x,y)\frac{t^n}{n!}. \tag{2.1}$$

Clearly that (2.1) contains as its special cases not only generalized truncated Euler polynomials $E_{n,m}^{(\alpha)}(x)$ (c.f.Eq.(1.2)) but also Kampé de Fériet generalization of the Hermite polynomials $H_n(x,y)$ (c.f.Eq.(1.1)). If we put $m = 0$, (2.1) gives a known result of Dattoli et al. [[4], p.386(1.6)] in the form

$$\frac{2}{e^t+1}e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HE_n(x,y)\frac{t^n}{n!}, \tag{2.2}$$

where ${}_HE_n(x,y)$ are called the Hermite-Euler polynomials.

In particular, in terms of truncated Euler numbers $E_{n-s,m}$ and Hermite polynomials $H_s(x,y)$, truncated Hermite-Euler polynomials ${}_HE_{n,m}(x,y)$ are represented as

$$ {}_HE_{n,m}(x,y)=\sum_{s=0}^{n}\left(\begin{array}{c} n \\ s \end{array}\right)E_{n-s,m}H_s(x,y). \tag{2.3}$$

It is possible to define truncated Hermite-Euler numbers ${}_HE_{n,m}$ assuming that

$$ {}_HE_{n,m}(0,0)={}_HE_{n,m}.$$

Taking $x=m\ =\ 0$, (2.1) gives the result

$$\sum_{s=0}^{[\frac{n}{2}]}\left(\begin{array}{c} n \\ s \end{array}\right)E_{n-s}y^s={}_HE_n(0,y). \tag{2.4}$$

If we take $m=0$ in (2.1), we can generalize some results involving summations of truncated Hermite-Euler polynomials and numbers by using $e^{it}=\cos t+i\sin t$ and the result

$$\sum_{n=0}^{\infty}f(n)\ =\ \sum_{n=0}^{\infty}f(2n)+\sum_{n=0}^{\infty}f(2n+1). \tag{2.5}$$

Since

$$\begin{aligned}\frac{2}{e^{it}+1}&=\frac{2i(\cos t+1-i\sin t)}{(\cos t+1+i\sin t)(\cos t+1-i\sin t)}=\frac{2i(\cos t+1-i\sin t)}{(\cos t+1)^2+(\sin t)^2}\\ &=\frac{(2i\sin t)+2(\cos t+1)}{\Omega}.\end{aligned} \tag{2.6}$$

Thus, by (2.1) and (2.6), we have

$$\begin{aligned}&e^{ixt-yt^2}\left(\frac{(2i\sin t)+2(\cos t+1)}{\Omega}\right)\\ &=\sum_{n=0}^{\infty}{}_HE_{2n}(x,y)\frac{(-1)^nt^{2n}}{(2n)!}+i\sum_{n=0}^{\infty}{}_HE_{2n+1}(x,y)\frac{(-1)^nt^{2n+1}}{(2n+1)!},\end{aligned} \tag{2.7}$$

where $\Omega=(\cos t+1)^2+(\sin t)^2$.

Therefore, we have

$$\sum_{n=0}^{\infty}{}_HE_{2n}(x,y)\frac{(-1)^nt^{2n}}{(2n)!}\ =\ \frac{e^{-yt^2}}{\Omega}2[\cos(t-xt)+\cos(xt)]. \tag{2.8}$$

$$\sum_{n=0}^{\infty}{}_HE_{2n+1}(x,y)\frac{(-1)^nt^{2n+1}}{(2n+1)!}\ =\ \frac{e^{-yt^2}}{\Omega}2[\sin(t-xt)-\cos(xt)], \tag{2.9}$$

where $\Omega=(\cos t+1)^2+(\sin t)^2$.

Theorem 2.2. *For $n\geq 1$, we have*

$$ {}_HE_{n,1}(x,y)=2nH_n((x-1),y). \tag{2.10}$$

Proof. For $m=1$ in (2.1) and using (1.1), we have

$$\sum_{n=0}^{\infty} {}_HE_{n,1}(x,y)\frac{t^{n-1}}{n!} = 2e^{(x-1)t+yt^2}$$

$$\sum_{n=1}^{\infty} {}_HE_{n,1}(x,y)\frac{t^{n}}{(n+1)!} = 2\sum_{n=1}^{\infty} H_n((x-1),y)\frac{t^n}{n!}.$$

Comparing the coefficients of equal powers of t on both sides, we get the required result. □

Theorem 2.3. *The following relation holds true:*

$$ {}_HE_{m,j}(x,y) = 2\begin{pmatrix} n+m \\ n \end{pmatrix} H_n(x,y) - \sum_{j=0}^{n}\begin{pmatrix} n+m \\ j \end{pmatrix} {}_HE_{m,j}(x,y), (n\geq 0), \tag{2.11}$$

where ${}_HE_{m,n}(x,y)=0, (n=0,1,\cdots,m-1)$.

Proof. From (2.1), we have

$$\frac{2t^m}{m!}e^{xt+yt^2} = \left(\sum_{n=0}^{\infty} {}_HE_{n,m}(x,y)\frac{t^n}{n!}\right)\left(1+\sum_{j=m}^{\infty}\frac{t^j}{j!}\right)$$

$$\frac{2t^m}{m!}\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} = \sum_{n=0}^{\infty} {}_HE_{n,m}(x,y)\frac{t^n}{n!} + \left(\sum_{n=0}^{\infty} {}_HE_{n,m}(x,y)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}\frac{t^{j+m}}{(j+m)!}\right)$$

$$\left(\sum_{n=0}^{\infty} {}_HE_{n,m}(x,y)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}\frac{t^{j+m}}{(j+m)!}\right) = \frac{2t^m}{m!}\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HE_{n,m}(x,y)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} H_n(x,y)\frac{2t^{n+m}}{n!m!} - \sum_{n=0}^{\infty} {}_HE_{n+m,m}(x,y)\frac{t^{n+m}}{(n+m)!} - \sum_{n=0}^{m-1} {}_HE_{n,m}(x,y)\frac{t^n}{n!}. \tag{2.12}$$

Also,

$$\left(\sum_{n=0}^{\infty} {}_HE_{n,m}(x,y)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty}\frac{t^{j+m}}{(j+m)!}\right) = \sum_{n=0}^{\infty}\sum_{j=0}^{n} {}_HE_{m,j}(x,y)\frac{t^j}{j!}\frac{t^{n-j+m}}{(n-j+m)!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\begin{pmatrix} n+m \\ j \end{pmatrix} {}_HE_{m,j}(x,y)\right)\frac{t^{n+m}}{(n+m)!}. \tag{2.13}$$

Comparing the coefficients of t^n in equation (2.12) and (2.13), we get the results (2.11). □

Theorem 2.4. *The following relation holds true:*

$$ {}_HE_{n,m}(x+z,y) = \sum_{s=0}^{n}\begin{pmatrix} n \\ s \end{pmatrix} E_{n-s,m}(z)H_s(x,y). \tag{2.14}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty} {}_HE_{n,m}(x+z,y)\frac{t^n}{n!} = \frac{\frac{2t^m}{m!}}{e^t+1-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}} e^{(x+z)t+yt^2}$$

$$= \frac{\frac{2t^m}{m!}}{e^t+1-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}} e^{zt}e^{xt+yt^2}$$

$$= \left(\sum_{n=0}^{\infty} E_{n,m}(z)\frac{t^n}{n!}\right)\left(\sum_{s=0}^{\infty} H_s(x,y)\frac{t^s}{s!}\right).$$

Replacing n by $n-s$ in above equation and comparing the coefficients of t^n, we get the desired result. □

Theorem 2.5. *The following relation holds true:*

$$ {}_HE_{n,m}(x+z,y) = \sum_{k=0}^{n}\sum_{s=k}^{n} \left\{ \begin{array}{c} s \\ k \end{array} \right\} \left(\begin{array}{c} n \\ s \end{array} \right) {}_HE_{n-s,m}(x,y)(x)_k. \tag{2.15}$$

Proof. By Theorem 2.3, we have

$$ {}_HE_{n,m}(x+z,y) = \sum_{s=0}^{n} \left(\begin{array}{c} n \\ s \end{array} \right) {}_HE_{n-s,m}(x,y)z^s.$$

Using the concept of (1.9), we have

$$= \sum_{s=0}^{n} \left(\begin{array}{c} n \\ s \end{array} \right) {}_HE_{n-s,m}(x,y) \sum_{k=0}^{s} \left\{ \begin{array}{c} s \\ k \end{array} \right\} (x)_k$$

$$= \sum_{k=0}^{n}\sum_{s=k}^{n} \left\{ \begin{array}{c} s \\ k \end{array} \right\} \left(\begin{array}{c} n \\ s \end{array} \right) {}_HE_{n-s,m}(x,y)(x)_k.$$

□

Thus, it completes the proof.

Theorem 2.6. *The following relation holds true:*

$$ {}_HE_{n,m}(x+z,y) = \sum_{\mu=0}^{n}\sum_{l=\mu}^{n} \left(\begin{array}{c} n \\ l \end{array} \right) \left\{ \begin{array}{c} l \\ \mu \end{array} \right\} {}_HE_{n-l,m}(x-\mu,y)(z)^{\mu}. \tag{2.16}$$

Proof. Consider (2.1), we have

$$\sum_{n=0}^{\infty} {}_HE_{n,m}(x+z,y)\frac{t^n}{n!} = \frac{\frac{2t^m}{m!}}{e^t+1-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}} e^{xt+yt^2}(e^{-t})^{-z}$$

$$= \frac{\frac{2t^m}{m!}}{e^t+1-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}} e^{xt+yt^2} \sum_{\mu=0}^{\infty} \left(\begin{array}{c} z+\mu-1 \\ \mu \end{array} \right) (1-e^{-t})^{-z}$$

$$= \frac{\frac{2t^m}{m!}}{e^t+1-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}} e^{xt+yt^2} \sum_{\mu=0}^{\infty} (z)^{\mu}\frac{(e^t-1)^{\mu}}{\mu!}e^{-\mu t}$$

$$= \sum_{\mu=0}^{\infty}(z)^{\mu}\left(\sum_{n=0}^{\infty}\left\{\begin{array}{c} n \\ \mu \end{array}\right\}\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}{}_HE_{n,m}(x-\mu,y)\frac{t^n}{n!}\right)$$

$$= \sum_{\mu=0}^{\infty}(z)^{\mu}\sum_{n=0}^{\infty}\left(\begin{array}{c} n \\ l \end{array}\right)\sum_{l=0}^{n}\left\{\begin{array}{c} l \\ \mu \end{array}\right\}{}_HE_{n-l,m}(x-\mu,y)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{\mu=0}^{n}\sum_{l=\mu}^{n}\left(\begin{array}{c} n \\ l \end{array}\right)\left\{\begin{array}{c} l \\ \mu \end{array}\right\}{}_HE_{n-l,m}(x-\mu,y)(z)^{\mu}\right)\frac{t^n}{n!}.$$

Comparing the coefficients of t^n, we obtain the required result. □

Theorem 2.7. *The following relation holds true:*

$$ {}_HE_{n,m}(x,y) = \sum_{\mu=0}^{n}\frac{1}{(1-\lambda)^r}\left(\begin{array}{c} n \\ \mu \end{array}\right)\sum_{i=0}^{r}\left(\begin{array}{c} r \\ i \end{array}\right)(-1)^{r-i}{}_HE_{n-\mu,m}(0,y)H_{\mu}^{(r)}(x|\lambda). \tag{2.17}$$

Proof. From (2.1), we have

$$\sum_{n=0}^{\infty}{}_HE_{n,m}(x,y)\frac{t^n}{n!} = \frac{\frac{2t^m}{m!}}{e^t+1-\sum_{h=0}^{m-1}\frac{t^h}{h!}}\left(\frac{e^t-\lambda}{1-\lambda}\right)^r\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt+yt^2}$$

$$= \left(\sum_{n=0}^{\infty}{}_HE_{n,m}(0,y)\frac{t^n}{n!}\right)\left(\sum_{i=0}^{r}\left(\begin{array}{c} r \\ i \end{array}\right)(e^{it})(-\lambda)^{r-i}\frac{1}{(1-\lambda)^r}\right)\left(\sum_{\mu=0}^{\infty}H_{\mu}^{(r)}(x|\lambda)\frac{t^{\mu}}{\mu!}\right)$$

$$= \sum_{i=0}^{r}\left(\begin{array}{c} r \\ i \end{array}\right)(-\lambda)^{r-i}\frac{1}{(1-\lambda)^r}(e^{it})\sum_{n=0}^{\infty}\sum_{\mu=0}^{n}\left(\begin{array}{c} n \\ \mu \end{array}\right){}_HE_{n-\mu,m}(0,y)H_{\mu}^{(r)}(x|\lambda)\frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty}\sum_{\mu=0}^{n}\frac{1}{(1-\lambda)^r}\left(\begin{array}{c} n \\ \mu \end{array}\right)\sum_{i=0}^{r}\left(\begin{array}{c} r \\ i \end{array}\right)(-1)^{r-i}{}_HE_{n-\mu,m}(0,y)H_{\mu}^{(r)}(x|\lambda)\frac{t^n}{n!}.$$

Equating the coefficients of equal powers of t on both sides, we obtain the desired result. □

Theorem 2.8. *The following relation holds true:*

$$\left(\begin{array}{c} n+m \\ n \end{array}\right)\sum_{j=0}^{n}\left(\begin{array}{c} n \\ j \end{array}\right)\left(B_{m,j}(x)H_{n-j}(x,y)-\frac{1}{2}{}_HE_{m,j}(x,y)x^{n-j}\right)$$

$$= \frac{1}{2}\sum_{j=0}^{n+m}\left(\begin{array}{c} n+m \\ j \end{array}\right){}_HE_{m,j}(x,y)B_{m,n+m-j}(x). \tag{2.18}$$

Proof. From (2.1), we have

$$\frac{t^m}{m!}e^{xt+yt^2} = \left(\sum_{n=0}^{\infty}\frac{1}{2}{}_HE_{n,m}(x,y)\frac{t^n}{n!}\right)\left(e^t+1-\sum_{h=0}^{m-1}\frac{t^h}{h!}\right)$$

$$= \left(\sum_{n=0}^{\infty}\frac{1}{2}{}_HE_{n,m}(x,y)\frac{t^n}{n!}\right)\left(e^t-\sum_{h=0}^{m-1}\frac{t^h}{h!}\right)+\left(\sum_{n=0}^{\infty}\frac{1}{2}{}_HE_{n,m}(x,y)\frac{t^n}{n!}\right)$$

Now,

$$\frac{t^m}{m!}e^{xt+yt^2}\sum_{n=0}^{\infty}B_{m,n}(x)\frac{t^n}{n!}=\left(\sum_{n=0}^{\infty}\frac{1}{2}{}_HE_{n,m}(x,y)\frac{t^n}{n!}\right)\frac{t^m}{m!}e^{xt}$$
$$+\left(\sum_{n=0}^{\infty}\frac{1}{2}{}_HE_{n,m}(x,y)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}B_{m,n}(x)\frac{t^n}{n!}\right),$$

which can be written as

$$\frac{t^m}{m!}\left(e^{xt+yt^2}\sum_{n=0}^{\infty}B_{m,n}(x)\frac{t^n}{n!}-e^{xt}\sum_{n=0}^{\infty}\frac{1}{2}{}_HE_{n,m}(x,y)\frac{t^n}{n!}\right)$$
$$=\left(\sum_{n=0}^{\infty}\frac{1}{2}{}_HE_{n,m}(x,y)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}B_{m,n}(x)\frac{t^n}{n!}\right).$$

Using Cauchy product rule, we get

$$\frac{t^m}{m!}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\left(B_{m,j}(x)H_{n-j}(x,y)-\frac{1}{2}{}_HE_{m,j}(x,y)x^{n-j}\right)\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\frac{1}{2}\binom{n}{j}{}_HE_{m,j}(x,y)B_{m,n-j}(x)\right)\frac{t^n}{n!}$$
$$=\sum_{n=m}^{\infty}\left(\sum_{j=0}^{n}\frac{1}{2}\binom{n}{j}{}_HE_{m,j}(x,y)B_{m,n-j}(x)\right)\frac{t^n}{n!}.$$

Here, we avoided the zero terms on the right-hand side. Thus

$$\frac{1}{m!}\sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j}\left(B_{m,j}(x)H_{n-j}(x,y)-\frac{1}{2}{}_HE_{m,j}(x,y)x^{n-j}\right)\right)\frac{t^{n+m}}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\left(\sum_{j=0}^{n+m}\frac{1}{2}\binom{n+m}{j}{}_HE_{m,j}(x,y)B_{m,n+m-j}(x)\right)\frac{t^{n+m}}{(n+m)!}\right).$$

Equating the coefficients of equal powers of t on both sides, we get the desired result. □

Theorem 2.9. *For non-negative integer n and m, we have*

$$2\sum_{j=0}^{n}\binom{n}{j}{}_HE_{m+1,n-j}(x,y)z^j-\frac{2n}{m+1}\sum_{j=0}^{n-1}\binom{n-1}{j}E_{m,n-j-1}(z)H_j(x,y)$$
$$=\sum_{j=0}^{n}\binom{n}{j}{}_HE_{m+1,n-j}(x,y)E_{m,j}(z). \tag{2.19}$$

Proof. From the definition (2.1), we have

$$\frac{2t^{m+1}}{(m+1)!}e^{xt+yt^2}=\left(\sum_{n=0}^{\infty}{}_HE_{m+1,n}(x,y)\frac{t^n}{n!}\right)\left(e^t+1-\sum_{j=0}^{m-1}\frac{t^j}{j!}-\frac{t^m}{m!}\right)$$

$$= \left(e^t + 1 - \sum_{j=0}^{m-1} \frac{t^j}{j!}\right)\left(\sum_{n=0}^{\infty} {}_HE_{m+1,n}(x,y)\frac{t^n}{n!}\right) - \frac{t^m}{m!}\sum_{n=0}^{\infty} {}_HE_{m+1,n}(x,y)\frac{t^n}{n!}$$

Now,

$$\frac{2t^{m+1}}{(m+1)!}e^{xt+yt^2}\sum_{n=0}^{\infty} E_{m,n}(z)\frac{t^n}{n!}$$

$$= \frac{2t^m}{m!}e^{zt}\sum_{n=0}^{\infty} {}_HE_{m+1,n}(x,y)\frac{t^n}{n!} - \frac{t^m}{m!}\sum_{n=0}^{\infty} {}_HE_{m+1,n}(x,y)\frac{t^n}{n!}\sum_{n=0}^{\infty} E_{m,n}(z)\frac{t^n}{n!}$$

$$\frac{2n}{m+1}\sum_{n=1}^{\infty}\left(\sum_{j=0}^{n-1}\binom{n-1}{j} E_{m,n-j-1}(z)H_j(x,y)\right)\frac{t^n}{n!}$$

$$= 2\sum_{n=1}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j} {}_HE_{m+1,n-j}(x,y)z^j\right)\frac{t^n}{n!}$$

$$-\sum_{n=1}^{\infty}\left(\sum_{j=0}^{n}\binom{n}{j} {}_HE_{m+1,n-j}(x,y)E_{m,j}(z)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of equal powers of t on both sides, we obtain the desired result. □

3. Summation formulae for Hermite-based truncated Euler polynomials

In this section, we prove the following results involving truncated Euler polynomials and Hermite polynomials.

Theorem 3.1. *The following implicit summation formula holds true:*

$$ {}_HE_{m,k+l}(z,y) = \sum_{n,p=0}^{k,l} \frac{k!l!(z-x)^{n+p}\,{}_HE_{m,k+l-p-n}(x,y)}{(k-n)!(l-p)!n!p!}. \tag{3.1}$$

Proof. Replacing t by $t+u$ and rewrite the generating function (2.1) as

$$\left(\frac{\frac{2t^m}{m!}}{e^{t+u}+1-\sum\limits_{h=0}^{m-1}\frac{(t+u)^h}{h!}}\right)e^{y(t+u)^2} = e^{-x(t+u)}\sum_{k,l=0}^{\infty} {}_HE_{m,k+l}(x,y)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{3.2}$$

Replacing x by z in the above equation and equating the resulting equation to the above equation, we get

$$e^{(z-x)(t+u)}\sum_{k,l=0}^{\infty} {}_HE_{m,k+l}(x,y)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_HE_{m,k+l}(x,y)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{3.3}$$

On expanding exponential function (3.3) gives

$$\sum_{N=0}^{\infty}\frac{[(z-x)(t+u)]^N}{N!}\sum_{k,l=0}^{\infty} {}_HE_{m,k+l}(x,y)\frac{t^k}{k!}\frac{u^l}{l!} = \sum_{k,l=0}^{\infty} {}_HE_{m,k+l}(x,y)\frac{t^k}{k!}\frac{u^l}{l!}, \tag{3.4}$$

which on using formula [[26],p.52(2)]

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{3.5}$$

in the left hand side becomes

$$\sum_{n,p=0}^{\infty}\frac{(z-x)^{n+p}}{n!p!}t^n u^p\sum_{k,l=0}^{\infty}{}_HE_{m,k+l}(x,y)\frac{t^k}{k!}\frac{u^l}{l!}=\sum_{k,l=0}^{\infty}{}_HE_{m,k+l}(z,y)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{3.6}$$

Now replacing k by $k-n$, l by $l-p$ and using the lemma [[26] ,p.100(1)] in the left hand side of (3.6), we get

$$\sum_{n,p=0}^{\infty}\sum_{k,l=0}^{\infty}\frac{(z-x)^{n+p}}{n!p!}{}_HE_{m,k+l-n-p}(x,y)\frac{t^k}{(k-n)!}\frac{u^l}{(l-p)!}=\sum_{k,l=0}^{\infty}{}_HE_{m,k+l}(z,y)\frac{t^k}{k!}\frac{u^l}{l!}. \tag{3.7}$$

Finally on equating the coefficients of the like powers of t and u in the above equation, we get the required result.
□

Remark 3.2. *By taking $l = 0$ in equation (3.1), we immediately deduce the following corollary.*

Corollary 3.3. *The following implicit summation formula holds true:*

$${}_HE_{m,k}(z,y)=\sum_{n=0}^{k}\binom{k}{n}(z-x)^n\,{}_HE_{m,k-n}^{[\alpha,m-1]}(x,y). \tag{3.8}$$

Remark 3.4. *On replacing z by $z+x$ and setting $y=0$ in Theorem 3.1, we get the following result involving truncated Hermite-Euler polynomials of one variable*

$${}_HE_{m,k+l}(z+x)=\sum_{n,m=0}^{k,l}\frac{k!l!(z)^{n+m}\,{}_HE_{m,k+l-m-n}(x)}{(k-n)!(l-m)!n!m!}, \tag{3.9}$$

whereas by setting $z=0$ in Theorem 3.1, we get another result involving truncated Hermite-Euler polynomials of one and two variables

$${}_HE_{m,k+l}(y)=\sum_{n,m=0}^{k,l}\frac{k!l!(-x)^{n+m}\,{}_HE_{m,k+l-m-n}(x,y)}{(k-n)!(l-m)!n!m!}. \tag{3.10}$$

Theorem 3.5. *The following implicit summation formula holds true:*

$${}_HE_{m,n}(x+z,y+u)=\sum_{m=0}^{n}\binom{n}{m}{}_HE_{m,n-s}(x,y)H_s(z,u). \tag{3.11}$$

Proof. We replace x by $x+z$ and y by $y+u$ in (2.1), use (1.2) and rewrite the generating function as

$$\sum_{n=0}^{\infty}{}_HE_{m,n}(x+z,y+u)\frac{t^n}{n!}=\left(\frac{\frac{2t^m}{m!}}{e^t+1-\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)e^{(xt+yt^2}e^{zt+ut^2}$$

$$=\sum_{n=0}^{\infty}{}_HE_{m,n}(x,y)\frac{t^n}{n!}\sum_{s=0}^{\infty}H_s(z,u)\frac{t^s}{s!}.$$

Now replacing n by $n-s$ and comparing the coefficients of t, we get the result (3.11).
□

Theorem 3.6. *The following implicit summation formula holds true:*

$$ {}_HE_{m,n}(y,x) = \sum_{k=0}^{[\frac{n}{2}]} E_{m,n-2k}(y)\frac{x^k}{(n-2k)!k!}. \tag{3.12} $$

Proof. We replace x by y and y by x in equation (2.1) to get

$$ \sum_{n=0}^{\infty} {}_HE_{m,n}(y,x)\frac{t^n}{n!} = \sum_{n=0}^{\infty} E_{m,n}(y)\frac{t^n}{n!}\sum_{k=0}^{\infty}\frac{x^k t^{2k}}{k!}. \tag{3.13} $$

Now replacing n by $n-2k$ and comparing the coefficients of t, we get the result (3.12).
□

Theorem 3.7. *The following implicit summation formula holds true:*

$$ \sum_{n=0}^{\infty} {}_HE_{m,n}(x,y) = \sum_{r=0}^{n}\binom{n}{r} {}_HE_{m,n-r}(x,y). \tag{3.14} $$

Proof. From (2.1), we have

$$ \sum_{n=0}^{\infty} {}_HE_{m,n}(x+1,y)\frac{t^n}{n!} - \sum_{n=0}^{\infty} {}_HE_{m,n}(x,y)\frac{t^n}{n!} $$

$$ = \frac{\frac{2t^m}{m!}}{e^t + 1 - \sum\limits_{h=0}^{m-1}\frac{t^h}{h!}} e^{xt+yt^2}(e^t-1) $$

$$ = \sum_{n=0}^{\infty} {}_HE_{m,n}(x,y)\frac{t^n}{n!}\sum_{r=0}^{\infty}\frac{t^r}{r!} - \sum_{n=0}^{\infty} {}_HE_{m,n}(x,y)\frac{t^n}{n!}. $$

Comparing the coefficients of t in both sides, we get the result.
□

4. Conclusion

In this paper, a new class of Hermite-based truncated Euler polynomials are introduced and their properties are explored using various generating function methods. Several elementary properties and implicit summation formulae are established for these polynomials. The generalization of these results may lead to other interesting results, which may be helpful to the theory of fractional calculus. Several techniques and methods are used which are applicable to the other fields of mathematics. The applicability of these techniques to the these polynomials can also be explored. These aspects will be undertaken in further investigation, as these results can be extended to Gould-Hopper-based truncated Euler polynomials defined by the following generating function:

$$ \frac{\frac{2t^m}{m!}}{e^t + 1 - \sum\limits_{h=0}^{m-1}\frac{t^h}{h!}} e^{xt+yt^r} = \sum_{n=0}^{\infty} {}_gE^{(r)}_{n,m}(x,y)\frac{t^n}{n!}. \tag{4.1} $$

Clearly, when $r=2$ equation (4.1) immediately gives the known result (2.1).

References

1. Araci, S, Acikgoz, M, em A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials, Adv. Stud. Contemp. Math. 22(3), 399-406 (2012)
2. Bell, E. T, *Exponential polynomials*, Ann. of Math., 35, 258-277, (1934).
3. Choi, J, Kim, D. S, Kim, T and Kim, Y. H, *A note on some identities of Frobenius-Euler numbers and polynomials*, Inter. J. Math. Math. Sci., 2012, 1-9, (2012).
4. Dattoli, G, Lorenzutta, S and Cesarano, C., *Finite sums and generalized forms of Bernoulli polynomials*, Rendiconti di Mathematica, 19, 385-391, (1999).
5. Duran, U, Acikgoz, M and Araci, S, *Hermite-based poly-Bernoulli polynomials with a q-paramenter*, Adv. Stud. Contemp. Math. (Kyungshang), 28(2), 285-296, (2018).
6. Duran, U and Acikgoz, M, *Truncated Fubini Polynomials*, Mathematics, 7, 431; doi:10.3390/math7050431, (2019).
7. Duran, U and Acikgoz, M, *On Degenerate Truncated special polynomials*, Mathematics, 8, 144; doi:10.3390/math8010144, (2020).
8. Hassen, A and Nguyen, H. D, *Hypergeometric Bernoulli polynomials and Appell sequences*, Int. J. Number Theory, 4, 767-774, (2008).
9. Hassen, A and Nguyen, H. D, *Hypergeometric zeta functions*, Int. J. Number Theory, 6, 99–126, (2010).
10. Jang, L. -C, Kim, D. S, Jang, G. -W and Kwon, J, *Some identities for q-Bernoulli numbers and polynomials arising form q-Bernstein polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), 28(4), 659-667, (2018).
11. Jang, G. -W and Kim, T, *A note on type 2 degenerate Euler and Bernoulli polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), 29(1), 147-159, (2019).
12. Kamano, K, *Sums of products of hypergeometric Bernoulli numbers*, J. Number Theory, 130, 2259–2271, (2010).
13. Komatsu, K and Ruiz C. P, *Truncated Euler polynomials*, Mathematica Slovaca, 68(3), 527-536, (2018).
14. Kim, D. S and Kim, T, *Some identities of Frobenius-Euler polynomials arising from umbral calculus*, Adv. Difference Equ., 2012, Article number: 196(2012).
15. Kim, D. S, Kim, T, Rim, S. -H and Dolgy, D. V, *Barne's multiple BErnoulli and Hermite mixed-type polynomials*, Proc. Jangjeon Math. Soc., 18(1), 7-19, (2015).
16. Khan, W. A, *Some properties of the generalized Apostol type Hermite-Based polynomials*, Kyungpook Math. J., 55, 597-614, (2015).
17. Khan, W. A and Ahmad, M, *Partially degenerate poly-Bernoulli polynomials associated with Hermite polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), 28(3), 487-496, (2018).
18. Khan, W. A, *A new class of heigher-order hypergeometric Bernoulli polynomials associated with Hermite polynomials*, Bol. Soc. Paran. Mat., Accepted (2020).
19. Kim, D. S and Kim, T, *Some new identities of Frobenius-Euler numbers and polynomials*, J. Ineq. Appl., 2012, 1-10, (2012).
20. Kim, T, Kwon, H. -I and Jang, G. -W, *Symmetric identities of heigher-order of degenerate q-Bernoulli polynomials*, Adv. Stud. Contemp. Math. (Kyungshang), 27(1), 31-41, (2017).
21. Kumam, W, Srivastava, H. M and Kumam, P, *Truncated-exponential-based Frobenius–Euler polynomials*, Advan. Diff. Eq., 2019:530, (2019).
22. Kurt, B and Simsek, Y, *On the generalized Apostol type Frobenius Euler polynomials*, Advances in Differences equations, (2013), 1-9, (2013).
23. Pathan, M. A and Khan, W. A, *Some implicit summation formulas and symmetric identities for the generalized Hermite-Euler polynomials*, East-West J. Maths., 16(1), 92-109, (2014).
24. Srivastava, H. M, Araci, S, Khan, W. A and Acikgoz, M, *A note on the truncated-exponential based Apostol-type polynomials*, Symmetry 11, Article ID 538,doi:10.3390/sym11040538, (2019).
25. Srivastava, H. M and Acikgoz, M, *A new approach to Legendre-truncated-exponential-based Sheffer sequences via Riordan arrays*, Applied Mathematics and Computation 369, 124683, (2020).
26. Srivastava, H. M and Manocha, H. L, *A treatise on generating functions*, Ellis Horwood Limited, New York, 1984.
27. Simsek, Y, *Generating functions for generalized Striling type numbers, Array type polynomials, Eulerian type polynomials and their applications*, Fixed Point Th. Appl., DOI: 1186/1687-1812-2013-87, (2013).

Waseem A. Khan, Department of Mathematics and Natural Sciences, Prince Mohammad Bin Fahd University, P.O Box 1664, Al Khobar 31952, Kingdom of Saudi Arabia
E-mail address: wkhan1@pmu.edu.sa, waseem08_khan@rediffmail.com

and

Divesh Srivastava, Department of Mathematics, Faculty of Science, Integral University, Lucknow-226026, India
E-mail address: divesh2712@gmail.com

and

Rifaqat Ali, Department of Mathematics, College of Science and Arts, Muhayil, King Khalid University, 61413 Abha, Kingdom of Saudi Arabia
E-mail address: rifaqat.ali1@gmail.com, rrafat@kku.edu.sa