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On New Difference Sequence Spaces via Cesàro Mean *

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abstract: In the present article, we define certain new classes of sequence spaces by using Cesàro mean and difference operator Δ^r, $r \in \mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$. Also, we study the topological structures of the defined classes and determine their $\alpha-, \beta-$ and $\gamma-$ duals. Matrix transformations of given classes with their basic sequence spaces are characterized.

Key Words: Cesàro operator $C(1,1)$, summation operator Φ, difference operator Δ^r, $\alpha-, \beta-$ and $\gamma-$ duals, matrix transformations.

Contents

1. Introduction

Let w be the space of all real or complex sequences $x = (x_k)$ and $\mathbb{N}_0$ be the set of all non negative integers. Any subspace of w is called a *sequence space*. By ℓ_∞, c and c_0, we denote the spaces of all bounded, convergent and null sequences, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$. Also by ℓ_1 and ℓ_p, we write the spaces of all absolutely and $p-$absolutely summable sequences, normed by $\sum_k |x_k|$ and $\left(\sum_k |x_k|^p\right)^{1/p}$ for $1 < p < \infty$, respectively.

Let $p = (p_k)$ be a bounded sequence of positive real numbers with $\sup_k p_k = H$ and M =max$(1, H)$. Then the basic sequence spaces $\ell_\infty(p), c_0(p), c(p)$ and $\ell(p)$ ([15]) are defined by

$$\ell_\infty(p) = \left\{x = (x_k) \in w : \sup_k |x_k|^{p_k} < \infty\right\},$$

$$c_0(p) = \left\{x = (x_k) \in w : \lim_{k\to\infty} |x_k|^{p_k} = 0\right\},$$

$$c(p) = \left\{x = (x_k) \in w : \lim_{k\to\infty} |x_k - l|^{p_k} = 0, \text{for some } l \in \mathbb{C}\right\}$$

and

$$\ell(p) = \left\{x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty\right\}.$$

It is known that the spaces $\ell_\infty(p), c_0(p)$ and $c(p)$ are complete spaces paranormed by

$$h_1(x) = \sup_k |x_k|^{p_k/M} \text{ if and only if } \inf_k p_k > 0,$$

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whereas, the space $\ell(p)$ is complete with the paranorm

$$h_2(x) = \left(\sum_k |x_k|^{p_k}\right)^{1/M}.$$

Let $A = (a_{nk})$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}_0$. For two sequence spaces X and Y we write a matrix mapping $A : X \to Y$ defined by

$$(Ax)_n = \sum_k a_{nk}x_k, \quad (n \in \mathbb{N}_0). \tag{1.1}$$

For every $x = (x_k) \in X$, we call Ax as the $A-$transform of x if the series $\sum_k a_{nk}x_k$ converges for each $n \in \mathbb{N}_0$. By (X, Y), we denote the class of all infinite matrices A such that $A : X \to Y$. Thus, $A \in (X, Y)$ if and only if the series in the right hand side of (1.1) converges for each $n \in \mathbb{N}_0$. The infinite matrix A is said to be bounded if $\|A\| = \sup_n \sum_{k=0}^{n} |a_{nk}| < \infty$. By bs and cs, we write the spaces of all bounded and convergent series, respectively. Now, with the help of matrix transformations, define the set $S(X, Y)$ by

$$S(X, Y) = \{z = (z_k) : xz = (x_kz_k) \in Y \text{ for all } x \in X\}. \tag{1.2}$$

With the notation of (1.2), we redefine the $\alpha-, \beta-$ and $\gamma-$ duals of a sequence space X, respectively as follows:

$$X^\alpha = S(X, \ell_1),\ X^\beta = S(X, cs) \text{ and } X^\gamma = S(X, bs).$$

Let $x = (x_k)$ be a sequence and $s_n(x)$ be its nth partial sum, defined by

$$s_n(x) = \sum_{k=0}^{n} x_k,\ (n \in \mathbb{N}_0). \tag{1.3}$$

Now, define a sequence $\Phi(x) = (\Phi_n(x))$ as

$$\Phi_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} s_j = \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=0}^{j} x_k,\ (n \in \mathbb{N}_0). \tag{1.4}$$

The sequence $(\Phi(x))$ is obtained by taking $\Phi - transform$ of the sequence x. In fact, the transform Φ is a linear operator and represents a lower triangular matrix $\Phi = (\phi_{nk})$, where

$$\phi_{nk} = \begin{cases} \dfrac{n-k+1}{n+1}, & (0 \le k \le n), \\ 0, & (k > n). \end{cases} \tag{1.5}$$

Eventually, the matrix $\Phi = (\phi_{nk})$ is being calculated by taking the product of Cesàro matrix $C(1, 1) = (c_{nk})$ and summation matrix $S = (s_{nk})$, i.e., $\Phi = C(1, 1).S$, where

$$c_{nk} = \begin{cases} \dfrac{1}{n+1}, & (0 \le k \le n), \\ 0, & (k > n), \end{cases} \quad \text{and } s_{nk} = \begin{cases} 1, & (0 \le k \le n), \\ 0, & (k > n). \end{cases}$$

It is easy to check that the inverse of the operator Φ is expressed as a lower triangular matrix $\Phi^{-1} = (\phi^{-1}_{nk})$, where

$$\phi^{-1}_{nk} = \begin{cases} n+1, & (k = n), \\ -2(k+1), & (k = n-1), \\ k+1 & (k = n-2), \\ 0, & (\text{otherwise}). \end{cases}$$

The generalized difference operator Δ^r , $(r \in \mathbb{N}_0)$ was introduced by Et and Çolak [9] which represents a lower triangular matrix as (δ^r_{nk}), where

$$\delta^r_{nk} = \begin{cases} (-1)^{n-k}\binom{r}{n-k}, & (0 \leq k \leq n), \\ 0, & \text{otherwise.} \end{cases}$$

It is noted that the inverse of the operator Δ^r can be directly found out as (δ^{-r}_{nk}), where

$$\delta^{-r}_{nk} = \begin{cases} \dfrac{(r)_{n-k}}{(n-k)!}, & (0 \leq k \leq n), \\ 0, & \text{otherwise.} \end{cases}$$

Here the symbol $(r)_k$ denotes the *Pochhammer* symbol or *shifted factorial* of a real number r and is being defined as

$$(r)_k = \begin{cases} 1, & (r = 0 \text{ or } k = 0), \\ r(r+1)(r+2)\dots(r+k-1), & (k \in \mathbb{N}). \end{cases}$$

Now, combining the operator Φ and the difference operator Δ^r, we define the rth difference summation operator $\Delta^r\Phi$ as

$$(\Delta^r\Phi x)_n = \sum_{k=0}^{r}(-1)^k\binom{r}{k}\Phi_{n-k}(x). \tag{1.6}$$

Throughout the text we use the convention that any term with negative subscript is equal to zero. On simplifying Eqn.(1.6) for $n \geq 1$, it is obtained that

$$\begin{aligned}
(\Delta^r\Phi x)_n &= \sum_{k=0}^{r}(-1)^k\binom{r}{k}\Phi_{n-k}(x) \\
&= \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=0}^{j}x_k - \frac{r}{n}\sum_{j=0}^{n-1}\sum_{k=0}^{j}x_k + \frac{r(r-1)}{2(n-1)}\sum_{j=0}^{n-2}\sum_{k=0}^{j}x_k + \dots + \frac{(-1)^r}{n-r+1}\sum_{j=0}^{n-r}\sum_{k=0}^{j}x_k \\
&= \frac{1}{n+1}\sum_{k=0}^{n}(n-k+1)x_k - \frac{r}{n}\sum_{k=0}^{n-1}(n-k)x_k + \frac{r(r-1)}{2(n-1)}\sum_{k=0}^{n-2}(n-k-1)x_k + \dots + \\
&\qquad \frac{(-1)^r}{n-r+1}\sum_{k=0}^{n-r}(n-r-k+1)x_k \\
&= \frac{1}{n+1}x_n + \left(\frac{2}{n+1} - \frac{r}{n}\right)x_{n-1} + \left(\frac{3}{n+1} - \frac{2r}{n} + \frac{r(r-1)}{2(n-1)}\right)x_{n-2} + \dots \\
&\quad + \left(\frac{n}{n+1} - \frac{r(n-2)}{n} + \dots + \frac{(-1)^r(n-r)}{n-r+1}\right)x_1 + \left(1 - r + \frac{r(r-1)}{2} + \dots + (-1)^r\right)x_0 \\
&= \sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^k\binom{r}{k}\frac{(j-k+1)}{n-k+1}x_{n-j}.
\end{aligned}$$

It is remarked that, for $n = 0$, we have $(\Delta^r\Phi x)_0 = x_0$ and similarly,

$$\begin{aligned}
(\Delta^r\Phi x)_1 &= \frac{x_1}{2} + (1-r)x_0, \\
(\Delta^r\Phi x)_2 &= \frac{x_2}{3} + \left(\frac{2}{3} - \frac{r}{2}\right)x_1 + \left(1 - r + \frac{r(r-1)}{2}\right)x_0, \\
(\Delta^r\Phi x)_3 &= \frac{x_3}{4} + \left(\frac{2}{4} - \frac{r}{3}\right)x_2 + \left(\frac{3}{4} - \frac{2r}{3} + \frac{r(r-1)}{4}\right)x_1 + \left(1 - r + \frac{r(r-1)}{2} - \frac{r(r-1)(r-2)}{6}\right)x_0, \\
&\vdots
\end{aligned}$$

The difference operator $\Delta^r\Phi$ is a linear operator. However, in matrix notations it is being calculated by taking of matrix product of Δ^r and Φ i.e. $\Delta^r\Phi = \Delta^r.\Phi$ and expressed by a lower triangular matrix $\Delta^r\Phi = (\Delta^r\phi_{nk})$, where

$$\Delta^r\phi_{nk} = \begin{cases} \dfrac{1}{n+1}, & (k=n), \\ \displaystyle\sum_{j=0}^{n-k}(-1)^j\binom{r}{j}\frac{(n-k-j+1)}{n-j+1} & (0\le k<n), \\ 0, & (k>n). \end{cases} \tag{1.7}$$

Next to, we state certain results concerning the inverse operator of $\Delta^r\Phi$ in the following theorems:

Theorem 1.1. *The inverse of the operator $\Delta^r\Phi$ is expressed as a lower triangular matrix $(\Delta^r\Phi)^{-1} = ((\Delta^r\phi)^{-1}_{nk})$, where*

$$(\Delta^r\phi)^{-1}_{nk} = \begin{cases} n+1, & (k=n), \\ \displaystyle\sum_{j=n-2}^{n}(-1)^{n-j}\binom{2}{n-j}\frac{(j+1)(r)_{j-k}}{(j-k)!}, & (0\le k<n), \\ 0, & (k>n). \end{cases}$$

Proof. The proof directly follows from the equation that

$$(\Delta^r\Phi)^{-1} = \Phi^{-1}.\Delta^{-r} = S^{-1}.C(1,1)^{-1}.\Delta^{-r},$$

where the inverse matrices $S^{-1} = s^{-1}_{nk}$ and $C(1,1)^{-1} = c^{-1}_{nk}$ respectively, are

$$s^{-1}_{nk} = \begin{cases} 1, & (k=n), \\ -1, & (k=n-1), \\ 0, & (\text{otherwise}), \end{cases} \text{ and } c^{-1}_{nk} = \begin{cases} n+1, & (k=n), \\ -(k+1), & (k=n-1), \\ 0, & (\text{otherwise}). \end{cases}$$

This completes the proof. □

Corollary 1.2. *In particular, if $r = 1$, then the inverse of the operator $\Delta^r\Phi$ can be easily calculated as $\Delta\Phi^{-1} = (\Delta\phi^{-1}_{nk})$ (see [17]), where*

$$\Delta\phi^{-1}_{nk} = \begin{cases} n+1, & (k=n), \\ -(n-1), & (k=n-1), \\ 0, & (\textit{otherwise}). \end{cases}$$

The idea of difference sequence spaces was initially introduced by Kızmaz [11]. Further the idea was generalized by several authors like Et and Colak [9], Baliarsingh [4,5]. For more detailed study on various difference sequence spaces and related topics, one may refer [1,2,3,6,7,8,12,13,14,16,17,16,18,19].

2. New sequence spaces

In the present section, we define new Cesàro type sequence spaces by combining the operators Φ and Δ^r. Some topological properties on these spaces are studied. Related matrix transformations on these spaces have been characterized.
If $x = (x_k)$ is a real sequence, then define

$$t_n(x) = \sum_{k=0}^{n}|x_k|, \quad (n\in\mathbb{N}_0)$$

and

$$T_n(x) = \frac{1}{n+1}\sum_{j=0}^{n} t_j = \frac{1}{n+1}\sum_{j=0}^{n}\sum_{k=0}^{j}|x_k|, \quad (n \in \mathbb{N}_0).$$

Now, we calculate

$$\Delta^r(T_n(x)) = \sum_{k=0}^{n}\sum_{j=k}^{n}(-1)^j\binom{r}{j-k}\frac{n-j+1}{n-j+k+1}|x_k|, \quad (n \geq 1).$$

Let $p = (p_k)$ be a bounded sequence as defined earlier. Then define the following classes of sequence spaces as

$$\overline{\ell}_\infty(\Delta^r, \Phi, p) = \left\{x = (x_k) \in w : \sup_n |\Delta^r(T_n(x))|^{p_n} < \infty\right\},$$

$$\overline{c}_0(\Delta^r, \Phi, p) = \left\{x = (x_k) \in w : \lim_{n\to\infty} |\Delta^r(T_n(x))|^{p_n} = 0\right\},$$

$$\overline{c}(\Delta^r, \Phi, p) = \left\{x = (x_k) \in w : \lim_{n\to\infty} |\Delta^r(T_n(x)) - l|^{p_n} = 0, \text{ for some } l \in \mathbb{R}\right\},$$

$$\overline{\ell}(\Delta^r, \Phi, p) = \left\{x = (x_k) \in w : \sum_{n=0}^{\infty} |\Delta^r(T_n(x))|^{p_n} < \infty\right\}.$$

It is noticed that if $r = 1$, then $\Delta^r(T_n(x)) = T_n(x) - T_{n-1}(x)$ and is being calculated as

$$T_n(x) - T_{n-1}(x) = \frac{1}{n(n+1)}\sum_{k=0}^{n} k|x_k|, \ (n \geq 1).$$

The above new classes generalize the following special cases:

(i) If $t_n(x) = x_n,\ n \in \mathbb{N}_0$ and $r = 0$, then the above classes reduce to the analog sequence spaces defined by Cesàro summability.

(ii) If $T_n(x) = x_n,\ n \in \mathbb{N}_0, r = 0$ and $p = e = (1, 1, \dots)$, then these classes reduce to the classical sequences spaces ℓ_∞, c_0, c and ℓ_1.

(iii) If $T_n(x) = x_n,\ n \in \mathbb{N}_0$ and $r = 0$, then these classes reduce to the classical Maddox spaces $\ell_\infty(p), c_0(p), c(p)$ and $\ell(p)$.

(iv) If $r = 1$, then these classes reduce to sequences spaces $\overline{\ell}_\infty(p), \overline{c}_0(p), \overline{c}(p), \overline{\ell}(p)$ and $\widehat{l}(p)$.

Theorem 2.1. *The spaces $\overline{\ell}_\infty(\Delta^r, \Phi, p), \overline{c}_0(\Delta^r, \Phi, p)$ and $\overline{c}(\Delta^r, \Phi, p)$ are linear spaces paranormed by g, defined by*

$$g(x) = \sup_n \left|\sum_{k=0}^{n}\sum_{j=k}^{n}(-1)^j\binom{r}{j-k}\frac{n-j+1}{n-j+k+1}x_k\right|^{p_n/M} \quad \text{if and only if } \inf_k p_k > 0. \tag{2.1}$$

Proof. Proof is a routine verification and hence omitted. □

Theorem 2.2. *The spaces $\overline{\ell}_\infty(\Delta^r, \Phi, p), \overline{c}_0(\Delta^r, \Phi, p)$ and $\overline{c}(\Delta^r, \Phi, p)$ are complete linear spaces under the paranorm g, defined in (2.1).*

Proof. We prove the theorem for the space $\overline{\ell}_\infty(\Delta^r, \Phi, p)$ under the paranorm defined in (2.1).

Consider a Cauchy sequence $\{x^n\}$ in the space $\overline{\ell}_\infty(\Delta^r,\Phi,p)$ and $x^n=\{x_0^{(n)},x_1^{(n)},x_2^{(n)}\dots\}$. Using the definition of the Cauchy sequence, we claim that for given $\epsilon>0$, there exists a positive integer $N_0(\varepsilon)$ such that

$$g(x^n-x^m)<\epsilon,\ \text{for all } m,n\geq N_0(\epsilon).$$

Keeping in view of (2.1), for each fixed $k\in\mathbf{N}_0$, we have

$$\begin{aligned}|\{\Delta^r\Phi(x^n)\}_k-&\{\Delta^r\Phi(x^m)\}_k|^{p_k/M}\\ &\leq\sup_k|\{\Delta^r\Phi(x^n)\}_k-\{\Delta^r\Phi(x^m)\}_k|^{p_k/M}\\ &<\epsilon,\ \text{for all } m,n\geq N_0(\epsilon),\end{aligned}$$

which suggests that $\{(\Delta^r\Phi(x^0))_k,(\Delta^r\Phi(x^1))_k,\dots\}$ is a Cauchy sequence in $\mathbb{R}$ for each fixed $k\in\mathbf{N}_0$. By completeness of $\mathbb{R}$, the sequence $\{\Delta^r\Phi(x^n)\}_k$ converges and suppose that

$$\{\Delta^r\Phi(x^n)\}_k\to\{\Phi(x)\}_k \text{ as } n\to\infty.$$

For each fixed $k\in\mathbf{N}_0$, $m\to\infty$ and $n\geq N_0(\epsilon)$, it is clear that

$$|\{\Delta^r\Phi(x^n)\}_k-\Delta^r\Phi(x)\}_k|^{p_k/M}\leq\epsilon. \tag{2.2}$$

As per the assumption, $x^n=\{x_k^{(n)}\in\overline{\ell}_\infty(\Delta^r,\Phi,p)$, therefore, we have

$$|\{\Delta^r\Phi(x^n)\}_k|^{p_k/M}<\infty \text{ for each fixed } k\in\mathbf{N}_0. \tag{2.3}$$

Therefore, from (2.2) and (2.3), we derive

$$\begin{aligned}|\{\Delta^r\Phi(x)\}_k|^{p_k/M}&\leq|\{\Delta^r\Phi(x^n)\}_k-\{\Delta^r\Phi(x)\}_k|^{p_k/M}+|\{\Delta^r\Phi(x^n)\}_k|^{p_k/M}\\ &<\infty \text{ for all } n\geq\mathbf{N}_0(\epsilon),\end{aligned}$$

which concludes that the sequence $\{\Delta^r\Phi(x)\}$ belongs to the space $\ell_\infty(p)$. This proves the theorem. □

Theorem 2.3. *The sequence space $\overline{\ell}(\Delta^r,\Phi,p)$ is a complete linear space under the paranorm g_1, where*

$$g_1(x)=\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{n}\sum_{j=k}^{n}(-1)^j\binom{r}{j-k}\frac{n-j+1}{n-j+k+1}x_k\right|^{p_n}\right)^{1/M}. \tag{2.4}$$

Proof. The proof is similar to that of Theorem 2.2. □

3. Dual spaces and related matrix transformations

In this section, the alpha-, beta- and gamma-duals of the proposed sequence spaces are defined. Finally, we characterize some matrix transformations related to the new defined spaces. We quote the following Lemmas for our next investigations.

Lemma 3.1. *$A\in(\ell_\infty(p),\ell(q))$ if and only if*

$$\sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}a_{nk}B^{1/p_k}\right|^{q_n}<\infty,\ \textit{for all integers } B>1. \tag{3.1}$$

Lemma 3.2. *Let $p_k>0$ for every k. Then $A\in(\ell_\infty(p),c(q))$ if and only if*

$$\sup_n\sum_k|a_{nk}|B^{1/p_k}<\infty,\quad \textit{for all integers } B>1 \textit{ and} \tag{3.2}$$

$$\lim_{n\to\infty}\left(\sum_k|a_{nk}-\alpha_k|B^{1/p_k}\right)^{q_n}=0,\ \exists(\alpha_k)\in w,\ \textit{for all integers } B>1. \tag{3.3}$$

Lemma 3.3. *Let $p_k > 0$ for every k. Then $A \in (\ell_\infty(p), \ell_\infty(q))$ if and only if*

$$\sup_{n\in\mathbf{N}} \left(\sum_k |a_{nk}| B^{1/p_k} \right)^{q_n} < \infty, \quad \textit{for all integers } B > 1. \tag{3.4}$$

Where $\mathcal{F}$ be the collection of all finite subsets of $\mathbf{N}_0$ and $K \in \mathcal{F}$ and $q = (q_n)$ be a bounded sequence of strictly positive real numbers(see [10]).

Theorem 3.4. *The alpha-, beta- and gamma-duals of the space $\overline{\ell}_\infty(\Delta^r, \Phi, p)$ are given by*

$$\begin{aligned}
\{\overline{\ell}_\infty(\Delta^r, \Phi, p)\}^\alpha &= D_1(p), \\
\{\overline{\ell}_\infty(\Delta^r, \Phi, p)\}^\beta &= D_2(p) \cap cs, \\
\{\overline{\ell}_\infty(\Delta^r, \Phi, p)\}^\gamma &= D_2(p),
\end{aligned}$$

where

$$D_1(p) = \bigcap_{B>1} \left\{ a = (a_n) : \sup_{K\in\mathcal{F}} \sum_n \left| \sum_{k\in K} \left[\sum_{j=n-2}^{n} (-1)^{n-j} \binom{2}{n-j} (j+1) \frac{(r)_{j-k}}{(j-k)!} \right] a_n \right| B^{1/p_k} < \infty \right\},$$

$$D_2(p) = \bigcap_{B>1} \left\{ a = (a_n) : \sup_n \sum_k \left| \sum_{i=k}^{n} \sum_{j=i-2}^{i} (-1)^{i-j} \binom{2}{i-j} (j+1) \frac{(r)_{j-k}}{(j-k)!} a_i \right| B^{1/p_k} < \infty \right\}.$$

Proof. Define the sequence $x = (x_n) \in \overline{\ell}_\infty(\Delta^r, \Phi, p)$ via the sequence $y \in \ell_\infty(p)$ as

$$x_n = \sum_{k=0}^{n} \sum_{j=n-2}^{n} (-1)^{n-j} \binom{2}{n-j} (j+1) \frac{(r)_{j-k}}{(j-k)!} y_k \ (n \in \mathbf{N}_0).$$

Clearly, for all $n \in \mathbf{N}_0$, we have

$$\begin{aligned}
x_n a_n &= \left[\sum_{k=0}^{n} \sum_{j=n-2}^{n} (-1)^{n-j} \binom{2}{n-j} (j+1) \frac{(r)_{j-k}}{(j-k)!} y_k \right] a_n \\
&= \sum_{k=0}^{n} \left[\sum_{j=n-2}^{n} (-1)^{n-j} \binom{2}{n-j} (j+1) \frac{(r)_{j-k}}{(j-k)!} a_n \right] y_k \\
&= (Cy)_n.
\end{aligned}$$

Note that matrix $C = (c_{nk})$ via the sequence $a = (a_n) \in w$ is being defined by

$$c_{nk} = \begin{cases} \displaystyle\sum_{j=n-2}^{n} (-1)^{n-j} \binom{2}{n-j} (j+1) \frac{(r)_{j-k}}{(j-k)!} a_n, & (0 \le k \le n), \\ 0, & (k > n). \end{cases}$$

Therefore, by Lemma 3.1 (with $q_n = 1$) we immediate obtain that

$$\{\overline{\ell}_\infty(\Delta^r, \Phi, p)\}^\alpha = D_1(p).$$

For the $\beta-$ dual, we consider

$$\begin{aligned}\sum_{k=0}^{n} x_k a_k &= \sum_{k=0}^{n}\left[\sum_{i=0}^{k}\sum_{j=k-2}^{k}(-1)^{k-j}\binom{2}{k-j}(j+1)\frac{(r)_{j-i}}{(j-i)!}y_i\right]a_k\\ &= \sum_{i=0}^{n}\left[\sum_{k=i}^{n}\sum_{j=k-2}^{k}(-1)^{k-j}\binom{2}{k-j}(j+1)\frac{(r)_{j-i}}{(j-i)!}a_k\right]y_i\\ &= \sum_{k=0}^{n}\left[\sum_{i=k}^{n}\sum_{j=i-2}^{i}(-1)^{i-j}\binom{2}{i-j}(j+1)\frac{(r)_{j-k}}{(j-k)!}a_i\right]y_k\\ &= (D_2 y)_n,\ \ (n\in \mathbf{N}_0),\end{aligned}$$

where the matrix $D_2 = (d_{nk})$ is defined by

$$d_{nk} = \begin{cases}\displaystyle\sum_{i=k}^{n}\sum_{j=i-2}^{i}(-1)^{i-j}\binom{2}{i-j}(j+1)\frac{(r)_{j-k}}{(j-k)!}a_i, & (0\le k\le n),\\ 0, & (\text{ otherwise}).\end{cases}$$

By Lemma 3.2 (with $q_n = 1$) we conclude that $ax = (a_n x_n) \in cs$ whenever $x \in \overline{\ell}_\infty(\Delta^r, \Phi, p)$ if and only if $D_2 y \in c$ whenever $y \in \ell_\infty(p)$, equivalently $a = (a_n) \in \{\overline{\ell}_\infty(p)\}^\beta$ if and only if $D_2 \in (\ell_\infty(p), c)$. Therefore,

$$\{\overline{\ell}_\infty(\Delta^r,\Phi,p)\}^\beta = D_2(p)\cap cs.$$

For $\gamma-$dual, the proof is similar, so it is omitted. □

Theorem 3.5. *The alpha-, beta- and gamma-duals of the space $\overline{c}_0(\Delta^r, \Phi, p)$ are given by*

$$\begin{aligned}\{\overline{c}_0(\Delta^r,\Phi,p)\}^\alpha &= D_3(p),\\ \{\overline{c}_0(\Delta^r,\Phi,p)\}^\beta &= D_4(p)\cap D_5,\\ \{\overline{c}_0(\Delta^r,\Phi,p)\}^\gamma &= D_6(p),\end{aligned}$$

where

$$D_3(p) = \bigcup_{B>1}\left\{a=(a_n) : \sup_{K\in\mathcal{F}}\sum_n\left|\sum_{k\in K}\left[\sum_{j=n-2}^{n}(-1)^{n-j}\binom{2}{n-j}\frac{(j+1)(r)_{j-k}}{(j-k)!}\right]a_n\right|B^{-1/p_k}<\infty\right\},$$

$$D_4(p) = \bigcup_{B>1}\left\{a=(a_n) : \sup_n\sum_k\left|\sum_{j=n-2}^{n}(-1)^{n-j}\binom{2}{n-j}\frac{(j+1)(r)_{j-k}}{(j-k)!}a_n-\alpha_k\right|B^{-1/p_k}<\infty,\ \textit{for } \alpha_k\in\mathbb{R}\right\},$$

$$D_5 = \left\{a=(a_n) : \lim_{n\to\infty}\left|\sum_{j=n-2}^{n}(-1)^{n-j}\binom{2}{n-j}\frac{(j+1)(r)_{j-k}}{(j-k)!}a_n-\alpha_k\right|=0,\ \textit{for all } \alpha_k\in\mathbb{R}\right\},$$

$$D_6(p) = \bigcup_{B>1}\left\{a=(a_n) : \sup_n\sum_k\left|\sum_{j=n-2}^{n}(-1)^{n-j}\binom{2}{n-j}\frac{(j+1)(r)_{j-k}}{(j-k)!}a_n\right|B^{-1/p_k}<\infty\right\}.$$

Theorem 3.6. *The alpha-, beta- and gamma-duals of the space $\overline{c}(\Delta^r, \Phi, p)$ are given by*

$$\begin{aligned}\{\overline{c}(\Delta^r,\Phi,p)\}^\alpha &= D_3(p)\cap D_7,\\ \{\overline{c}(\Delta^r,\Phi,p)\}^\beta &= D_4(p)\cap D_5\cap D_8,\\ \{\overline{c}(\Delta^r,\Phi,p)\}^\gamma &= D_6(p)\cap D_9,\end{aligned}$$

where

$$D_7 = \left\{ a = (a_n) : \sum_n \left| \sum_k \left[\sum_{j=n-2}^{n} (-1)^{n-j} \binom{2}{n-j} \frac{(j+1)(r)_{j-k}}{(j-k)!} \right] a_n \right| < \infty \right\},$$

$$D_8 = \left\{ a = (a_n) : \lim_{n\to\infty} \left| \sum_k \sum_{j=n-2}^{n} (-1)^{n-j} \binom{2}{n-j} \frac{(j+1)(r)_{j-k}}{(j-k)!} a_n - \alpha \right| = 0, \text{ for some } \alpha \in \mathbb{R} \right\},$$

$$D_9 = \left\{ a = (a_n) : \sup_n \left| \sum_k \sum_{j=n-2}^{n} (-1)^{n-j} \binom{2}{n-j} \frac{(j+1)(r)_{j-k}}{(j-k)!} a_n \right| < \infty \right\}.$$

Theorem 3.7. *The alpha-, beta- and gamma-duals of the space $\overline{\ell}(\Delta^r, \Phi, p)$ are given by*

$$\{\overline{\ell}(\Delta^r, \Phi, p)\}^\alpha = D_{10}(p),$$
$$\{\overline{\ell}(\Delta^r, \Phi, p)\}^\beta = D_4(p) \cap D_5 \cap D_{11}(p),$$
$$\{\overline{\ell}(\Delta^r, \Phi, p)\}^\gamma = D_6(p) \cap D_{12}(p) \cup D_{13},$$

where

$$D_{10}(p) = \bigcup_{B>1} \left\{ a = (a_n) : \sum_k \sum_n \left| \left[\sum_{j=n-2}^{n} (-1)^{n-j} \binom{2}{n-j} \frac{(j+1)(r)_{j-k}}{(j-k)!} \right] a_n \right| B^{-1/p_k} < \infty \right\},$$

$$D_{11}(p) = \left\{ a = (a_n) : \sup_{n,k\in K} \left| \sum_{j=n-2}^{n} (-1)^{n-j} \binom{2}{n-j} \frac{(j+1)(r)_{j-k}}{(j-k)!} a_n \right|^{p_k} < \infty \right\}.$$

$$D_{12}(p) = \bigcup_{B>1} \left\{ a = (a_n) : \sup_{k,n} \left| \left[\sum_{j=n-2}^{n} (-1)^{n-j} \binom{2}{n-j} \frac{(j+1)(r)_{j-k}}{(j-k)!} \right] a_n \right| B^{-1/p_k} < \infty \right\},$$

$$D_{13} = \left\{ a = (a_n) : \lim_{n\to\infty} \left| \sum_{j=n-2}^{n} (-1)^{n-j} \binom{2}{n-j} \frac{(j+1)(r)_{j-k}}{(j-k)!} a_n \right| = 0 \right\}.$$

Now, we characterize some matrix mapping on the spaces $\overline{\ell}_\infty(\Delta^r, \Phi, p)$, $\overline{c}(\Delta^r, \Phi, p)$, $\overline{c}_0(\Delta^r, \Phi, p)$ and $\overline{\ell}(\Delta^r, \Phi, p)$. We discuss the matrix transformations of these spaces to the spaces $\ell_\infty(q), c(q), c_0(q)$ and $\ell(q)$. For our investigation, we need the following lemmas due to [10]. Suppose (q_n) is a non decreasing bounded sequence of positive real numbers and K be the finite subset of $\mathbf{N}$, the set of non negative integers. The set K^* is defined by $K^* = \{k \in \mathbf{N} : p_k \le 1\}$.

Lemma 3.8. *$A \in (\ell_\infty(p), c_0(q))$ if and only if*

$$\lim_{n\to\infty} \left(\sum_k |a_{nk}| B^{1/p_k} \right)^{q_n} = 0, \qquad (\forall B \in \mathbf{N}, B > 1). \tag{3.5}$$

Lemma 3.9. *$A \in (c_0(p), \ell_\infty(q))$ if and only if*

$$\sup_n \left(\sum_k |a_{nk}| B^{-1/p_k} \right)^{q_n} < \infty, \qquad (\exists B \in \mathbf{N}, B > 1). \tag{3.6}$$

Lemma 3.10. *$A \in (c_0(p), c_0(q))$ if and only if*

$$\sup_n B^{*1/q_n} \sum_k |a_{nk}| B^{-1/p_k} < \infty, \qquad (\exists B \in \mathbf{N}, \forall B^* \in \mathbf{N}, B, B^* > 1), \tag{3.7}$$

$$\lim_{n\to\infty} |a_{nk}|^{q_n} = 0, \qquad (\forall k \in \mathbb{N}). \tag{3.8}$$

Lemma 3.11. *$A \in (c_0(p), c(q))$ if and only if*

$$\sup_n \sum_k |a_{nk}| B^{-1/p_k} < \infty, \qquad (\exists B \in \mathbf{N}, B > 1), \tag{3.9}$$

$$\sup_n {B^*}^{1/q_n} \sum_k |a_{nk}| B^{-1/p_k} < \infty, \qquad (\exists B \in \mathbf{N}, \forall B^* \in \mathbf{N}, B, B^* > 1), \tag{3.10}$$

$$\lim_{n\to\infty} |a_{nk} - \alpha_k|^{q_n} = 0, \qquad (\alpha_k \in \mathbb{R}). \tag{3.11}$$

Lemma 3.12. *$A \in (c_0(p), \ell(q))$ if and only if*

$$\sup_{K \in \mathcal{F}} \sum_n \left| \sum_{k \in K} a_{nk} B^{-1/p_k} \right|^{q_n} < \infty, \qquad (\exists B \in \mathbf{N}, B > 1, q_n \geq 1). \tag{3.12}$$

Lemma 3.13. *$A \in (c(p), \ell_\infty(q))$ if and only if (3.6) holds and*

$$\sup_n \left| \sum_k a_{nk} \right|^{q_n} < \infty. \tag{3.13}$$

Lemma 3.14. *$A \in (c(p), c_0(q))$ if and only if (3.7) and (3.8) holds and*

$$\lim_{n\to\infty} \left| \sum_k a_{nk} \right|^{q_n} = 0. \tag{3.14}$$

Lemma 3.15. *$A \in (c(p), c(q))$ if and only if (3.9), (3.10), (3.11) hold and*

$$\lim_{n\to\infty} \left| \sum_k a_{nk} - \alpha \right|^{q_n} = 0, \qquad (\alpha \in \mathbb{R}). \tag{3.15}$$

Lemma 3.16. *$A \in (c(p) : \ell(q))$ if and only if (3.12) holds and*

$$\sum_n \left| \sum_k a_{nk} \right|^{q_n} < \infty, \qquad (q_n \geq 1). \tag{3.16}$$

Lemma 3.17. *$A \in (\ell(p), \ell_\infty(q))$ if and only if*

$$\sup_{n,k} \left| a_{nk} B^{*-1/q_n} \right|^{p_k} < \infty, \qquad (\exists B^* \in \mathbf{N}, B^* > 1) \tag{3.17}$$

$$\sup_n \sum_k \left| a_{nk} B^{*-1/q_n} \right|^{p_k} < \infty, \qquad (\exists B^* \in \mathbf{N}, B^* > 1) \tag{3.18}$$

Lemma 3.18. *$A \in (\ell(p), c_0(q))$ if and only if (3.8) holds and*

$$\sup_{n,k} \left| a_{nk} B^{-1/p_k} \right|^{q_n} < \infty, \qquad (\exists B \in \mathbf{N}, B > 1) \tag{3.19}$$

$$\sup_n \sum_k \left| a_{nk} B^{*-1/q_n} \right|^{p_k} < \infty, \qquad (\exists B^* \in \mathbf{N}, B^* > 1) \tag{3.20}$$

Lemma 3.19. *$A \in (\ell(p), c(q))$ if and only if (3.11),(3.17) and (3.18) hold and*

$$\sup_{n,k} |a_{nk}|^{p_k} < \infty, \tag{3.21}$$

$$\sup_n \sum_k |a_{nk}|^{p_k} < \infty. \tag{3.22}$$

Lemma 3.20. *$A \in (\ell(p) : \ell(q))$ if and only if*

$$\sup_k \sum_n \left| a_{nk} B^{-1/p_k} \right|^{q_n} < \infty, \qquad (\exists B \in \mathbf{N}, B > 1) \tag{3.23}$$

Consider an infinite matrix $\tilde{A} = (\tilde{a}_{nk})$ via the matrix $A = (a_{nk})$ as

$$\tilde{a}_{nk} = \sum_{j=k}^{\infty} \sum_{i=k-2}^{k} (-1)^{k-i} \binom{2}{k-i} (i+1) \frac{(r)_{i-j}}{(i-j)!} a_{nj}.$$

$\tilde{A}$ is called the associated matrix of A. Using Lemmas 3.1-3.20, we state the following theorems without proof.

Theorem 3.21. **(i)** *$A \in (\overline{\ell}_\infty(\Delta^r, \Phi, p), \ell_\infty(q))$ if and only if (3.4) holds with $a_{nk} = \tilde{a}_{nk}$.*

(ii) *$A \in (\overline{\ell}_\infty(\Delta^r, \Phi, p), c_0(q))$ if and only if (3.5) holds with $a_{nk} = \tilde{a}_{nk}$.*

(iii) *$A \in (\overline{\ell}_\infty(\Delta^r, \Phi, p), c(q))$ if and only if (3.2) and (3.3) hold with $a_{nk} = \tilde{a}_{nk}$.*

(iv) *$A \in (\overline{\ell}_\infty(\Delta^r, \Phi, p), \ell(q))$ if and only if (3.1) holds with $a_{nk} = \tilde{a}_{nk}$.*

Theorem 3.22. **(i)** *$A \in (\overline{c}_0(\Delta^r, \Phi, p), \ell_\infty(q))$ if and only if (3.6) holds with $a_{nk} = \tilde{a}_{nk}$.*

(ii) *$A \in (\overline{c}_0(\Delta^r, \Phi, p), c_0(q))$ if and only if (3.7) and (3.8) hold with $a_{nk} = \tilde{a}_{nk}$.*

(iii) *$A \in (\overline{c}_0(\Delta^r, \Phi, p), c(q))$ if and only if (3.9), (3.10) and (3.11) hold with $a_{nk} = \tilde{a}_{nk}$.*

(iv) *$A \in (\overline{c}_0(\Delta^r, \Phi, p), \ell(q))$ if and only if (3.12) holds with $a_{nk} = \tilde{a}_{nk}$.*

Theorem 3.23. **(i)** *$A \in (\overline{c}(\Delta^r, \Phi, p), \ell_\infty(q))$ if and only if (3.6) and (3.13) hold with $a_{nk} = \tilde{a}_{nk}$.*

(ii) *$A \in (\overline{c}(\Delta^r, \Phi, p), c_0(q))$ if and only if (3.7), (3.8) and (3.14) hold with $a_{nk} = \tilde{a}_{nk}$.*

(iii) *$A \in (\overline{c}(\Delta^r, \Phi, p), c(q))$ if and only if (3.9), (3.10), (3.11) and (3.15) hold with $a_{nk} = \tilde{a}_{nk}$.*

(iv) *$A \in (\overline{c}(\Delta^r, \Phi, p), \ell(q))$ if and only if (3.12) and (3.16) hold with $a_{nk} = \tilde{a}_{nk}$.*

Theorem 3.24. **(i)** *$A \in (\overline{\ell}(\Delta^r, \Phi, p), \ell_\infty(q))$ if and only if (3.17) and (3.18) hold with $a_{nk} = \tilde{a}_{nk}$.*

(ii) *$A \in (\overline{\ell}(\Delta^r, \Phi, p), c_0(q))$ if and only if (3.6), (3.19) and (3.20) hold with $a_{nk} = \tilde{a}_{nk}$.*

(iii) *$A \in (\overline{\ell}(\Delta^r, \Phi, p), c(q))$ if and only if (3.11),(3.17),(3.18),(3.21) and (3.22) hold with $a_{nk} = \tilde{a}_{nk}$.*

(iv) *$A \in (\overline{\ell}(\Delta^r, \Phi, p), \ell(q))$ if and only if (3.23) holds with $a_{nk} = \tilde{a}_{nk}$.*

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