Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–7.
ISSN-0037-8712 in press
doi:10.5269/bspm.52990

Some Calculations on Kaluza-Klein Metric with Respect to Lifts in Tangent Bundle

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ABSTRACT: In the present paper, a Riemannian metric on the tangent bundle, which is another generalization of Cheeger-Gromoll metric and Sasaki metric, is considered. This metric is known as Kaluza-Klein metric in literature which is completely determined by its action on vector fields of type X^H and Y^V. We obtain the covarient and Lie derivatives applied to the Kaluza-Klein metric with respect to the horizontal and vertical lifts of vector fields, respectively on tangent bundle TM.

Key Words: Kaluza-Klein metric, horizontal lift, vertical lift, tangent bundle.

Contents

1. Introduction

The research in the topic of the differential geometry of the tangent bundle TM of a Riemannian manifold (M, g) started with the work of S. Sasaki in 1958, [14]. Although the Sasaki metric is *naturally* defined, it has been shown in many papers that a lot of geometric properties (like locally symmetricity, having constant scalar curvature, being Einstein manifold etc.) of tangent bundle with the Sasaki metric can not be ensured unless the base manifold is local flat. Recall that when the base manifold is local flat, the tangent bundle with the Sasaki metric is local flat too. This rigidity leads mathematicians to search for other metrics. One of them is Cheeger-Gromoll metric [5]. For geometric properties of Cheeger-Gromoll metric, see [2,8,15,16]. Later, some different metrics are defined with generalize both of Sasaki and Cheeger-Gromoll metrics. For these metrics, we refer to [3,7,9,11].

In the present paper, a Riemannian metric on the tangent bundle, which is another generalization of Cheeger-Gromoll metric and Sasaki metric, is considered. This metric is known as Kaluza-Klein metric in literature which is completely determined by its action on vector fields of type X^H and Y^V. We obtain the covarient and Lie derivatives applied to the Kaluza-Klein metric with respect to the horizontal and vertical lifts of vector fields, respectively.

Let TM be the tangent bundle over an $n-$dimensional manifold M and π the natural projection $\pi : TM \longrightarrow M$. Suppose the manifold M be covered by a system of coordinate neighborhoods (U, x^i), where (x^i), $i = 1, ..., n$ is a local coordinate system defined in the neighborhood U. Let (u^i) be the Cartesian coordinates in each tangent space T_pM at $P \in M$ with respect to the natural base $\left\{\frac{\partial}{\partial x^i}\mid_p\right\}$, P being an arbitrary point in U whose coordinates are (x^i). Then we can introduce local coordinates (x^i, u^i) on open set $\pi^{-1}(U) \subset TM$. We call them *induced coordinates* on $\pi^{-1}(U)$ from (U, x^i). The projection π is represented by $(x^i, u^i) \longrightarrow (x^i)$. The indices I, J,... run from 1 to $2n$, the indices $\bar{i}$, $\bar{j}$... run from $n + 1$ to $2n$. Summation over repeated indices in always implied.

Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M. Then the vertical lift X^V and the horizontal lift X^H of X are given, with respect to the induced coordinates, by

$$X^V = X^i \partial_{\bar{i}}, \tag{1.1}$$

$$X^H = X^i \partial_i - u^j \Gamma^i_{jk} X^k \partial_{\bar{i}}, \tag{1.2}$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$ and Γ^i_{jk} are the coefficients of the Levi-Civita connection ∇ of g [17].

2010 *Mathematics Subject Classification:* 15A72, 53A45, 47B47, 53C15.
Submitted April 05, 2020. Published April 27, 2021

In particular, we have the vertical spray u^V and the horizontal spray u^H on TM defined by

$$u^V = u^i(\partial_i)^V = u^i\partial_{\bar{i}},\ u^H = u^i(\partial_i)^H = u^i\delta_i,$$

where $\delta_i = \partial_i - u^j\Gamma^s_{ji}\partial_{\bar{s}}$. u^V is also called the canonical or Liouville vector field on TM.

For any $(x,u) \in TM$, the energy density on TM in direction of u is defined to be $t = g(u,u)/2$. Let f be any smooth function of R to R, we have

$$X^H(f(t)) = 0 \tag{1.3}$$

$$X^V(f(t)) = f'(t)g(X,u) \tag{1.4}$$

and in particular, we get

$$X^H(t) = 0. \tag{1.5}$$

$$X^V(t) = g(X,u). \tag{1.6}$$

Let X, Y and Z be any vector fields on M, then we have

$$X^H(g(Y,u)) = g((\nabla_X Y), u), \tag{1.7}$$

$$X^V(g(Y,u)) = g(X,Y), \tag{1.8}$$

$$X^H((g(Y,Z))^V) = X(g(Y,Z)) \tag{1.9}$$

$$X^V((g(Y,Z))^V) = 0 \ [1]. \tag{1.10}$$

Proposition 1.1. *Let X and Y be any vector fields on a Riemannian manifold (M,g), we have [17]*

$$\begin{aligned} [X^H, Y^H] &= [X,Y]^H - (R(X,Y)u)^V, \\ [X^H, Y^V] &= (\nabla_X Y)^V, \\ [X^V, Y^V] &= 0, \end{aligned} \tag{1.11}$$

where R is the Riemannian curvature tensor of g defined by

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}. \tag{1.12}$$

Definition 1.2. *[4] Let (M,g) be a Riemannian manifold. The Kaluza-Klein metric ${}^{KK}g$ is defined for any vector fields X and Y with these three equations below:*

$$\begin{aligned} {}^{KK}g(X^H, Y^H) &= cg(X,Y), \\ {}^{KK}g(X^H, Y^V) &= 0, \\ {}^{KK}g(X^V, Y^V) &= ag(X,Y) + bg(X,u)g(Y,u) \end{aligned} \tag{1.13}$$

where a, b and c are smooth functions of t over $[0,+\infty]$ and $t = g(u,u)/2$.

Remark 1.3. *Special cases of the Kaluza-Klein metric are listened below:*

i) If $a(t) = b(t) = (1+2t)^{-1}$, $c = 1$, we get the Cheeger-Gromoll metric ${}^{CG}g$,
ii) If $a(t) = 1$, $b(t) = 0$, $c = 1$, we get the Sasaki metric ${}^{S}g$,
iii) If $a(t) = b(t)$, $c = 1$, we get metric g_a which depends on one parameter,
iv) If $c = 1$, we get the metric $g_{a,b}$ which depend on two parameters.

2. Main Results

Definition 2.1. *Let M be an $n-$dimensional differentiable manifold of class C^{∞}. Differantial transformation, defined by $D = L_X$, is called as Lie derivation with respect to vector field X if*

$$\begin{aligned} L_X f &= Xf,\ \forall f \in \Im_0^0(M), \\ L_X Y &= [X,Y],\ \forall X,Y \in \Im_0^1(M). \end{aligned} \tag{2.1}$$

$[X,Y]$ is called by Lie bracked. The Lie derivative $L_X F$ of a tensor field F of type $(1,1)$ with respect to a vector field X is defined by [17]

$$(L_X F)Y = [X, FY] - F[X,Y].$$

Proposition 2.2. *For any $X,Y \in \Im_0^1(M_n)$ [17]*

$$\begin{aligned} i)\ [X^V, Y^H] &= [X,Y]^V - (\nabla_X Y)^V = -(\hat{\nabla}_Y X)^V, \\ ii)\ [X^C, Y^H] &= [X,Y]^H - \gamma(L_X Y), \\ iii)\ [X^H, Y^V] &= [X,Y]^V + (\nabla_Y X)^V, \\ iv)\ [X^H, Y^H] &= [X,Y]^H - \gamma\hat{R}(X,Y), \end{aligned} \tag{2.2}$$

where $\hat{R}$ denotes the curvature tensor of the affine connection $\hat{\nabla}$.

Theorem 2.3. *Let ${}^{KK}g$ be Kaluza-Klein metric is defined by (1.13) and L_X the operator Lie derivation with respect to X. From (1.11),(2.2) and Definition (2.1), we get the following results*

$$\begin{aligned} i)\ (L_{X^V}\,{}^{KK}g)(Y^V, Z^V) &= a'(t)g(X,u)g(Y,Z) + b'(t)g(X,u)g(Y,u)g(Z,u) \\ &\quad + bg(X,Y)g(Z,u) + bg(Y,u)g(X,Z), \\ ii)\ (L_{X^V}\,{}^{KK}g)(Y^V, Z^H) &= ag(Y,(\hat{\nabla}_Z X)) + bg(Y,u)g((\hat{\nabla}_Z X),u), \\ iii)\ (L_{X^V}\,{}^{KK}g)(Y^H, Z^V) &= ag((\hat{\nabla}_Y X),Z) + bg((\hat{\nabla}_Y X),u)g(Z,u), \\ iv)\ (L_{X^H}\,{}^{KK}g)(Y^V, Z^V) &= a((\hat{\nabla}_X g)(Y,Z) + bg((\nabla_X Y),u)g(Z,u) + bg(Y,u)g((\nabla_X Z),u) \\ &\quad - bg((\hat{\nabla}_X Y),u)g(Z,u) - bg(Y,u)g((\hat{\nabla}_X Z),u) \\ v)\ (L_{X^H}\,{}^{KK}g)(Y^H, Z^V) &= ag((R(X,Y)u),Z) + bg((R(X,Y),u)g(Z,u), \\ vi)\ (L_{X^V}\,{}^{KK}g)(Y^H, Z^H) &= c'(t)g(X,u)g(Y,Z), \\ vii)\ (L_{X^H}\,{}^{KK}g)(Y^V, Z^H) &= ag(Y,(R(X,Z)u) + bg(Y,u)g(R(X,Z)u,u), \\ viii)\ (L_{X^H}\,{}^{KK}g)(Y^H, Z^H) &= c((L_X g)(Y,Z)), \end{aligned}$$

where the vertical lift $X^V \in \Im_0^1(TM)$ of $X \in \Im_1^0(M)$ and the horizontal lifts $X^H \in \Im_0^1(TM)$ of $X \in \Im_0^1(M)$ defined by (1.1) and (1.2), respectively.

Proof. i)

$$\begin{aligned} (L_{X^V}\,{}^{KK}g)(Y^V, Z^V) &= L_{X^V}\,{}^{KK}g(Y^V, Z^V) - {}^{KK}g(L_{X^V}Y^V, Z^V) - {}^{KK}g(Y^V, L_{X^V}Z^V) \\ &= X^V\,{}^{KK}g(Y^V, Z^V) \\ &= X^V(ag(Y,Z) + bg(Y,u)g(Z,u)) \\ &= a'(t)g(X,u)g(Y,Z) + b'(t)g(X,u)g(Y,u)g(Z,u) \\ &\quad + bg(X,Y)g(Z,u) + bg(Y,u)g(X,Z) \end{aligned}$$

ii)

$$
\begin{aligned}
(L_{X^V}\ {}^{KK}g)(Y^V,Z^H) &= L_{X^V}\ {}^{KK}g(Y^V,Z^H) -{}^{KK} g(L_{X^V}Y^V,Z^H) -{}^{KK} g(Y^V,L_{X^V}Z^H)\\
&= -{}^{KK}g(Y^V,[X,Z]^V-(\nabla_X Z)^V)\\
&= -{}^{KK}g(Y^V,(\hat{\nabla}_Z X)^V)\\
&= ag(Y,(\hat{\nabla}_Z X))+bg(Y,u)g((\hat{\nabla}_Z X),u)
\end{aligned}
$$

iii)

$$
\begin{aligned}
(L_{X^V}\ {}^{KK}g)(Y^H,Z^V) &= L_{X^V}\ {}^{KK}g(Y^H,Z^V) -{}^{KK} g(L_{X^V}Y^H,Z^V) -{}^{KK} g(Y^H,L_{X^V}Z^V)\\
&= -{}^{KK}g([X,Y]^V-(\nabla_X Y)^V-,Z^V)\\
&= -{}^{KK}g(-(\hat{\nabla}_Y X)^V,Z^V)\\
&= ag((\hat{\nabla}_Y X),Z)+bg((\hat{\nabla}_Y X),u)g(Z,u)
\end{aligned}
$$

iv)

$$
\begin{aligned}
(L_{X^H}\ {}^{KK}g)(Y^V,Z^V) &= L_{X^H}\ {}^{KK}g(Y^V,Z^V) -{}^{KK} g(L_{X^H}Y^V,Z^V) -{}^{KK} g(Y^V,L_{X^H}Z^V)\\
&= X^H(ag(Y,Z)+bg(Y,u)g(Z,u)) +{}^{KK} g(L_{Y^V}X^H,Z^V)\\
&\quad +{}^{KK}g(Y^V,L_{Z^V}X^H)\\
&= aX(g(Y,Z))+bg((\nabla_X Y),u)g(Z,u)+bg(Y,u)g((\nabla_X Z),u)\\
&\quad +{}^{KK}g(-(\hat{\nabla}_X Y)^V,Z^V) +{}^{KK} g(Y^V,-(\hat{\nabla}_X Z)^V)\\
&= aX(g(Y,Z))+bg((\nabla_X Y),u)g(Z,u)+bg(Y,u)g((\nabla_X Z),u)\\
&\quad -{}^{KK}g((\hat{\nabla}_X Y)^V,Z^V) -{}^{KK} g(Y^V,(\hat{\nabla}_X Z)^V)\\
&= aX(g(Y,Z))+bg((\nabla_X Y),u)g(Z,u)+bg(Y,u)g((\nabla_X Z),u)\\
&\quad -ag((\hat{\nabla}_X Y),Z)-bg((\hat{\nabla}_X Y),u)g(Z,u)-ag(Y,(\hat{\nabla}_X Z))\\
&\quad -bg(Y,u)g((\hat{\nabla}_X Z),u)\\
&= a((\hat{\nabla}_X g)(Y,Z)+bg((\nabla_X Y),u)g(Z,u)+bg(Y,u)g((\nabla_X Z),u)\\
&\quad -bg((\hat{\nabla}_X Y),u)g(Z,u)-bg(Y,u)g((\hat{\nabla}_X Z),u)
\end{aligned}
$$

v)

$$
\begin{aligned}
(L_{X^H}\ {}^{KK}g)(Y^H,Z^V) &= L_{X^H}\ {}^{KK}g(Y^H,Z^V) -{}^{KK} g(L_{X^H}Y^H,Z^V) -{}^{KK} g(Y^H,L_{X^H}Z^V)\\
&= -{}^{KK}g([X,Y]^H-(R(X,Y)u)^V,Z^V) +{}^{KK} g(Y^H,L_{Z^V}X^H)\\
&= +{}^{KK}g((R(X,Y)u)^V,Z^V) -{}^{KK} g(Y^H,(\hat{\nabla}_X Z)^V)\\
&= ag((R(X,Y)u),Z)+bg((R(X,Y),u)g(Z,u)
\end{aligned}
$$

vi)

$$
\begin{aligned}
(L_{X^V}\ {}^{KK}g)(Y^H,Z^H) &= L_{X^V}\ {}^{KK}g(Y^H,Z^H) -{}^{KK} g(L_{X^V}Y^H,Z^H) -{}^{KK} g(Y^H,L_{X^V}Z^H)\\
&= X^V(cg(Y,Z)) +{}^{KK} g((\hat{\nabla}_Y X)^V,Z^H) +{}^{KK} g(Y^H,(\hat{\nabla}_Z X)^V)\\
&= c'(t)g(X,u)g(Y,Z)
\end{aligned}
$$

vii)

$$
\begin{aligned}
(L_{X^H}\ {}^{KK}g)(Y^V,Z^H) &= L_{X^H}\ {}^{KK}g(Y^V,Z^H) -{}^{KK} g(L_{X^H}Y^V,Z^H) -{}^{KK} g(Y^V,L_{X^H}Z^H)\\
&= {}^{KK}g(L_{Y^V}X^H,Z^H) -{}^{KK} g(Y^V,[X,Z]^H-(R(X,Z)u)^V)\\
&= {}^{KK}g(-(\hat{\nabla}_X Y)^V,Z^H) -{}^{KK} g(Y^V,[X,Z]^H) +{}^{KK} g(Y^V,(R(X,Z)u)^V)\\
&= ag(Y,(R(X,Z)u)+bg(Y,u)g(R(X,Z)u,u)
\end{aligned}
$$

viii)

$$\begin{aligned}(L_{X^H}\,{}^{KK}g)(Y^H,Z^H) &= L_{X^H}\,{}^{KK}g(Y^H,Z^H) -{}^{KK}g(L_{X^H}Y^H,Z^H) -{}^{KK}g(Y^H,L_{X^H}Z^H)\\ &= X^H(cg(Y,Z)) -{}^{KK}g([X,Y]^H-(R(X,Y)u)^V,Z^H)\\ &\quad -{}^{KK}g(Y^H,[X,Z]^H-(R(X,Z)u)^V)\\ &= cX(g(Y,Z)) -{}^{KK}g([X,Y]^H,Z^H) -{}^{KK}g(Y^H,[X,Z]^H)\\ &= cX(g(Y,Z)) - cg([X,Y],Z) - cg(Y,[X,Z])\\ &= c((L_Xg)(Y,Z))\end{aligned}$$

□

Definition 2.4. *Let M be an $n-$dimensional diferentiable manifold. Differantial transformation of algebra TM, defined by*

$$D=\nabla_X : T(M)\to T(M),\ X\in\Im_0^1(M)$$

is called as covariant derivation with respect to vector field X if

$$\begin{aligned}\nabla_{fX+gY}t &= f\nabla_Xt+g\nabla_Yt,\\ \nabla_Xf &= Xf,\end{aligned}\tag{2.3}$$

where $\forall f,g\in\Im_0^0(M)$, $\forall X,Y\in\Im_0^1(M)$, $\forall t\in\Im(M)$ (see [10], p.123).

On the other hand, a transformation defined by

$$\nabla:\Im_0^1(M)\times\Im_0^1(M)\to\Im_0^1(M)$$

is called as an affin connection (see for details [10,13]).

Proposition 2.5. *The horizontal lift of an affine connection ∇ in M to $T(M)$, denoted by ∇^H, defined by*

$$\begin{aligned}\nabla^H_{X^V}Y^V &= 0,\ \nabla^H_{X^V}Y^H=0,\\ \nabla^H_{X^H}Y^V &= (\nabla_XY)^V,\nabla^H_{X^H}Y^H=(\nabla_XY)^H\end{aligned}\tag{2.4}$$

for any $X,Y\in\Im_0^1(M)$ [17].

Theorem 2.6. *Let ${}^{CG}g$ be Kaluza-Klein metric, is defined by (1.13) and the horizontal lift ∇^H of a symetric affine connection ∇ in M to $T(M)$. From (2.4) and Definition (2.4), we get the following results*

$$\begin{aligned}i)\ (\nabla^H_{X^V}\,{}^{KK}g)(Y^V,Z^V) &= a'(t)g(X,u)g(Y,Z)+b'(t)g(X,u)g(Y,u)g(Z,u)\\ &\quad +bg(X,Y)g(Z,u)+bg(Y,u)g(X,Z),\\ ii)\ (\nabla^H_{X^H}\,{}^{KK}g)(Y^V,Z^V) &= a((\nabla_Xg)(Y,Z)),\\ iii)\ (\nabla^H_{X^V}\,{}^{KK}g)(Y^H,Z^H) &= c'g(X,u)g(Y,Z),\\ iv)\ (\nabla^H_{X^H}\,{}^{KK}g)(Y^H,Z^H) &= c((\nabla_Xg)(Y,Z)),\\ v)\ (\nabla^H_{X^V}\,{}^{KK}g)(Y^V,Z^H) &= 0,\\ vi)\ (\nabla^H_{X^V}\,{}^{KK}g)(Y^H,Z^V) &= 0,\\ vii)\ (\nabla^H_{X^H}\,{}^{KK}g)(Y^H,Z^V) &= 0,\\ viii)\ (\nabla^H_{X^H}\,{}^{KK}g)(Y^V,Z^H) &= 0,\end{aligned}$$

where the vertical lift $X^V \in \Im_0^1(TM)$ of $X \in \Im_1^0(M)$ and the horizontal lifts $X^H \in \Im_0^1(TM)$ of $X \in \Im_0^1(M)$ defined by (1.1) and (1.2), respectively.

Proof. *i)*

$$\begin{aligned}
(\nabla^H_{X^V}\ {}^{KK}g)(Y^V,Z^V) &= \nabla^H_{X^V}\ {}^{KK}g(Y^V,Z^V) - {}^{KK}g(\nabla^H_{X^V}Y^V,Z^V) - {}^{KK}g(Y^V,\nabla^H_{X^V}Z^V) \\
&= X^V(ag(Y,Z)+bg(Y,u)g(Z,u)) \\
&= a'(t)g(X,u)g(Y,Z)+b'(t)g(X,u)g(Y,u)g(Z,u) \\
&\quad +bg(X,Y)g(Z,u)+bg(Y,u)g(X,Z)
\end{aligned}$$

ii)

$$\begin{aligned}
(\nabla^H_{X^H}\ {}^{KK}g)(Y^V,Z^V) &= \nabla^H_{X^H}\ {}^{KK}g(Y^V,Z^V) - {}^{KK}g(\nabla^H_{X^H}Y^V,Z^V) - {}^{KK}g(Y^V,\nabla^H_{X^H}Z^V) \\
&= X^H(ag(Y,Z)+bg(Y,u)g(Z,u)) - {}^{KK}g((\nabla_X Y)^V,Z^V) \\
&\quad - {}^{KK}g(Y^V,(\nabla_X Z)^V) \\
&= aXg(Y,Z)+bg((\nabla_X Y),u)g(Z,u)+bg(Y,u)g((\nabla_X Z),u) \\
&\quad -ag((\nabla_X Y),Z)-bg((\nabla_X Y),u)g(Z,u) \\
&\quad -ag(Y,(\nabla_X Z))-bg(Y,u)g((\nabla_X Z),u) \\
&= a((\nabla_X g)(Y,Z))
\end{aligned}$$

iii)

$$\begin{aligned}
(\nabla^H_{X^V}\ {}^{KK}g)(Y^H,Z^H) &= \nabla^H_{X^V}\ {}^{KK}g(Y^H,Z^H) - {}^{KK}g((\nabla^H_{X^V}Y^H),Z^H) - {}^{KK}g(Y^H,(\nabla^H_{X^V}Z^H)) \\
&= X^V(cg(Y,Z)) \\
&= c'g(X,u)g(Y,Z)
\end{aligned}$$

iv)

$$\begin{aligned}
(\nabla^H_{X^H}\ {}^{KK}g)(Y^H,Z^H) &= \nabla^H_{X^H}\ {}^{KK}g(Y^H,Z^H) - {}^{KK}g(\nabla^H_{X^H}Y^H,Z^H) - {}^{KK}g(Y^H,\nabla^H_{X^H}Z^H) \\
&= X^H(cg(Y,Z)) - {}^{KK}g((\nabla_X Y)^H,Z^H) - {}^{KK}g(Y^H,(\nabla_X Z)^H) \\
&= cXg(Y,Z)-cg((\nabla_X Y),Z)-cg(Y,(\nabla_X Z)) \\
&= c((\nabla_X g)(Y,Z))
\end{aligned}$$

v)

$$\begin{aligned}
(\nabla^H_{X^V}\ {}^{KK}g)(Y^V,Z^H) &= \nabla^H_{X^V}\ {}^{KK}g(Y^V,Z^H) - {}^{KK}g(\nabla^H_{X^V}Y^V,Z^H) - {}^{KK}g(Y^V,\nabla^H_{X^V}Z^H) \\
&= \nabla^H_{X^V}\ {}^{KK}g(Y^V,Z^H) \\
&= 0
\end{aligned}$$

vi)

$$\begin{aligned}
(\nabla^H_{X^V}\ {}^{KK}g)(Y^H,Z^V) &= \nabla^H_{X^V}\ {}^{KK}g(Y^H,Z^V) - {}^{KK}g(\nabla^H_{X^V}Y^H,Z^V) - {}^{KK}g(Y^H,\nabla^H_{X^V}Z^V) \\
&= \nabla^H_{X^V}\ {}^{KK}g(Y^H,Z^V) \\
&= 0
\end{aligned}$$

vii)

$$\begin{aligned}
(\nabla^H_{X^H}\ {}^{KK}g)(Y^H,Z^V) &= \nabla^H_{X^H}\ {}^{KK}g(Y^H,Z^V) - {}^{KK}g(\nabla^H_{X^H}Y^H,Z^V) - {}^{KK}g(Y^H,\nabla^H_{X^H}Z^V) \\
&= -{}^{KK}g((\nabla_X Y)^H,Z^V) - {}^{KK}g(Y^H,(\nabla_X Z)^V) \\
&= 0
\end{aligned}$$

viii)

$$\begin{aligned}(\nabla^H_{X^H}\ {}^{KK}g)(Y^V,Z^H) &= \nabla^H_{X^H}\ {}^{KK}g(Y^V,Z^H) -{}^{KK} g((\nabla^H_{X^H}Y^V),Z^H) -{}^{KK} g(Y^V,(\nabla^H_{X^H}Z^H)) \\ &= -{}^{KK}g((\nabla_X Y)^V,Z^H) -{}^{KK} g(Y^V,(\nabla_X Z)^H) \\ &= 0\end{aligned}$$

□

Acknowledgments

The authors are grateful to the referee for his/her valuable comments and suggestions.

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