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Results of Singular Direchelet Problem Involving the $p(x)$-laplacian with Critical Growth

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ABSTRACT: In this paper, we study the existence and multiplicity of solutions for Dirichlet singular elliptic problems involving the $p(x)$-Laplace equation with critical growth. The technical approach is mainly based on the variational method combined with the genus theory.

Key Words: $p(x)$-laplacian, critical growth, multiple solutions, critical point theory.

Contents

1. Introduction

In this work, we want to study the following nonlinear Dirichlet problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) - a(x)|u|^{p(x)-2}u = \mu\frac{|u|^{r(x)-2}u}{|x|^{s(x)}} + f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $0 \in \Omega$ is an open bounded subset of $\mathbb{R}^N (N \geq 2)$, with smooth boundary $\partial\Omega$, $\mu > 0$ are a reals parameters, $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is of Carathéodory function which satisfies somes growth conditions, $p \in C_+(\overline{\Omega})$ with $1 < p^- := \inf_{\overline{\Omega}} p(x) \leq p^+ := \sup_{\overline{\Omega}} p(x) < N$, $r \in C_+(\overline{\Omega})$ with $1 < r^- := \inf_{\overline{\Omega}} r(x) \leq r^+ := \sup_{\overline{\Omega}} r(x) < N$ and $s \in C_+(\overline{\Omega})$ with $1 < s^- := \inf_{\overline{\Omega}} s(x) \leq s^+ := \sup_{\overline{\Omega}} s(x) < N$.

The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years, we can for example refer to [9,15,20]. This great interest may be justified by their various physical applications. In fact, there are applications concerning image restoration [7], dielectric breakdown, electrical resistivity and polycrystal plasticity [3,4] and continuum mechanics [2].

The study of differential equations and variational problems involving the $p(x)$-Laplace operator $-\Delta_{p(x)}u := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, which is a natural generalization of the p-Laplace operator. The $p(x)$-Laplacian operator possesses more complicated nonlinearities than the p-Laplacian operator, mainly due to the fact that it is not homogeneous.

In the last decades, several authors have focused on Quasilinear elliptic problems involving the Hardy potential driven by the p-Laplacian. The main interest of this kind of problems is the presence of the singular potential $\frac{1}{|x|^s}$, $0 \leq s \leq p$, $\frac{1}{|x|^s}$ relating to the Hardy inequality and this equation arise in the context of geophysical and industrial contents; see Callegari and Nachman [6].

In this context, with the critical Sobolev-Hardy exponent, we mention the paper of N. Ghoussoub and C. Yuan [16] studied the following elliptic problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \mu\frac{|u|^{q-2}u}{|x|^s} + \lambda|u|^{r-2}u & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

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where $\lambda, \mu > 0$ and f verified some conditions. By variational methods, in [[16],Theorem 8.1] are consider the problem ((1.2)) with the critical Sobolev-Hardy exponent and with the critical Sobolev exponent proved the existence of solutions under somes conditions but in [[16],Theorem 10.1] under condition $1 < p \leq q < p^*(s)$, $r = p^*$ are obtained some results under differ cases of the problem ((1.2)).

When $p = 2$, A. Ferrero and F. Gazzola [13] with the critical Sobolev exponent are studied ((1.2)) with the assumptions of perturbed fonctional $g(x,s)$ and proved the result when, roughly speaking, $g(x,s)$ stays below $\lambda_1 s$ in a neighborhood of $s = 0$.

Under $p > 1$, Y. Li, Q. Guo and P. Niu [17] are investigated the quasilinear elliptic equations with Dirichlet boundary conditions and combined critical Sobolev–Hardy terms on bounded smooth domains and proved the existence and multiplicity of solutions by employed Ekeland's variational principle.

If the singular problem is driven by $p(x)$-laplacian operator, we refer to [18] and [21] where further bibliographical references can be found. R. M. Khanghahia, A. Razania [18] are considered the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \frac{|u|^{s-2}u}{|x|^s} = \lambda f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is an open bounded subset of $\mathbb{R}^N (N \geq 2)$, with smooth boundary and $p \in C_+(\overline{\Omega})$ with

$$1 < p^- := \inf_{\overline{\Omega}} p(x) \leq p^+ := \sup_{\overline{\Omega}} p(x) < +\infty.$$

where $\lambda > 0$ is a real parameter and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is of Carathéodory function which satisfies Ambrosetti-Rabinowitz's type condition, and are proveded the existence of two weak solutions .

With the critical Sobolev-Hardy exponent, Y. Mei, F. Yongqiang and L. Wang [21] are considered the following $p(x)$-Laplacian problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = \frac{h(x)|u|^{p_s^*(x)-2}u}{|x|^{s(x)}} + f(x,u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.4}$$

where $0 \in \overline{\Omega}$ is an open bounded subset of $\mathbb{R}^N (N \geq 2)$, with smooth boundary and $p \in C_+(\overline{\Omega})$ Lipschitz and radially symmetric on $\overline{\Omega}$, $s(x)$ is Lipschitz and radially symmetric on $\overline{\Omega}$ with

$$1 < p^- := \inf_{\overline{\Omega}} p(x) \leq p^+ := \sup_{\overline{\Omega}} p(x) < N,$$

$$0 \leq s(x) \ll p(x),$$

and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is of Carathéodory function which satisfies somes conditions.

In this paper, the author's inspired of the of X.Fan [10] and based on the Theorem 2.10 and 2.11 established a principle of concentration compactness and obtained the existence of solutions for the problem (1.4).

Motivated by this interest and inspired by the works cited above, the main contribution of the manuscript is the existence and multiplicity of sulutions for ((1.1)) under the assumptions that the nonlinearity g is superlinear and satisfies some subcritical growth conditions.

The aim of the article is to consider the problem ((1.1)) and we also considerably generalize the results in [[16],Theorem 10.1] and the results in [[18],Theorem 2.2] under assumptions

$$f(x,u) = g(x,u) + \lambda |u|^{q(x)-2}u$$

where $\mathcal{A} = \{q(x) = p^*(x)\}$ is nonempty and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is of Carathéodory function which satisfies somes growth conditions.

The difficulty in our work, is due to the lack of compactness of the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*(x)}(\Omega)$ and the Palais–Smale condition for the corresponding energy functional could not be checked directly. To deal with this difficulty, we use a version of the concentration compactness lemma due to Lions for variable exponents [5].

Throughout this work, we make the following assumptions on the Dirichlet problem (1.1):

$(H1)$

$$1 < s(x) < p^- < N, \tag{1.5}$$

$(H2)$

$$1 < p^+ < r^- < p_s^*(x), \tag{1.6}$$

where by [[10], Remark 2.1] we defined

$$p_s^*(x) = p(x)\frac{N - s(x)}{N - p(x)}.$$

$(H3)$

$$p^+ < q^- := \inf_{\overline{\Omega}} q(x) \le q(x) \le p^*(x), \tag{1.7}$$

where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \ge N. \end{cases}$$

(A_1) $a(x) \in L^\infty(\Omega)$ and there exists $\alpha > 0$ such that

$$\int_\Omega \left(\frac{|\nabla u|^{p(x)}}{p(x)} - a(x)\frac{|u|^{p(x)}}{p(x)} \right) dx \ge \alpha \int_\Omega \frac{|u|^{p(x)}}{p(x)} dx, \forall u \in W_0^{1,p(x)}(\Omega)$$

(A_2) $p(x) = p^+$ for all x in $\Gamma^+ := \{x \in \Omega : a(x) > 0\}$;

(G_1) $g \in C(\in \overline{\Omega} \times \mathbb{R}, \mathbb{R})$, odd with respect to t and

$$g(x,t) = o(|t|^{p(x)-1}), |t| \to 0 \text{ uniformly } x \text{ in } \overline{\Omega}$$

$$g(x,t) = o(|t|^{q(x)-1}), |t| \to +\infty \text{ uniformly } x \text{ in } \overline{\Omega}$$

(G_2) $G(x,t) \le \frac{1}{p^+} g(x,t)$ for all $t > 0$ and a.e in Ω, where $G(x,t) = \int_0^t gx, s)ds$.

to illustrate these conditions, you can consult example 1.1 in [19].

Our main results are the following

Theorem 1.1. *Assume that $(A_1) - (A_2)$,$(G_1) - (G_2)$ hold. Then, there exist a sequence $(\lambda_k) \subset (0, +\infty)$ with $\lambda_k > \lambda_{k+1}$, such that for any $\lambda \in (\lambda_{k+1}, \lambda_k]$, problem (1.1) has at least k pairs of nontrivial solutions.*

The remainder of this paper is organized as follows, in Section 2 we state some basic properties of the variable exponent Lebesgue-Sobolev in order to solve our problem, finally, in Section 3 we prove the main results of this work.

2. Preliminaries

In the sequel, let $p(x) \in C_+(\overline{\Omega})$, where

$$C_+(\overline{\Omega}) = \left\{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\right\}.$$

The variable exponent Lebesgue space is defined by

$$L^{p(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable and} \int_\Omega |u(x)|^{p(x)}\, dx < +\infty\}$$

furnished with the Luxemburg norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\sigma > 0 : \int_\Omega |\frac{u(x)}{\sigma}|^{p(x)}\, dx \leq 1\},$$

Remark 2.1. Variable exponent Lebesgue spaces resemble to classical Lebesgue spaces in many respects, they are separable Banach spaces and the Hölder inequality holds. The inclusions between Lebesgue spaces are also naturally generalized, that is, if $0 < mes(\Omega) < \infty$ and p, q are variable exponents such that $p(x) < q(x)$ a.e. in Ω, then there exists a continuous embedding $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$.

The variable exponent Sobolev space is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.2. *[11,12] The spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable, uniformly convex, reflexive Banach spaces. The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $q(x)$ is the conjugate function of $p(x)$; i.e.,*

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

for all $x \in \Omega$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$ we have

1. $$\Big|\int_\Omega u(x)v(x)dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)}.$$

2. *If $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$ $L^{p_2(x)} \hookrightarrow L^{p_1(x)}$ and the embedding is continuous.*

Proposition 2.3 ([23]). *Let $\rho(u) = \int_\Omega |u|^{p(x)}dx$. For $u, u_n \in L^{p(\cdot)}(\Omega)$, we have*

(1) $|u|_{p(\cdot)} < \ (=;>); 1 \Leftrightarrow \rho(u) < \ (=;>)\ 1;$

(2) $|u|_{p(\cdot)} > 1 \Rightarrow |u|^{p^-}_{p(\cdot)} \leq \rho(u) \leq |u|^{p^+}_{p(\cdot)};$

(3) $|u|_{p(\cdot)} < 1 \Rightarrow |u|^{p^+}_{p(\cdot)} \leq \rho(u) \leq |u|^{p^-}_{p(\cdot)};$

(4) $|u_n|_{p(\cdot)} \to 0 \Leftrightarrow \rho(u_n) \to 0;$

(5) $|u_n|_{p(\cdot)} \to \infty \Leftrightarrow \rho(u_n) \to \infty.$

Moreover, if $h_1, h_2, h_3 : \overline{\Omega} \to (1, \infty)$ are Lipschitz continuous functions such that $\frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} = 1$, then for any $u \in L^{h_1(x)}(\Omega)$, $v \in L^{h_2(x)}(\Omega)$, $w \in L^{h_3(x)}(\Omega)$, the following inequality holds see [9, Proposition 2.5]

$$\int_\Omega |uvw|dx \leq (\frac{1}{h_1^-} + \frac{1}{h_2^-} + \frac{1}{h_3^-})|u|_{h_1(x)}|v|_{h_2(x)}|w|_{h_3(x)}. \tag{2.1}$$

Proposition 2.4. *[8] Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^\infty(\Omega)$ and $1 \leq p(x)q(x) \leq \infty$, for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$|u|_{p(x)q(x)} \leq 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)},$$
$$|u|_{p(x)q(x)} \geq 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \leq ||u|^{p(x)}|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}.$$

In particular if $p(x) = p$ is a constant, then

$$|\,|u|^p|_{q(x)} = |u|^p_{pq(x)}.$$

Proposition 2.5 ([12]). *In $W_0^{1,p(x)}(\Omega)$ the Poincaré inequality holds; that is, there exists a positive constant C_0 such that*

$$|u|_{L^{p(x)}(\Omega)} \leq C_0|\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

So, $|\nabla u|_{L^{p(x)}(\Omega)}$ is a norm equivalent to the norm $\|u\|$ in the space $W_0^{1,p(x)}(\Omega)$. We will use the equivalent norm in the following discussion and write $\|u\|_p = |\nabla u|_{L^{p(x)}(\Omega)}$ for simplicity.

Proposition 2.6. *[11,12] Assume that the boundary of Ω possesses the cone property and $p, q \in C_+(\overline{\Omega})$ such that $q(x) \leq p^*(x)$ $(q(x) < p^*(x))$ for all $x \in \overline{\Omega}$, then there is a continuous (compact) embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega),$$

We write

$$I(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\, dx.$$

Proposition 2.7 ([12]). *The functional $I : X \to \mathbb{R}$ is convex. The mapping $I' : X \to X^*$ is a strictly monotone, bounded homeomorphism, and is of (S_+) type, namely*

$$u_n \rightharpoonup u \text{ and } \limsup_{n\to+\infty} I'(u_n)(u_n - u) \leq 0 \text{ implies } u_n \to u,$$

where $X = W_0^{1,p(x)}(\Omega)$, X^* is the dual space of X.

Let us now consider the weighted variable exponent Lebesgue space. Let a measurable function $c : \Omega \to \mathbb{R}$. Define

$$L^{p(x)}_{c(x)}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ measurable and} \int_\Omega c(x)|u(x)|^{p(x)}\, dx < +\infty\}$$

furnished with the Luxemburg norm

$$|u|_{L^{p(x)}_{c(x)}(\Omega)} = |u|_{(p(x),c(x))} = \inf\{\sigma > 0 : \int_\Omega c(x)|\frac{u(x)}{\sigma}|^{p(x)}\, dx \leq 1\}.$$

Then, $L^{p(x)}_{c(x)}(\Omega)$ is a Banach space.

Proposition 2.8 ([10]). *Set $\rho(u) = \int_\Omega b(x)|u(x)|^{p(x)}dx$. For $u \in L^{p(x)}_{b(x)}(\Omega)$, we have*

(i) $|u|_{(p(x),b(x))} < 1 (= 1; > 1) \Leftrightarrow \rho(u) < 1(= 1; > 1)$;

(ii) *If* $|u|_{(p(x),b(x))} < 1 \Rightarrow |u|^{p^+}_{(p(x),b(x))} \leq \rho(u) \leq |u|^{p^-}_{(p(x),b(x))}$;

(iii) *If* $|u|_{(p(x),b(x))} > 1 \Rightarrow |u|^{p^-}_{(p(x),b(x))} \leq \rho(u) \leq |u|^{p^+}_{(p(x),b(x))}$.

Proposition 2.9 ([10], Corollary 2.1). *Assume that $0 \in \overline{\Omega}$ and the boundary of Ω) possesses the cone property.Suppose that $p, s, r \in C(\overline{\Omega})$, $0 \leq s(x) \leq N$.If r satisfies the condition $1 \leq r(x) \leq \frac{N-s(x)}{N} p^*(x)$ $\forall\ x \in \overline{\Omega}$ then there is a compact embedding $W^{1,p(x)}(\Omega) \to L^{r(x)}_{|x|^{-s(x)}}(\Omega)$.*

Lemma 2.10 ([1]). *Let $h, r \in L^\infty(\Omega)$ with $h(x) \leq r(x)$ a.e in Ω and $u \in L^{r(x)}(\Omega)$. Then, $|u|^{h(x)} \in L^{\frac{r(x)}{h(x)}}(\Omega)$ and*

$$\||u|^{h(x)}\|_{L^{\frac{r(x)}{h(x}}(\Omega)} \leq \|u\|^{h^+}_{L^{r(x)}(\Omega)} + \|u\|^{h^-}_{L^{r(x)}(\Omega)}$$

or

$$\||u|^{h(x)}\|_{L^{\frac{r(x)}{h(x}}(\Omega)} \leq \max\left(\|u\|^{h^+}_{L^{r(x)}(\Omega)}, \|u\|^{h^-}_{L^{r(x)}(\Omega)}\right)$$

Lemma 2.11 ([1]). *For each $z \in \overline{\Omega}$ and $u \in L^{p(x)}$,*

$$\int_\Omega |u(x)\nabla\phi_{j,\epsilon}(x-z)|^{p(x)} dx \leq C_5 \left\{\|u\|^{p^+}_{L^{p^*}(B_\epsilon(z)))} + \|u\|^{p^-}_{L^{p^*(x)}(B_\epsilon(z))}\right\}$$

Proposition 2.12 ([21], Lemma 3.1). *let $\{u_n\} \subset L^{r(x)}_{|x|^{-s(x)}}(\Omega)$ be bounded, and $u_n \to u \in L^{r(x)}_{|x|^{-s(x)}}(\Omega)$ a.e on Ω, then*

$$\lim_{n\to\infty} \int_\Omega \left[\frac{|u_n|^{r(x)}}{|x|^{s(x)}} - \frac{|u_n - u|^{r(x)}}{|x|^{s(x)}}\right] dx = \int_\Omega \frac{|u|^{r(x)}}{|x|^{s(x)}} dx$$

Proposition 2.13 ([5], Theorem 1.1). *Let $q(x)$ and $p(x)$ be two continuous functions such that*

$$1 < \inf_{x\in\Omega} p(x) \leq \sup_{x\in\Omega} p(x) < N \quad \text{and} \quad 1 \leq q(x) \leq p^*(x) \quad \text{in } \Omega.$$

Let $\{u_j\}_{j\in\mathbb{N}}$ be a weakly convergent sequence in $W_0^{1,p(x)}(\Omega)$ with weak limit u, and such that:

- $|\nabla u_j|^{p(x)} \rightharpoonup \mu$ *weakly-* in the sense of measures.*
- $|u_j|^{q(x)} \longrightarrow \nu$ *weakly-* in the sense of measures.*

Also assume that $\mathcal{A} = \{x \in \Omega \colon q(x) = p^*(x)\}$ is nonempty. Then, for some countable index set I, we have:

$$\nu = |u|^{q(x)} + \sum_{i\in I} \nu_i \delta_{x_i} \quad \nu_i > 0 \tag{2.2}$$

$$\mu \geq |\nabla u|^{p(x)} + \sum_{i\in I} \mu_i \delta_{x_i} \quad \mu_i > 0 \tag{2.3}$$

$$S\nu_i^{1/p^*(x_i)} \leq \mu_i^{1/p(x_i)} \quad \forall i \in I. \tag{2.4}$$

where $\{x_i\}_{i\in I} \subset \mathcal{A}$ and S is the best constant in the Gagliardo-Nirenberg-Sobolev inequality for variable exponents, namely

$$S = S_q(\Omega) := \inf_{\phi\in C_0^\infty(\Omega)} \frac{\||\nabla\phi|\|_{L^{p(x)}(\Omega)}}{\|\phi\|_{L^{q(x)}(\Omega)}}.$$

3. Proof of Main result

Associated with the problem (1.1), we have the energy functional $I : X := W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}dx - \int_\Omega \frac{a(x)}{p(x)}|u(x)|^{p(x)}dx - \mu\int_\Omega \frac{|u(x)|^{r(x)}}{|x|^{s(x)}r(x)}dx$$
$$- \lambda\int_\Omega \frac{1}{q(x)}|u(x)|^{q(x)}dx - \int_\Omega G(x,u)dx. \tag{3.1}$$

By conditions $(A_1)-(G_1)$, $I \in C(X,\mathbb{R})$ with

$$I^{'}(u)v = \int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx - \int_\Omega a(x)|u|^{p(x)-2}uvdx$$
$$- \mu\int_\Omega \frac{|u(x)|^{r(x)-2}u}{|x|^{s(x)}}vdx - \lambda\int_\Omega |u(x)|^{q(x)-2}uvdx - \int_\Omega g(x,u)vdx \tag{3.2}$$

Definition 3.1. *We say that $u \in X$ is a weak solution of problem (1.1) if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla v dx - \int_\Omega a(x)|u|^{p(x)-2}uvdx$$
$$= \mu\int_\Omega \frac{|u(x)|^{r(x)-2}u}{|x|^{s(x)}}vdx + \lambda\int_\Omega |u(x)|^{q(x)-2}uvdx + \int_\Omega g(x,u)vdx$$

for all $v \in X$

We recall a version of the Mountain Pass Theorem for even functional involving genus theory, which will be used in proof of Theorem (1.1). For details of the proof, see [22] or [14] .

Theorem 3.1. *Let E be an infinite dimensional Banach space with $E = V \oplus X$, where V is finite dimensional and let $I \in C^1(E,\mathbb{R})$ be a even function with $I(0) = 0$ and satisfying*

(i) There are constants $\beta, \varrho > 0$ such that $I(u) \geq \beta$ for all $u \in \partial B_\varrho \cap X$;

(ii) There is $\tau > 0$ such that I satisfies the $(PS)_c$) condition $0 < c < \tau$;

(iii) For each finite dimensional subspace $\tilde{E} \subset E$, there is $R = R(\tilde{E}) > 0$ such that $I(u) \leq 0$ for all $u \in \tilde{E}\backslash B_R(0)$.

Suppose V is k dimensional and $V = span\{e_1, ..., e_k\}$. For $n \geq k$, inductively choose $e_{n+1} \notin E_n := span\{e_1, ..., e_n\}$. Let $R_n = R(E_n)$ and $D_n = B_{R_n} \cap E_n$. Define

$$G_n := \{h \in C(D_n, E) : h \text{ is odd and } h(u) = u, \forall\, \partial B_{R_n} \cap E_n\}.$$

$$\Gamma_j = \left\{h\left(\overline{D_n\backslash Y}\right) : h \in G_n, n \geq j, Y \in \Sigma, \text{ and} \gamma(Y) \leq n-j\right\},$$

where

$$\Sigma = \{Y \subset E\backslash\{0\} : Y \text{ is closed in } E \text{ and } Y = -Y,\}$$

and $\gamma(Y)$ is the genus of $Y \in \Sigma$. For each $j \in \mathbb{N}$, let

$$c_j = \inf_{k\in\Gamma_j}\max_{u\in K} I(u)$$

Then $0 < \beta \leq c_j \leq c_{j+1}$ for $j > k$, and if $j > k$ and $c_j \leq \tau$, we have that c_j is the critical value of I. Moreover, if $c_j = c_{j+1} = \cdots = c_{j+l} = c < \tau$ for $j > k$, then $\gamma(K_c) \geq l+1$, where

$$K_c = \{u \in E : I(u) = c \text{ and } I^{'}(u) = 0\}.$$

In the sequel, we derive some results related to the above theorem and Palais-Smale compactness condition.

Lemma 3.2. *Assume that* (A_1) *and* (G_1) *hold. Then for each* $\lambda > 0$, I *satisfies condition* (i) *given in Theorem* (3.1).

Proof. . Let $\delta > 0$. By (A_1), we have

$$\begin{aligned}
&\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}dx - \int_\Omega \frac{a(x)}{p(x)}|u(x)|^{p(x)}dx \\
= \;& \frac{1}{\delta+1}\left(\int_\Omega \frac{|\nabla u(x)|^{p(x)}}{p(x)}dx - \int_\Omega \frac{a(x)}{p(x)}|u(x)|^{p(x)}dx\right) \\
+ \;& \frac{\delta}{\delta+1}\int_\Omega\left(\frac{|\nabla u(x)|^{p(x)}}{p(x)}dx - \int_\Omega \frac{a(x)}{p(x)}|u(x)|^{p(x)}dx\right) \\
= \;& \frac{1}{\delta+1}\int_\Omega\left(\frac{|\nabla u(x)|^{p(x)}}{p(x)}dx - \int_\Omega \frac{a(x)}{p(x)}|u(x)|^{p(x)}dx\right) \\
+ \;& \frac{\delta}{\delta+1}\int_\Omega\left(\frac{|\nabla u(x)|^{p(x)}}{p(x)}dx\right) - \frac{\delta}{\delta+1}\int_\Omega \frac{a(x)}{p(x)}|u(x)|^{p(x)}dx \\
\geq \;& \frac{\alpha}{\delta+1}\int_\Omega \frac{a(x)}{p(x)}|u(x)|^{p(x)}dx + \frac{\delta}{\delta+1}\int_\Omega\left(\frac{|\nabla u(x)|^{p(x)}}{p(x)}dx\right) - \frac{\delta\|a\|_\infty}{(1+\delta)p^-}\int_\Omega u(x)|^{p(x)}dx \\
\geq \;& \frac{\delta}{\delta+1}\int_\Omega\left(\frac{|\nabla u(x)|^{p(x)}}{p(x)}dx\right) + \frac{1}{\delta+1}\left(\frac{\alpha}{p^+} - \frac{\delta\|a\|_\infty}{p^-}\right)\int_\Omega u(x)|^{p(x)}dx.
\end{aligned}$$

We can choose $\delta > 0$ such that $C_0 := \frac{1}{\delta+1}\left(\frac{\alpha}{p^+} - \frac{\delta\|a\|_\infty}{p^-}\right) > 0$. So

$$\int_\Omega\left(\frac{|\nabla u(x)|^{p(x)}}{p(x)}\right)dx - \int_\Omega \frac{a(x)}{p(x)}|u(x)|^{p(x)}dx \geq \frac{\delta}{\delta+1}\int_\Omega\left(\frac{|\nabla u(x)|^{p(x)}}{p(x)}dx\right) \tag{3.3}$$

$$+C_0\int_\Omega |u(x)|^{p(x)}dx. \tag{3.4}$$

On the other hand, form (A_1) and (G_1), given $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|G(x,t)| \leq \frac{\epsilon}{p(x)}|t|^{p(x)} + \frac{C_\epsilon}{q(x)}|t|^{q(x)}. \tag{3.5}$$

Combining (3.3) and (3.5),

$$\begin{aligned}
I(u) \;\geq\;& \frac{\delta}{\delta+1}\int_\Omega\left(\frac{|\nabla u(x)|^{p(x)}}{p(x)}\right)dx - \lambda\int_\Omega \frac{1}{q(x)}|u(x)|^{q(x)}dx \\
+\;& \left(C_0 - \frac{\epsilon}{p^-}\right)\int_\Omega |u(x)|^{p(x)}dx - C_\epsilon\int_\Omega \frac{1}{q(x)}|u(x)|^{q(x)}dx - \mu\int_\Omega \frac{|u(x)|^{r(x)}}{|x|^{s(x)}r(x)}dx \\
\geq\;& \frac{\delta}{(\delta+1)p^+}\int_\Omega\left(|\nabla u(x)|^{p(x)}\right)dx + \left(C_0 - \frac{\epsilon}{p^-}\right)\int_\Omega |u(x)|^{p(x)}dx \\
-\;& \frac{\lambda + C_\epsilon}{q^-}\int_\Omega \frac{1}{q(x)}|u(x)|^{q(x)}dx - \mu\int_\Omega \frac{|u(x)|^{r(x)}}{|x|^{s(x)}r(x)}dx.
\end{aligned}$$

Hence for ϵ sufficiently small,

$$I(u) \geq \frac{\delta}{(\delta+1)p^+}\int_\Omega\left(|\nabla|u(x)|^{p(x)}\right)dx - \frac{\lambda + C_\epsilon}{q^-}\int_\Omega \frac{1}{q(x)}|u(x)|^{q(x)}dx - \mu\int_\Omega \frac{|u(x)|^{r(x)}}{|x|^{s(x)}r(x)}dx.$$

By the continuous embedding see Proposition (2.6) $X \hookrightarrow L^{q(x)}(\Omega)$, there exists $C_0 > 0$ such that

$$|u|_{q^-} \leq C_0\|u\|,$$

and by Proposition (2.9) $W^{1,p(x)}(\Omega) \to L^{r(x)}_{|x|^{-s(x)}}(\Omega)$, there exists $C_1 > 0$ such that

$$|u|_{r(x)|x|^{-s(x)}} \leq C_1\|u\|.$$

Consequently, by Proposition (2.3), for $\|u\| = \varrho$, with $0 < \varrho < 1$,

$$I(u) \geq \frac{\delta}{(\delta+1)p^+}\|u\|^{p^+} - \frac{(\lambda + C_\epsilon)C_0}{q^-}\|u\|^{q^-} - \frac{\mu C_1}{r^-}\|u\|^{r^-}.$$

Since $\lambda, \mu > 0$, $p^+ < q^-$ and $p^+ < r^-$, there exists $\beta > 0$ such that $I(u) \geq \beta$ for $\|u\| = \varrho$, where ϱ is chosen sufficiently small.

Lemma 3.3. *Assume that (A_1) and (G_1) hold. Then for each $\lambda > 0$, I satisfies condition (iii) given in Theorem (3.1).*

A direct computation shows that given $\epsilon > 0$, there is $M_\epsilon > 0$ such that

$$G(x,t) \geq -M_\epsilon - \epsilon|t|^{q(x)}\forall(x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{3.6}$$

Consequently,

$$\begin{aligned} I(u) &\leq \frac{1}{p^-}\int_\Omega \left(|\nabla u(x)|^{p(x)}\right)dx + \frac{\|a\|_\infty}{p^-}\int_\Omega |u(x)|^{p(x)}dx + \left(\epsilon - \frac{\lambda}{q^+}\right)\int_\Omega |u(x)|^{q(x)}dx \\ &+ M_\epsilon|\Omega| - \frac{\mu}{r^+}\int_\Omega \frac{|u(x)|^{r(x)}}{|x|^{s(x)}}dx. \end{aligned}$$

By choosing $\epsilon = \frac{\lambda}{2q^+}$

$$\begin{aligned} I(u) &\leq \frac{1}{p^-}\int_\Omega \left(|\nabla u(x)|^{p(x)}\right)dx + \frac{\|a\|_\infty}{p^-}\int_\Omega |u(x)|^{p(x)}dx + \left(\epsilon - \frac{\lambda}{q^+}\right)\int_\Omega |u(x)|^{q(x)}dx \\ &+ M_\epsilon|\Omega| - \frac{\mu}{r^+}\int_\Omega \frac{|u(x)|^{r(x)}}{|x|^{s(x)}}dx \\ &\leq \frac{1}{p^-}\int_\Omega |\nabla u(x)|^{p(x)}dx + \|a\|_\infty\int_\Omega |u(x)|^{p(x)}dx - \frac{\lambda}{2q^+}\int_\Omega |u(x)|^{q(x)}dx \\ &+ M_\epsilon|\Omega| - \frac{\mu}{r^+}\int_\Omega \frac{|u(x)|^{r(x)}}{|x|^{s(x)}}dx. \end{aligned}$$

Since $dimE < \infty$, the norms $\|.\|$ and $|.|_{q(x)}$ and $|u|_{r(x)|x|^{-s(x)}}$ are equivalent in E. According to Proposition (2.3) and Proposition (2.8), for $\left(\|u\|, |u|_{p(x)}, |u|_{q(x)}, |u|_{r(x)|x|^{-s(x)}}\right) > 1$,

$$\begin{aligned} I(u) &\leq \frac{1}{p^-}\int_\Omega |\nabla u(x)|^{p(x)}dx + \|a\|_\infty\int_\Omega |u(x)|^{p(x)}dx - \frac{\lambda}{2q^+}\int_\Omega |u(x)|^{q(x)}dx \\ &+ M_\epsilon|\Omega| - \frac{\mu}{r^+}\int_\Omega \frac{|u(x)|^{r(x)}}{|x|^{s(x)}}dx \\ &\leq \frac{1}{p^-}\left(\|u\|^{p^+} + \|a\|_\infty C_1\|u\|^{p^+}\right) - \frac{\lambda C_2}{2q^+}\|u\|^{q^-} - \frac{\mu C_3}{r^+}\|u\|^{r^-} + M_\epsilon|\Omega|. \end{aligned}$$

By $\lambda, \mu > 0$, $p^+ < q^-$ and $p^+ < r^-$, we conclude that $I(u) < 0$ for $\|u\| \geq R > 1$, where R is chosen large enough. $\square$

Lemma 3.4. *Assume that $(A_1)-(A_2)$,$(G_1)-(G_2$ hold. Then any (PS) sequence of I is bounded in X.*

Let $\{u_n\}$ be a $(PS)_d$ sequence of I. Then,

$$I(u_n)\to d \text{ and } I^{'}(u_n)\to 0 \text{ as } n\to+\infty.$$

$$\begin{aligned}
d+1+\|u_n\| &\geq I(u_n)-\frac{1}{p^+}\langle I^{'}(u_n),u_n\rangle\\
&= \int_\Omega\left(\frac{1}{p(x)}-\frac{1}{p^+}\right)|\nabla u_n|^{p(x)}dx+\int_\Omega\left(\frac{1}{p^+}-\frac{1}{p(x)}\right)a(x)|u_n|^{p(x)}dx\\
&+ \lambda\int_\Omega\left(\frac{1}{p^+}-\frac{1}{q(x)}\right)|u_n|^{q(x)}dx+\mu\int_\Omega\left(\frac{1}{p^+}-\frac{1}{r(x)}\right)\frac{|u_n|^{r(x)}}{|x|^{s(x)}}dx\\
&+ \int_\Omega\left(\frac{1}{p^+}g(x,u_n)u_n-G(x,u_n)\right)dx\\
&\geq \int_\Omega\left(\frac{1}{p(x)}-\frac{1}{p^+}\right)|\nabla u_n|^{p(x)}dx+\int_\Omega\left(\frac{1}{p^+}-\frac{1}{p(x)}\right)a(x)|u_n|^{p(x)}dx\\
&+ \lambda\int_\Omega\left(\frac{1}{p^+}-\frac{1}{q(x)}\right)|u_n|^{q(x)}dx+\mu\int_\Omega\left(\frac{1}{p^+}-\frac{1}{r(x)}\right)\frac{|u_n|^{r(x)}}{|x|^{s(x)}}dx.
\end{aligned}$$

Then, for n sufficiently large

$$\begin{aligned}
&\lambda\int_\Omega\left(\frac{1}{p^+}-\frac{1}{q^-}\right)|u_n|^{q(x)}dx\\
\leq\ &\lambda\int_\Omega\left(\frac{1}{p^+}-\frac{1}{q(x)}\right)|u_n|^{q(x)}dx\\
\leq\ &d+1+\|u_n\|+\int_\Omega\left(\frac{1}{p(x)}-\frac{1}{p^+}\right)a(x)|u_n|^{p(x)}dx+\mu\int_\Omega\left(\frac{1}{r(x)}-\frac{1}{p^+}\right)\frac{|u_n|^{r(x)}}{|x|^{s(x)}}dx\\
\leq\ &d+1+\|u_n\|+\|a\|_\infty\left(\frac{1}{p^-}-\frac{1}{p^+}\right)\int_\Omega|u_n|^{p(x)}dx+\mu\left(\frac{1}{r^-}-\frac{1}{p^+}\right)\int_\Omega\frac{|u_n|^{r(x)}}{|x|^{s(x)}}dx.
\end{aligned}$$

On the other hand, by (1.5), for any $\epsilon>0$ there exists $C_\epsilon>0$ such that

$$|t|^{p(x)}<\epsilon|t|^{q(x)}+C_\epsilon \text{ for all } (x,t)\in\Omega\times\mathbb{R}. \tag{3.7}$$

Sine $p^+<r^-<p_s^*(x)$, it follows that

$$\begin{aligned}
&\lambda\left(\frac{1}{p^+}-\frac{1}{q^-}\right)\int_\Omega|u_n|^{q(x)}dx\leq d+1+\|u_n\|\\
+\ &\epsilon\|a\|_\infty\left(\frac{1}{p^-}-\frac{1}{p^+}\right)\int_\Omega|u_n|^{q(x)}dx+\|a\|_\infty\left(\frac{1}{p^-}-\frac{1}{p^+}\right)C_\epsilon|\Omega|.
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\lambda\left(\frac{1}{p^+}-\frac{1}{q^-}\right)-\epsilon\|a\|_\infty\left(\frac{1}{p^-}-\frac{1}{p^+}\right)\right)\int_\Omega|u_n|^{q(x)}dx&\leq d+1+\|u_n\|\\
\epsilon\|a\|_\infty\left(\frac{1}{p^-}-\frac{1}{p^+}\right)\int_\Omega|u_n|^{q(x)}dx&+\|a\|_\infty\left(\frac{1}{p^-}-\frac{1}{p^+}\right)C_\epsilon|\Omega|.
\end{aligned}$$

Choosing $\epsilon=\frac{\lambda}{2\|a\|_\infty}\frac{\frac{1}{p^+}-\frac{1}{q^-}}{\frac{1}{p^-}-\frac{1}{p^+}}$, we obtain

$$\frac{\lambda}{2}\left(\frac{1}{p^+}-\frac{1}{q^-}\right)\int_\Omega|u_n|^{q(x)}dx\leq d+1+\|u_n\|+\|a\|_\infty\left(\frac{1}{p^-}-\frac{1}{p^+}\right)C_\epsilon|\Omega|.$$

Thus

$$\int_\Omega |u_n|^{q(x)}dx \leq C_2(1+\|u_n\|). \tag{3.8}$$

Similarly , for n sufficiently large

$$\begin{aligned}
& \mu\left(\frac{1}{p^+}-\frac{1}{r^-}\right)\int_\Omega \frac{|u_n|^{r(x)}}{|x|^{s(x)}}dx \\
\leq\ & \mu\int_\Omega\left(\frac{1}{p^+}-\frac{1}{r(x)}\right)\frac{|u_n|^{r(x)}}{|x|^{s(x)}}dx \\
\leq\ & d+1+\|u_n\|+\int_\Omega\left(\frac{1}{p(x)}-\frac{1}{p^+}\right)a(x)|u_n|^{p(x)}dx+\lambda\int_\Omega\left(\frac{1}{q(x)}-\frac{1}{p^+}\right)|u_n|^{q(x)}dx \\
\leq\ & d+1+\|u_n\|+\|a\|_\infty\left(\frac{1}{p^-}-\frac{1}{p^+}\right)\int_\Omega |u_n|^{p(x)}dx+\lambda\int_\Omega\left(\frac{1}{q^-}-\frac{1}{p^+}\right)|u_n|^{q(x)}dx.
\end{aligned}$$

With $p^+ < q^-$ and (3.7), it follows that

$$\mu\left(\frac{1}{p^+}-\frac{1}{r^-}\right)\int_\Omega \frac{|u_n|^{r(x)}}{|x|^{s(x)}}dx \leq d+1+\|u_n\|$$
$$+\epsilon\|a\|_\infty\left(\frac{1}{p^-}-\frac{1}{p^+}\right)\int_\Omega |u_n|^{q(x)}dx+\|a\|_\infty\left(\frac{1}{p^-}-\frac{1}{p^+}\right)C_\epsilon|\Omega|.$$

By (3.11), we obtain

$$\mu\left(\frac{1}{p^+}-\frac{1}{r^-}\right)\int_\Omega \frac{|u_n|^{r(x)}}{|x|^{s(x)}}dx \leq d+1+\|u_n\|$$
$$+\epsilon\|a\|_\infty\left(\frac{1}{p^-}-\frac{1}{p^+}\right)C_2(1+\|u_n\|)+\|a\|_\infty\left(\frac{1}{p^-}-\frac{1}{p^+}\right)C_\epsilon|\Omega|.$$

Thus

$$\int_\Omega \frac{|u_n|^{r(x)}}{|x|^{s(x)}}dx \leq C_2(1+\|u_n\|). \tag{3.9}$$

By (3.3)

$$\begin{aligned}
\frac{\delta}{(\delta+1)p^+}\int_\Omega |\nabla|u_n|^{p(x)}dx &\leq I(u_n)+\lambda\int_\Omega |u_n|^{q(x)}dx+\int_\Omega G(x,u_n)dx \\
&\leq d+o_n(1)+\frac{\lambda}{q^-}\int_\Omega |u_n|^{q(x)}dx+\epsilon\int_\Omega |u_n|^{q(x)}dx+C_\epsilon|\Omega| \\
&= d+o_n(1)+\left(\frac{\lambda}{q^-}+\epsilon\right)\int_\Omega |u_n|^{q(x)}dx+\frac{\mu}{r^-}\int_\Omega \frac{|u_n|^{r(x)}}{|x|^{s(x)}}dx+C_\epsilon|\Omega|.
\end{aligned}$$

Therefore,for n sufficient large

$$\int_\Omega |\nabla|u_n|^{p(x)}dx \leq C_3(1+\|u_n\|),$$

and so

$$\min\left(\|u_n\|^{p^+},\|u_n\|^{p^-}\right) \leq C_4(1+\|u_n\|).$$

Consequently (u_n) is bounded in X.

In view of the last result, if (u_n) is (PS) sequence of I, we can extract a subsequence of (u_n),still denoted by (u_n), and $u \in X$. such that

- $u_n \rightharpoonup u$ in X,

- $u_n \rightharpoonup u$ in $L^{q(x)}(\Omega)$,
- $u_n \to u$ in $L^{m(x)}(\Omega)$, $m \in C_+(\overline{\Omega})$, $m(x) < p^*(x) \, \forall \, x \in \overline{\Omega}$.

By (1.7), from the concentration compactness lemma [5], there exist two nonnegative measures $\mu, \nu \in \mathcal{M}(\Omega)$, a countable set $\mathcal{J}$, points $\{x_j\}_{j\in\mathcal{J}}$ in Ω and sequence $\{\mu_j\}_{j\in\mathcal{J}}, \{\nu_j\}_{j\in\mathcal{J}} \subset [0,+\infty)$, such that

$$\begin{aligned} |\nabla u_n|^{p(x)} \rightharpoonup \mu &\geq |\nabla u_n|^{p(x)} + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j} \text{ in } \Omega, \\ |u_n|^{q(x)} \to \mu &= |u_n|^{q(x)} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j} \text{ in } \Omega, \\ S\nu_j^{\frac{1}{p^*(x_j)}} &\leq \mu_j^{\frac{1}{p^*(x_j)}} \text{ for all } j \in \mathcal{J}, \end{aligned} \tag{3.10}$$

where

$$S = S_q(\Omega) = \inf_{\phi \in C_0^\infty(\Omega)} \frac{\|\phi\|}{\|\phi\|_{L^{q(x)}(\Omega)}}.$$

Let $\phi \in C_0^\infty(\mathbb{R}^{\mathbb{N}})$ such that $\phi \equiv 1$ in $B_1(0)$, $\phi \equiv 0$ on $\Omega \backslash B_1(0)$

For $\epsilon > 0$ and $j \in \mathcal{J}$ consider $\phi_{j,\epsilon} \in C^\infty(\mathbb{R}^{\mathbb{N}})$ such that

$$\phi_{j,\epsilon}(x) = \phi(\frac{x - x_j}{\epsilon}), \text{ for all } x \in \mathbb{R}^{\mathbb{N}},$$

and $|\nabla \phi_{j,\epsilon}|_\infty \leq \frac{2}{\epsilon}$ where $x_j \in \overline{\Omega}$ belongs to the support of ν.

Lemma 3.5. *Under the conditions of Lemma (3.4), if $\{u_n\}$ is a (PS) sequence for I and $\{\nu_j\}$ as above, then for each $j \in \mathcal{J}$*

$$\nu_j = 0 \text{ or } \nu_j > \frac{S^N}{\lambda^{\frac{N}{p^*(x)}}}.$$

Let $\phi_{j,\epsilon}$ as above. By Lemma (3.4), we see that for each $j \in \mathcal{J}$, $\{u_n \phi_{j,\epsilon}\}$ is bounded in X. Since $I'(u_n) \to 0$, $\langle I'(u_n), u_n \phi_{j,\epsilon} \rangle = o_n(1)$. Then

$$\begin{aligned} & \int_\Omega |\nabla u_n|^{p(x)} \phi_{j,\epsilon} dx + \int_\Omega u_n |\nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi_{j,\epsilon} + o_n(1) \\ = \; & \lambda \int_\Omega |u_n|^{q(x)} \phi_{j,\epsilon} dx + \mu \int_\Omega \frac{|u_n|^{r(x)}}{|x|^{s(x)}} \phi_{j,\epsilon} dx + \int_\Omega a(x) |u_n|^{p(x)} \phi_{j,\epsilon} dx + \int_\Omega g(x, u_n) u_n \phi_{j,\epsilon} dx. \end{aligned} \tag{3.11}$$

For each $\delta > 0$, applying Young's inequality

$$\int_\Omega u_n |\nabla u_n|^{p(x)} \nabla u_n \nabla \phi_{j,\epsilon} \leq \delta \int_\Omega u_n |\nabla u_n|^{p(x)} dx + C_8(\delta) \int_\Omega |u_n \nabla \phi_{j,\epsilon}|^{p(x)}, \tag{3.12}$$

passing to the limit of $n \to +\infty$ in (3.13), we get

$$\limsup_{n\to\infty} \int_\Omega |u_n| \nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi_{j,\epsilon}| dx \leq \delta C_9 + C_8(\delta) \int_\Omega |u \nabla \phi_{j,\epsilon}|^{p(x)}. \tag{3.13}$$

Now,using Lemma (2.10)

$$\limsup_{n\to\infty} \int_\Omega |u_n| \nabla u_n|^{p(x)-2} \nabla u_n \nabla \phi_{j,\epsilon}| dx \leq \delta C_9 + C_{11} C_6 \left\{ \|u\|^{p^+}_{L^{p^*}(B_\epsilon(x_j)))} + \|u\|^{p^-}_{L^{p^*(x)}(B_\epsilon(x_j))} \right\}. \tag{3.14}$$

On the other hand, by the compactness lemma of strauss [24] and Sobolev embedding:

$$\lim_{n\to\infty} \int_\Omega g(x, u_n) u_n \phi_{j,\epsilon} dx = \int_\Omega g(x, u) u \phi_{j,\epsilon} dx, \tag{3.15}$$

$$\lim_{n\to\infty}\int_\Omega a(x)|u_n|^{p(x)}\phi_{j,\epsilon}dx=\int_\Omega a(x)|u|^{p(x)}\phi_{j,\epsilon}dx \tag{3.16}$$

Since $u_n\to u$ in $L^{m(x)}(\Omega)$, $m\in C_+(\overline{\Omega})$, $m(x)<p^*(x)\,\forall\, x\in\overline{\Omega}$ and by Proposition (2.9) $W^{1,p(x)}(\Omega)\to L^{r(x)}_{|x|^{-s(x)}}(\Omega)$, there exists $C_1>0$ such that

$$|u_n-u|^{r(x)}_{r(x)|x|^{-s(x)}}\le C_1\min\left(\|u_n-u\|^{r^-},\|u_n-u\|^{r^+}\right)$$

Then by least result and Lemma (2.10) and Proposition (2.12), we have

$$\lim_{n\to\infty}\int_\Omega\frac{|u_n|^{r(x)}}{|x|^{s(x)}}dx\quad=\quad\lim_{n\to\infty}\int_\Omega\left[\frac{|u_n|^{r(x)}}{|x|^{s(x)}}-\frac{|u_n-u|^{r(x)}}{|x|^{s(x)}}+\frac{|u_n-u|^{r(x)}}{|x|^{s(x)}}\right]dx.$$

Then

$$\lim_{n\to\infty}\int_\Omega\frac{|u_n|^{r(x)}}{|x|^{s(x)}}\phi_{j,\epsilon}dx=\int_\Omega\frac{|u|^{r(x)}}{|x|^{s(x)}}\phi_{j,\epsilon}dx. \tag{3.17}$$

By (3.11) and (3.13)-(3.17)

$$\begin{aligned}
&\int_\Omega|\nabla u_n|^{p(x)}\phi_{j,\epsilon}dx\le\lim_{n\to\infty}\lambda\int_\Omega|u_n|^{q(x)}\phi_{j,\epsilon}dx\\
+\;&\int_\Omega g(x,u)u\phi_{j,\epsilon}dx+\int_\Omega a(x)|u|^{p(x)}\phi_{j,\epsilon}dx+\delta C_9\\
+\;&C_{11}C_6\left\{\|u\|^{p^+}_{L^{p^*}(B_\epsilon(x_j)))}+\|u\|^{p^-}_{L^{p^*(x)}(B_\epsilon(x_j))}\right\}\\
+\;&\delta^{'}C_{13}+C_{15}\|\frac{1}{\epsilon}|\nabla\phi_{j,\epsilon}|\|+\delta^{"}C_{16}+C_{17}\left\{\|u\|^{p^+}_{L^{p^*}(B_\epsilon(x_j)))}+\|u\|^{p^-}_{L^{p^*(x)}(B_\epsilon(x_j))}\right\}.
\end{aligned}$$

Now, as

$$|\nabla u_n|^{p(x)}\to\mu,\ \text{ and } |u|^{q(x)}\to\nu \text{ in } \mathcal{M}(\Omega).$$

Then

$$\begin{aligned}
&\int_\Omega\phi_{j,\epsilon}d\mu\le\lim_{n\to\infty}\lambda\int_\Omega\phi_{j,\epsilon}d\nu+\mu\int_\Omega\frac{|u_n|^{r(x)}}{|x|^{s(x)}}\phi_{j,\epsilon}dx\\
+\;&\int_{B_\epsilon(x_j))}g(x,u)u\phi_{j,\epsilon}dx+\int_{B_\epsilon(x_j))}a(x)|u|^{p(x)}\phi_{j,\epsilon}dx+\delta C_9\\
+\;&C_{11}C_6\left\{\|u\|^{p^+}_{L^{p^*}(B_\epsilon(x_j)))}+\|u\|^{p^-}_{L^{p^*(x)}(B_\epsilon(x_j))}\right\}\\
+\;&\delta^{'}C_{13}+C_{15}\|\frac{1}{\epsilon}|\nabla\phi_{j,\epsilon}|\|+\delta^{"}C_{16}+C_{17}\left\{\|u\|^{p^+}_{L^{p^*}(B_\epsilon(x_j)))}+\|u\|^{p^-}_{L^{p^*(x)}(B_\epsilon(x_j))}\right\}.
\end{aligned}$$

Letting $\epsilon\to0$, $\delta\to0$, $\delta^{'}\to0$ and $\delta^{"}\to0$ we obtain

$$\mu_j\le\lambda\nu_j.$$

Therefore

$$\nu_j>\frac{S^N}{\lambda^{\frac{N}{p^*(x)}}}.$$

Lemma 3.6. *Assume that $(A_1)-(A_2)$, $(G_1)-(G_2)$ and (1.5) are satisfied. If $\lambda<1$, then I satisfies $(PS)_d$ for $d<\lambda^{1-\frac{N}{p^+}}\left(\frac{1}{p^+}-\frac{1}{q^-}\right)S^N$.*

Proof. Let $\{u_n\} \subset X$ such that

$$I(u_n) \to d \text{ and } I^{'}(u_n) \to 0.$$

Then

$$\begin{aligned}
d &= I(u_n) - \frac{1}{p^+}\langle I^{'}(u_n), u_n\rangle + o_{n(1)} \\
&= \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right)|\nabla u_n|^{p(x)}dx + \int_\Omega \left(\frac{1}{p^+} - \frac{1}{p(x)}\right) a(x)|u_n|^{p(x)}dx \\
&+ \lambda \int_\Omega \left(\frac{1}{p^+} - \frac{1}{q(x)}\right)|u_n|^{q(x)}dx + \mu \int_\Omega \left(\frac{1}{p^+} - \frac{1}{r(x)}\right)\frac{|u_n|^{r(x)}}{|x|^{s(x)}}dx \\
&+ \int_\Omega \left(\frac{1}{p^+}g(x,u_n)u_n - G(x,u_n)\right)dx + o_n(1).
\end{aligned}$$

By (A_2) and (G_2), we have

$$d \geq \lambda\left(\frac{1}{p^+} - \frac{1}{q^-}\right)\int_\Omega |u_n|^{q(x)}dx + o_{n(1)}.$$

By (3.10), we obtain

$$\begin{aligned}
d &\geq \lambda\left(\frac{1}{p^+} - \frac{1}{q^-}\right)\lim_{n\to+\infty}\int_\Omega |u_n|^{q(x)}dx \\
&\geq \lambda\left(\frac{1}{p^+} - \frac{1}{q^-}\right)\left(\int_\Omega |u_n|^{q(x)}dx + \sum_{j\in\mathcal{J}}\nu_j\right) \\
&\geq \lambda\left(\frac{1}{p^+} - \frac{1}{q^-}\right)\nu_j \text{ for } j \in \mathcal{J}.
\end{aligned}$$

If $\nu_j > 0$ for some $j \in \mathcal{J}$, by Lemma (3.5)we get

$$d \geq \lambda\left(\frac{1}{p^+} - \frac{1}{q^-}\right)\frac{S^N}{\lambda^{\frac{N}{p(x_j)}}}.$$

Hence $\lambda < 1$,

$$d \geq \lambda\left(\frac{1}{p^+} - \frac{1}{q^-}\right)\frac{S^N}{\lambda^{\frac{N}{p^+}}} = \lambda^{1-\frac{N}{p^+}}\left(\frac{1}{p^+} - \frac{1}{q^-}\right)S^N.$$

Which is impossible, and $\nu_j = 0$ for $j \in \mathcal{J}$. Then

$$\lim_{n\to+\infty}\int_\Omega |u_n|^{q(x)}dx = \int_\Omega |u|^{q(x)}dx$$

By the least limit we have

$$\lim_{n\to+\infty}\int_\Omega |u_n - u|^{q(x)}dx = 0$$

Thanks to Proposition (2.3),we deduce

$$u_n \to u \text{ in } L^{q(x)}(\Omega)$$

On the other hand, we obtain

$$\begin{aligned} o_n(1) &= \langle I^{'}(u_n) - I^{'}(u), u_n - u\rangle \\ &= \int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\nabla(u_n - u)dx \\ &+ \int_\Omega a(x)\left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right)(u_n-u)dx \\ &+ \lambda\int_\Omega a(x)\left(|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\right)(u_n-u)dx \\ &+ \mu\int_\Omega \left(\frac{|u_n|^{r(x)-2}u_n}{|x|^{s(x)}} - \frac{|u|^{r(x)-2}u}{|x|^{s(x)}}\right)(u_n-u)dx \\ &- \int_\Omega (g(x,u_n) - g(x,u))(u_n-u)dx. \end{aligned}$$

By standard argument, we see that

$$\begin{aligned} \lim_{n\to+\infty}\int_\Omega a(x)\left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right)(u_n-u)dx = 0, \\ \lim_{n\to+\infty}\lambda\int_\Omega a(x)\left(|u_n|^{q(x)-2}u_n - |u|^{q(x)-2}u\right)(u_n-u)dx = 0, \\ \lim_{n\to+\infty}\int_\Omega \left(\frac{|u_n|^{r(x)-2}u_n}{|x|^{s(x)}} - \frac{|u|^{r(x)-2}u}{|x|^{s(x)}}\right)(u_n-u)dx = 0 \\ \text{and} \lim_{n\to+\infty}\int_\Omega (g(x,u_n) - g(x,u))(u_n-u)dx = 0. \end{aligned}$$

It follows that

$$\lim_{n\to+\infty}\int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\nabla(u_n-u)dx = 0. \tag{3.18}$$

Let us consider the sets

$$\Omega_+ = \{x \in \Omega / p(x) \geq 2\} \text{ in } \Omega_- = \{x \in \Omega / p(x) \leq 2\}.$$

We recall the following well-known inequalities, which hold any three real x, y and p

$$\left(x|x|^{p-2}|y|^{p-2}\right)(x-y) \geq c(p)\begin{cases} |x-y|^p, & \text{if } p \geq 2, \\ \frac{|x-y|^2}{(|x|+|y|)^{2-p}}, & \text{if } 1 < p < 2, \end{cases} \tag{3.19}$$

where $c(p) = 2^{2-p}$ when $p \geq 2$ and $c(p) = p-1$ when $1 < p < 2$.

By (3.19) and (3.18) we have

$$\lim_{n\to+\infty}\int_{\Omega_+} |\nabla u_n - \nabla u|^{p(x)}dx = 0. \tag{3.20}$$

Put

$$\beta_n = \int_\Omega \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\nabla(u_n-u)dx,$$
$$\text{and } \delta_n = |\nabla u_n| + |\nabla u|.$$

Let $\{u_n\} \subset X$ and $u_n \rightharpoonup u$ in X. By (3.19) we have for $p(x) \geq 2$

For $1 < p(x) < 2$ by (3.19) we obtain

$$\begin{aligned} \beta_n &\geq \int_{\Omega_-} \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\Delta u|^{p(x)-2}\nabla u \right) \nabla(u_n - u)dx \\ &\geq (p^- - 1)\frac{|\nabla u_n - \nabla u|^2}{\delta_n^{2-p(x)}} \end{aligned} \tag{3.21}$$

On the other hand, by (3.21) and Holder's inequality,

$$\begin{aligned} \int_{\Omega_-} |\nabla u_n - \nabla u|^{p(x)}dx &\leq \frac{1}{p^- - 1}\int_{\Omega_-} (p(x)-1)|\nabla u_n - \nabla u|^{p(x)}dx \\ &\leq \frac{1}{p^- - 1}\int_{\Omega_-} \beta_n^{\frac{p(x)}{2}}\delta_n^{\frac{2-p(x)}{2}}dx. \end{aligned}$$

Since $\{u_n\}$ is bounded in X we have

$$\int_{\Omega_-} |\nabla u_n - \nabla u|^{p(x)}dx \leq C_p^{'} \parallel \beta_n^{\frac{p(x)}{2}} \parallel_{L^{\frac{2}{p(x)}}(\Omega)} .$$

It result that

$$\lim_{n\to+\infty}\int_{\Omega_-} |\nabla u_n - \nabla u|^{p(x)}dx = 0. \tag{3.22}$$

Using again results; Proposition (2.7), (3.20) and (3.22) we get

$$\lim_{n\to+\infty}\int_{\Omega} |\nabla u_n - \nabla u|^{p(x)}dx = 0.$$

and hence

$$u_n \to u \text{ in} X.$$

□

The next Lemma is similar to [[25],Lemma5]

Lemma 3.7. *Under assumptions of Theorem (1.1), there exists a sequence $\{\mathcal{M}_n\} \subset (0, +\infty)$ independent of λ, with $\mathcal{M}_n \subset \mathcal{M}_{n+1}$ such that for any $\lambda > 0$,*

$$c_n^\lambda = \inf_{K\in\Gamma_n}\max_{u\in K} I(u) < \mathcal{M}_n.$$

Proof. (Proof of Theorem (1.1)) By choosing for each $k \geq 1$, λ_k sufficiently small, we construct a sequence (λ_k), with $\lambda_k > \lambda_{k+1}$ such that $\mathcal{M}_k < \lambda_k^{1-\frac{N}{p^+}}\left(\frac{1}{p^+} - \frac{1}{q^+}\right)S^N$. Thus for $\lambda \in (\lambda_k, \lambda_{k+1}]$,

$$0 < c_1^\lambda \leq c_2^\lambda \leq \cdots \leq c_k^\lambda < \lambda^{1-\frac{N}{p^+}}\left(\frac{1}{p^+} - \frac{1}{q^+}\right)S^N.$$

Thanks to Theorem (3.1), the levels $c_1^\lambda \leq c_2^\lambda \leq \cdots \leq c_k^\lambda$ are critical values of I. So, if

$$c_1^\lambda < c_2^\lambda < \cdots < c_k^\lambda,$$

I has at least k critical points. Now, if $c_j^\lambda < c_{j+1}^\lambda$ for some $j = 1, \cdots, k-1$, again Theorem (3.1) implies that $K_{c_j^\lambda}$ is an infinite set [[22],Cap.7].Then in this case, Problem (1.1) has infinitely many solutions. Then Problem (1.1) has at least k pair solutions. □

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