Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 ON LINE
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–14.
ISSN-0037-8712 IN PRESS
doi:10.5269/bspm.51654

Finding the Closest Efficient Targets in DEA by a Numeration Method: The FDH Non-Convex Technology

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ABSTRACT: Satisfying the Production Possibility Set (PPS) in Free Disposability Hull (FDH) property, there is only a few approaches which discuss on identifying the closest efficient targets of Decision Making Units (DMUs) in Data Envelopment Analysis (DEA). In this paper, without solving any optimization problem, a successful numeration method is proposed to compute the minimum distance of units from the strong efficient frontier of the FDH non-convex PPS. In fact, by some ratios obtained from a linear mixed-integer bi-level programming problem, the closest efficient targets of units are calculated. Moreover, there is an interesting discuss about simplifying a linear mixed-integer bi-level programming problem to reach to the ratios. Finally, the applicability of the proposed method to a real-world problem is illustrated through a numerical example.

Key Words: Data envelopment analysis (DEA), closest targets Free disposal hull (FDH).

Contents

1. Introduction

Data Envelopment Analysis (DEA), originally developed by Charnes et al. [6] and later extended by Banker et al. [4], is a non-parametric linear programming-based method to evaluate the relative efficiency of a set of homogeneous Decision Making Units (DMUs) which use some inputs to produce some outputs. The relative comparison in DEA is made with reference to a set of the inputs-outputs vectors as the Production Possibility Set (PPS) constructed from all the points which can convert their inputs into their outputs by assuming several postulates. Afterwards, based on both the information about existing data on the performance of the units and some preliminary assumptions, DEA forms an empirical efficient frontier in the PPS. The points on the efficient boundary (the efficient points) result in the maximum achievable outputs from a given input set (or alternatively the minimum inputs necessary to produce the given outputs). In other words, an efficient point has no potential improvement, whereas an inefficient point can become efficient by deleting its input excess and augmenting the output shortfall. Therefore, the input-output vectors on the efficient boundary can be patterns and targets for the inefficient DMUs to imitate and they can also point out keys for inefficient DMUs to improve their performance by an appropriate movement towards the efficient boundary. In fact, the closer an inefficient DMU is to the efficient boundary, the easier it is to remove its inefficiency i.e., less variation in its inputs and/or outputs is needed. DEA models not only try to obtain an approximation of the distance between the DMUs and

2010 *Mathematics Subject Classification:* 90C08.
Submitted December 29, 2019. Published May 06, 2021

the efficient boundary of the PPS but also determine an appropriate efficient target for the DMU under assessment. In other words, by projecting the DMU under assessment to the efficient boundary, these models try to find an appropriate efficient projection for this DMU among an infinite number of efficient points and a suitable way among an infinite number of ways to reach to the efficient boundary. However, many traditional DEA models can not compute the exact distance of an inefficient DMU from the efficient boundary and therefore, can not obtain the closest efficient target; as a result these models may project an inefficient unit onto the furthest efficient projection to the evaluated DMU, which makes the attainment of this efficient projection more difficult. Calculating the least distance measures of units to the efficient frontier and finding the closest efficient targets have attracted increasing interest of researchers in recent DEA literature ([1-3,6,8-10,12-14]).
On the other hand, one of the assumed postulates in constructing the PPSs of the conventional DEA is the convexity assumption. By relaxing the convexity assumption, Deprins et al. [8] proposed an extension of the conventional technologies called Free Disposal Hull (FDH) non-convex technology. In fact, the FDH technology is based on a representation of the production technology given by observed production plans, imposing strong disposability of inputs and outputs without the convexity assumption. Most of the papers obtain the minimum distances of units to the efficient boundary for convex PPSs and in fact, there are a few papers which discuss about these distances for nonconvex PPSs.
The paper unfolds as follows: Some preliminaries and basic concepts on DEA in the next section are reviewed. Section 3 discusses about some existing methods to obtain the closest efficient targets. In Section 4, a numeration approach for finding the closest efficient targets for a given unit is proposed. Section 5 includes an empirical illustration of the proposed approach. The last section is the conclusion.

2. Preliminaries

The following definitions and notations from DEA are needed for the next discussions. First of all and as usual, it is assumed that we have observed n decision making units (DMUs) where each DMU$_j$ ($j \in J = \{1, 2, \cdots, n\}$) produces s outputs y_{rj} ($r = 1, 2, \cdots, s$), using m inputs x_{ij} ($i = 1, 2, \cdots, m$). Define $\boldsymbol{x}_j = (x_{1j}, x_{2j}, \cdots, x_{mj})^t \in \mathbb{R}^m_+$ and $\boldsymbol{y}_j = (y_{1j}, y_{2j}, \ldots, y_{sj})^t \in \mathbb{R}^s_+$ as the input and output vectors of DMU$_j$, respectively. All components of the vectors $\boldsymbol{x}_j$ and $\boldsymbol{y}_j$ for all the DMUs are non-negative and each DMU has at least one positive input and output, that is: $\boldsymbol{x}_j \geq \boldsymbol{0}$, $\boldsymbol{x}_j \neq \boldsymbol{0}$ and $\boldsymbol{y}_j \geq \boldsymbol{0}$, $\boldsymbol{y}_j \neq \boldsymbol{0}$ for any $j = 1, 2, \ldots, n$. Also, assume that matrices $\boldsymbol{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n]$ and $\boldsymbol{Y} = [\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n]$ are $m \times n$ and $s \times n$ matrices of inputs and outputs, respectively. The production possibility set (PPS) T is generally defined as:

$$T = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \mid \boldsymbol{x} \text{ can produce } \boldsymbol{y}\}.$$

The mathematical definition of the PPS depends on the conditions of the problem and the postulates selected by the manager. The traditional FDH technology introduced by Deprins et al. [8] for the variable returns to scale (VRS) is as follows:

$$T_{\text{FDH-VRS}} = \cup_{j=1}^{n} \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \mid \boldsymbol{x}_j \leq \boldsymbol{x},\ \boldsymbol{y}_j \geq \boldsymbol{y}\}.$$

Moreover,, in the case of constant returns to scale (CRS), another FDH technology is introduced by Kerstens and Vanden Eeckaut [12] as follows:

$$T_{\text{FDH-CRS}} = \cup_{j=1}^{n} \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^m_+ \times \mathbb{R}^s_+ \mid \lambda_j \boldsymbol{x}_j \leq \boldsymbol{x},\ \lambda_j \boldsymbol{y}_j \geq \boldsymbol{y},\ \lambda_j \geq 0\}.$$

Definition 2.1. *Let* $(\boldsymbol{x}, \boldsymbol{y})$ *and* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$, *which* $(\boldsymbol{x}, \boldsymbol{y}) \neq (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$, *be in* T. *If* $(\bar{\boldsymbol{x}}, -\bar{\boldsymbol{y}}) \leq (\boldsymbol{x}, -\boldsymbol{y})$, *we say* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}})$ *dominates* $(\boldsymbol{x}, \boldsymbol{y})$.

Definition 2.2. $(\boldsymbol{x}, \boldsymbol{y}) \in T$ *is called a weak efficient point of* T *if there is no other* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in T$ *such that:*

$$(\bar{\boldsymbol{x}}, -\bar{\boldsymbol{y}}) < (\boldsymbol{x}, -\boldsymbol{y}).$$

If a point is not weak efficient, then it is called an inefficient point. $\partial^{\mathrm{W}}(T)$ (the weak efficient frontier of T) shows the set of all weak efficient points of T. This means that:

$$\partial^{\mathrm{W}}(T) = \{(\boldsymbol{x}, \boldsymbol{y}) \in T \mid \nexists (\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in T;\ (\bar{\boldsymbol{x}}, -\bar{\boldsymbol{y}}) < (\boldsymbol{x}, -\boldsymbol{y})\}.$$

Definition 2.3. $(\boldsymbol{x},\boldsymbol{y})\in T$ *is called a strong efficient point (a nondominated point) in* T *if there is no other* $(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})\in T$ *such that dominates* $(\boldsymbol{x},\boldsymbol{y})$*. In the other words, there is no other* $(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})\in T$ *such that:*

$$(\bar{\boldsymbol{x}},-\bar{\boldsymbol{y}})\leq(\boldsymbol{x},-\boldsymbol{y})$$

and strict inequality holds in at least one component.

The set of all strong efficient points of T is called the strong efficient frontier and denoted by $\partial^{\mathrm{S}}(T)$. That is:

$$\partial^{\mathrm{S}}(T)=\{(\boldsymbol{x},\boldsymbol{y})\in T\mid\nexists(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})\in T;\ (\bar{\boldsymbol{x}},-\bar{\boldsymbol{y}})\leq(\boldsymbol{x},-\boldsymbol{y}),\ (\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})\neq(\boldsymbol{x},\boldsymbol{y})\}.$$

Also, it is worth to mention that the strong efficient frontier is a part of the weak efficient frontier of T, i.e., $\partial^{\mathrm{S}}(T)\subseteq\partial^{\mathrm{W}}(T)$.

3. Existing Methods

Having mentioned previously, there is a few papers which discuss about the distances of units to the efficient frontier of nonconvex PPSs. Along this line of researches, the paper of Silva et al. [14] analyses the issue of finding the closest efficient targets for both non-convex and convex technologies. Then, using the BRZW efficiency measure (Brockett et al. [5]), it explains the approach developed to find the closer targets in the $T_{FDH-VRS}$ technology. To find this unit, they consider a set of points consisting the units dominating the unit being assessed. Then, for each inefficient unit, its closest peer in that set is determined through the BRWZ efficiency measure. However, the exact distances of units to the efficient frontier by any norm have not been discussed in this method. To obtain the minimum distance from DMUs to the weak efficient frontier for the FDH nonconvex PPS, Vakili [16] solve the following problem

$$\begin{array}{ll}
\min\limits_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{s})} & (\parallel(\boldsymbol{x},\boldsymbol{y})-(\boldsymbol{x}_o,\boldsymbol{y}_o)\parallel+M\boldsymbol{s})\\
\text{s.t.} & (\boldsymbol{x},\boldsymbol{y})\in T_{\text{FDH-VRS}}\\
& \max\limits_{\boldsymbol{s}}\ \boldsymbol{s}\\
& \quad\text{s.t.}\ \sum\limits_{j=1}^{n}\boldsymbol{x}_j\lambda_j\leq\boldsymbol{x}-\boldsymbol{s}1_m,\\
& \sum\limits_{j=1}^{n}\boldsymbol{y}_j\lambda_j\geq\boldsymbol{y}+\boldsymbol{s}1_s,\\
& \boldsymbol{y}+\boldsymbol{s}1_s\geq\boldsymbol{0},\\
& \boldsymbol{x}-\boldsymbol{s}1_m\geq\boldsymbol{0},\\
& \sum\limits_{j=1}^{n}\lambda_j=1,\\
& \lambda_j\in\{0,1\},\ j=1,2,...,n,\\
& \boldsymbol{s}\geq\boldsymbol{0}.
\end{array}$$

This method can be applied to the strong efficient frontier by changing the lower level problem to Additive model. No matter of the kind of PPSs and norms, the model is a linear mixed-integer bi-level programming problem which is very expensive to solve. Ebrahimnejad et al. [9] proposed a new approach for $T_{FDH-VRS}$ to find the least distance from the strong efficient frontier. Additive model is solved first to determine the strong efficient units and then the assessed $DMU_o=(\boldsymbol{x}_o,\boldsymbol{y}_o)$ is projected on frontier. By assumption that $(\boldsymbol{x}_c,\boldsymbol{y}_c)\in E$ is the reference point of DMU_o, the method finds all defining hyperplanes of PPS binding at $(\boldsymbol{x}_c,\boldsymbol{y}_c)$ as follows:

$$\begin{cases} x_i=x_{ic},\ i=1,2,\ldots,m,\\ y_r=y_{rc},\ r=1,2,\ldots,s.\end{cases}$$

Then, they try to find the distance between DMU_o and each of these defining hyperplanes. Depending on the definition of hyperplanes, if the hyperplane is as $x_k=x_{kc}$, then this entry is kept fixed and the other entries are changed such that it remains on the frontier giving the least distance to the frontier in Euclidean norm. Thus it is enough to solve the problem

$$
\begin{aligned}
\min \quad & \sum_{i=1, i\neq k}^{m} (x_{io} - (x_i + d_{1i}))^2 + \sum_{r=1}^{s} (y_{ro} - (y_r - d_{2r}))^2 + (x_k - x_{ko})^2 \\
\text{s.t.} \quad & \sum_{j\in E} \lambda_j x_{ij} + \lambda_{L+1}(x_i + d_{1i}) = (x_i + d_{1i}),\ i = 1, \ldots, m, i \neq k, \\
& \sum_{j\in E} \lambda_j y_{rj} + \lambda_{L+1}(y_r - d_{2r}) = (y_r - d_{2r}),\ r = 1, \ldots, s, \\
& \sum_{j\in E} \lambda_j x_{kj} + \lambda_{L+1} x_k = x_k, \\
& x_k = x_{kc}, \\
& x_i + d_{1i} \geq x_{ic},\ i = 1, 2, \ldots, m, i \neq k, \\
& y_r - d_{2r} \leq y_{rc},\ r = 1, 2, \ldots, s \\
& x_i + d_{1i} \leq x_{it},\ i = 1, 2, \ldots, m, i \neq k, \\
& y_r - d_{2r} \geq y_{rt},\ r = 1, 2, \ldots, s, \\
& \lambda_j \in \{0, 1\},\ j = 1, 2, \ldots, n, \\
& x_{ij} \geq 0,\ y_{rj} \geq 0,\ j = 1, \ldots, n,\ i = 1, 2, \ldots, m,\ r = 1, \ldots, s,
\end{aligned}
$$

where

$$x_{it} \in \{x_{ij} | x_{ij} > x_{ic},\ j = 1, \ldots, n\},\ y_{rt} \in \{y_{rj} | y_{rj} < y_{rc},\ j = 1, \ldots, n\}$$

and

$$|E| = L.$$

The same process is done for $y_k = y_{kc}$. It is worth to note that some mixed-integer quadratic programming problems should be solved to determine the closest efficient target. Also, Mehdiloozad et al. [13] proposed a numeration method based on directional FDH measures of efficiency to find the closest efficient target in $T_{FDH-VRS}$. In fact, they consider the weights $\boldsymbol{g}^-$ and $\boldsymbol{g}^+$ and try to solve the problem

$$
\begin{aligned}
\beta^* = \max \quad & \beta \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{io} - \beta g_i^-,\ i = 1, 2, \ldots, m, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_i^+ = y_{ro} + \beta g_r^+,\ r = 1, 2, \ldots, s, \\
& \sum_{j=1}^{n} \lambda_j = 1, \\
& \lambda_j \in \{0, 1\},\ j = 1, 2, \ldots, n, \\
& s_i^- \geq 0,\ s_i^+ \geq 0,\ i = 1, 2, \ldots, m,\ r = 1, 2, \ldots, s.
\end{aligned}
$$

In general, β^* cannot be interpreted as an efficiency index for any arbitrary direction vector. Thus, to overcome this problem, they restricted $(\boldsymbol{g}^-, \boldsymbol{g}^+) \in \mathbb{R}_+^m \times \mathbb{R}_+^s$ as follows.

$$\max_{j=1,2,\ldots,n} \left\{ \frac{x_{io}}{g_i^-} \right\} \leq 1$$

$$\max_{j=1,2,\ldots,n} \left\{ \frac{\overline{x}_i - \underline{x}_i}{g_i^-} \right\} \leq 1,\ \overline{x}_i = \max_{j=1,2,\ldots,n} \{x_{ij}\},\ \underline{x}_i = \max_{j=1,2,\ldots,n} \{x_{ij}\}.$$

They considered E_o as the set of all efficient DMUs dominating assessed DMU_o and then they proposed the numeration method

$$\min_{j\in E_o} \left\{ \frac{1}{m} \sum_{i=1}^{m} \frac{1}{g_i^-}(x_{io} - x_{ij}) + \frac{1}{s} \sum_{r=1}^{s} \frac{1}{g_r^+}(y_{rj} - y_{ro}) \right\},$$

or

$$\max_{j\in E_o}\left\{\frac{1-\dfrac{1}{m}\displaystyle\sum_{i=1}^{m}\frac{1}{g_i^-}(x_{io}-x_{ij})}{1+\dfrac{1}{s}\displaystyle\sum_{r=1}^{s}\frac{1}{g_r^+}(y_{rj}-y_{ro})}\right\}$$

to calculate the closest efficient observed DMU. This method leads to the closest efficient target along with the direction $(-\boldsymbol{g}^-,\boldsymbol{g}^+)$ which may not be the closest distance to the efficient frontier by a norm generally. Actually, obtaining a direction leading to the least distance to the efficient frontier by an arbitrary norm is not easy. Moreover, it is possible to attain to the furthest efficient unit when a direction is not suitable. Also, the obtained point may be a weak efficient point which is not desirable.
A common drawback of models proposed in the literature is achieving at the closest weak efficient targets or the closest strong efficient targets among DMUs dominating assessed DMU. However, it is shown in Section 3 that the closest strong efficient may not dominate assessed DMU. Moreover, as far as the author know, there is no numeration method in the literature to find the closest strong efficient target on both $T_{FDH-VRS}$ and $T_{FDH-CRS}$. So, to overcome to this problems, this paper develops a numeration method to find the closest strong efficient targets in the FDH nonconvex PPSs which are associated with the least distance to the strong efficient frontier.

4. Numeration method for obtaining the closest efficient targets in the FDH PPSs

In this section, in order to obtain the minimum distance of DMUs to the efficient frontier of the FDH nonconvex PPSs and therefore find the closest efficient targets of the inefficient DMUs, a distance minimization numeration method is going to be presented. Now, consider $\text{DMU}_o=(\boldsymbol{x}_o,\boldsymbol{y}_o)\in T$ as the unit under assessment. The distance of DMU_o to the strong efficient frontier of T by norm $\|\ .\ \|$ is shown by $d_{\|.\|}((\boldsymbol{x}_o,\boldsymbol{y}_o),\partial^{\mathrm{S}}(T))$ and defined as below:

$$d_{\|.\|}((\boldsymbol{x}_o,\boldsymbol{y}_o),\partial^{\mathrm{S}}(T))=\begin{array}{ll}\min\limits_{(\boldsymbol{x},\boldsymbol{y})} & \|(\boldsymbol{x},\boldsymbol{y})-(\boldsymbol{x}_o,\boldsymbol{y}_o)\|\\ \text{s.t.} & (\boldsymbol{x},\boldsymbol{y})\in\partial^{\mathrm{S}}(T).\end{array}\tag{4.1}$$

Regarding the paper of Jahanshahloo et al. [11], Problem (4.1) can be converted to the following bi-level programming problem:

$$d_{\|.\|}((\boldsymbol{x}_o,\boldsymbol{y}_o),\partial^{\mathrm{S}}(T))=\begin{array}{ll}\min\limits_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{s}^-,\boldsymbol{s}^+)} & \|(\boldsymbol{x},\boldsymbol{y})-(\boldsymbol{x}_o,\boldsymbol{y}_o)\|+M(\mathbf{1}_m^t\boldsymbol{s}^-+\mathbf{1}_s^t\boldsymbol{s}^+)\\ \text{s.t.} & (\boldsymbol{x},\boldsymbol{y})\in T\\ & \quad\max\limits_{(\boldsymbol{s}^-,\boldsymbol{s}^+)}\ \mathbf{1}_m^t\boldsymbol{s}^-+\mathbf{1}_s^t\boldsymbol{s}^+\\ & \quad\text{s.t. } (\boldsymbol{x}-\boldsymbol{s}^-,\boldsymbol{y}+\boldsymbol{s}^+)\in T\\ & \quad\boldsymbol{s}^-\geq\mathbf{0},\boldsymbol{s}^+\geq\mathbf{0},\end{array}\tag{4.2}$$

where $M\gg 0$ is a sufficiently large positive real number and $\mathbf{1}_k$ is a column vector in $\mathbb{R}^k$ with all components equal to 1. Note that in the bi-level programming problem (4.2), the set of all variables is partitioned between the vectors $(\boldsymbol{s}^-,\boldsymbol{s}^+)$ and $(\boldsymbol{x},\boldsymbol{y})$, while $(\boldsymbol{s}^-,\boldsymbol{s}^+)$ is to be chosen as an optimal solution to the second optimization problem parameterized in $(\boldsymbol{x},\boldsymbol{y})$. In other words, $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{s}^-,\boldsymbol{s}^+)$ is a feasible solution to the bi-level problem (4.2) if $(\boldsymbol{s}^-,\boldsymbol{s}^+)$ is an optimal solution to the lower-level problem corresponding to the parameter $(\boldsymbol{x},\boldsymbol{y})\in T$. Note that in the objective function of the upper level problem, $\mathbf{1}_m^t\boldsymbol{s}^-+\mathbf{1}_s^t\boldsymbol{s}^+$ is the objective value of the lower level problem for parameter $(\boldsymbol{x},\boldsymbol{y})$. Additionally, it has been proved by Jahanshahloo et al. [11] that if $(\boldsymbol{x}^*,\boldsymbol{y}^*,\boldsymbol{s}^{-*},\boldsymbol{s}^{+*})$ is an optimal solution to the Problem (4.2), then $(\boldsymbol{s}^{-*},\boldsymbol{s}^{+*})=(\mathbf{0},\mathbf{0})$. In other words, the optimal value of the lower level problem is zero in optimality.

4.1. Distance to the strong efficient frontier of $T_{\text{FDH-VRS}}$

This subsection tries to simplify Problem (4.1) by considering $T_{\text{FDH-VRS}}$ instead of T i.e., the following problem:

$$d_{\|.\|}((\boldsymbol{x}_o,\boldsymbol{y}_o),\partial^{\mathrm{S}}(T_{\text{FDH-VRS}})) = \begin{array}{ll} \min\limits_{(\boldsymbol{x},\boldsymbol{y})} & \|(\boldsymbol{x},\boldsymbol{y})-(\boldsymbol{x}_o,\boldsymbol{y}_o)\| \\ \text{s.t.} & (\boldsymbol{x},\boldsymbol{y})\in\partial^{\mathrm{S}}(T_{\text{FDH-VRS}}). \end{array} \tag{4.3}$$

At first, to simplify Problem (4.3), the following theorem is presented.

Theorem 4.1. *If $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ is an optimal solution to the Problem (4.3), then $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ is an observed DMU.*

Proof: To prove this theorem, assume that $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ is an optimal solution to the Problem (4.3). Now, by contradiction, assume that $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ is not an observed DMU. At first, since $(\boldsymbol{x}^*,\boldsymbol{y}^*)\in T_{FDH-VRS}$, there exists an index $j\in J$ such that:

$$(\boldsymbol{x}^*,\boldsymbol{y}^*)\in\{(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{x}\geq\boldsymbol{x}_j,\boldsymbol{y}\leq\boldsymbol{y}_j\}.$$

Since $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ is not an observed DMU, so there exists $(\tilde{\boldsymbol{s}}^-,\tilde{\boldsymbol{s}}^+)\gneqq(\mathbf{0},\mathbf{0})$ such that $\boldsymbol{x}^*-\tilde{\boldsymbol{s}}^-=\boldsymbol{x}_j,\ \boldsymbol{y}^*+\tilde{\boldsymbol{s}}^+=\boldsymbol{y}_j$, which is a contradiction with the efficiency of $(\boldsymbol{x}^*,\boldsymbol{y}^*)$. Therefore, $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ is one of the observed DMUs. $\square$

Now, if $T_{\text{FDH-VRS}}$ is considered in problem (4.2) as PPS as well, we have the following problem ($d=d_{\|.\|}((\boldsymbol{x}_o,\boldsymbol{y}_o),\partial^{\mathrm{S}}(T_{\text{FDH-VRS}}))$).

$$\begin{array}{ll} d= & \min\limits_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{s}^-,\boldsymbol{s}^+)} \ \|\,(\boldsymbol{x},\boldsymbol{y})-(\boldsymbol{x}_o,\boldsymbol{y}_o)\,\|+M(\mathbf{1}_m^t\boldsymbol{s}^-+\mathbf{1}_s^t\boldsymbol{s}^+) \\ & \text{s.t. } (\boldsymbol{x},\boldsymbol{y})\in T_{\text{FDH-VRS}} \\ & \max\limits_{(\boldsymbol{s}^-,\boldsymbol{s}^+)} \ \mathbf{1}_m^t\boldsymbol{s}^-+\mathbf{1}_s^t\boldsymbol{s}^+ \\ & \quad \text{s.t. } (\boldsymbol{x}-\boldsymbol{s}^-,\boldsymbol{y}+\boldsymbol{s}^+)\in\cup_{j=1}^n\{(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{x}\geq\boldsymbol{x}_j,\boldsymbol{y}\leq\boldsymbol{y}_j\} \\ & \boldsymbol{s}^-\geq\mathbf{0},\boldsymbol{s}^+\geq\mathbf{0}. \end{array} \tag{4.4}$$

In what follows it is clear that Problem (4.4) is equivalent to the bi-level Problem (4.5):

$$\begin{array}{ll} d= & \min\limits_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{s}^{j-},\boldsymbol{s}^{j+})} \ \|\,(\boldsymbol{x},\boldsymbol{y})-(\boldsymbol{x}_o,\boldsymbol{y}_o)\,\|+M(\max\limits_{j\in J}\{\max\limits_{(\boldsymbol{s}^{j-},\boldsymbol{s}^{j+})}\ \mathbf{1}_m^t\boldsymbol{s}^{j-}+\mathbf{1}_s^t\boldsymbol{s}^{j+}\}) \\ & \text{s.t. } (\boldsymbol{x},\boldsymbol{y})\in T_{\text{FDH-VRS}} \\ & \max\limits_{j\in J}\ \max\limits_{(\boldsymbol{s}^{j-},\boldsymbol{s}^{j+})} \ \mathbf{1}_m^t\boldsymbol{s}^{j-}+\mathbf{1}_s^t\boldsymbol{s}^{j+} \\ & \text{s.t. } (\boldsymbol{x}-\boldsymbol{s}^{j-},\boldsymbol{y}+\boldsymbol{s}^{j+})\in\{(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{x}\geq\boldsymbol{x}_j,\boldsymbol{y}\leq\boldsymbol{y}_j\} \\ & \boldsymbol{s}^{j-}\geq\mathbf{0},\boldsymbol{s}^{j+}\geq\mathbf{0}. \end{array} \tag{4.5}$$

Since $(\boldsymbol{x},\boldsymbol{y})$ is a parameter for the lower level problem of the Problem (4.5), therefore, Problem (4.6):

$$\begin{array}{ll} \max\limits_{(\boldsymbol{s}^{j-},\boldsymbol{s}^{j+})} & \mathbf{1}_m^t\boldsymbol{s}^{j-}+\mathbf{1}_s^t\boldsymbol{s}^{j+} \\ \text{s.t.} & \boldsymbol{x}-\boldsymbol{s}^{j-}\geq\boldsymbol{x}_j \\ & \boldsymbol{y}+\boldsymbol{s}^{j+}\leq\boldsymbol{y}_j \\ & \boldsymbol{s}^{j-}\geq\mathbf{0},\boldsymbol{s}^{j+}\geq\mathbf{0}, \end{array} \tag{4.6}$$

for some $j\in J$ has the following optimal solution:

$$s_i^{j-}=x_i-x_{ij}, i=1,2,\ldots,m$$

$$s_r^{j+}=y_{rj}-y_r, r=1,2,\ldots,s,$$

where $x_i \geq x_{ij}$ and $y_{rj} \geq y_r$ for each i, r. Therefore, for $i = 1, 2, \ldots, m$ and $r = 1, 2, \ldots, s$, Problem (4.5) can be written as:

$$\begin{aligned} \min_{(\boldsymbol{x},\boldsymbol{y})} \quad & \| (\boldsymbol{x},\boldsymbol{y}) - (\boldsymbol{x}_o,\boldsymbol{y}_o) \| + M(\max_{j\in J}\{\sum_{i=1}^{m}(x_i - x_{ij}) + \sum_{r=1}^{s}(y_{rj} - y_r) | x_i \geq x_{ij}, y_r \leq y_{rj}\}) \\ \text{s.t.} \quad & (\boldsymbol{x},\boldsymbol{y}) \in T_{\text{FDH-VRS}}. \end{aligned} \tag{4.7}$$

Moreover, by attention to Theorem 4.1, since the optimal solution of Problem (4.3) (problem(4.7)) occurs in one observed DMU and the coefficient of M must be zero in the optimality of problem(4.7); therefore, Problem (4.7) can be written as:

$$\begin{aligned} & \min_{(\boldsymbol{x}_k,\boldsymbol{y}_k)} \| (\boldsymbol{x}_k,\boldsymbol{y}_k) - (\boldsymbol{x}_o,\boldsymbol{y}_o) \| \\ & \text{s.t.} \\ & \max_{j\in J}\{\sum_{i=1}^{m}(x_{ik} - x_{ij}) + \sum_{r=1}^{s}(y_{rj} - y_{rk}) | x_{ik} \geq x_{ij}, y_{rk} \leq y_{rj}, i = 1, ..., m, r = 1, ..., s\} = 0 \\ & k = 1, 2, \ldots, n. \end{aligned} \tag{4.8}$$

Problem (4.8) presents a numeration method for obtaining the minimum distance of $(\boldsymbol{x}_o, \boldsymbol{y}_o)$ from the strong efficient frontier of $T_{\text{FDH-VRS}}$ by norm $\|.\|$, which can be summarized as the below algorithm.

Algorithm 1.

1. Set $k = 1$ and go to Step 2.

2. If $k \leq n$, compute:

   $$\omega_k = \max_{j\in J}\{\sum_{i=1}^{m}(x_{ik} - x_{ij}) + \sum_{r=1}^{s}(y_{rj} - y_{rk}) | x_{ik} \geq x_{ij}, y_{rk} \leq y_{rj}, i = 1, 2, \ldots, m, r = 1, 2, \ldots, s\},$$

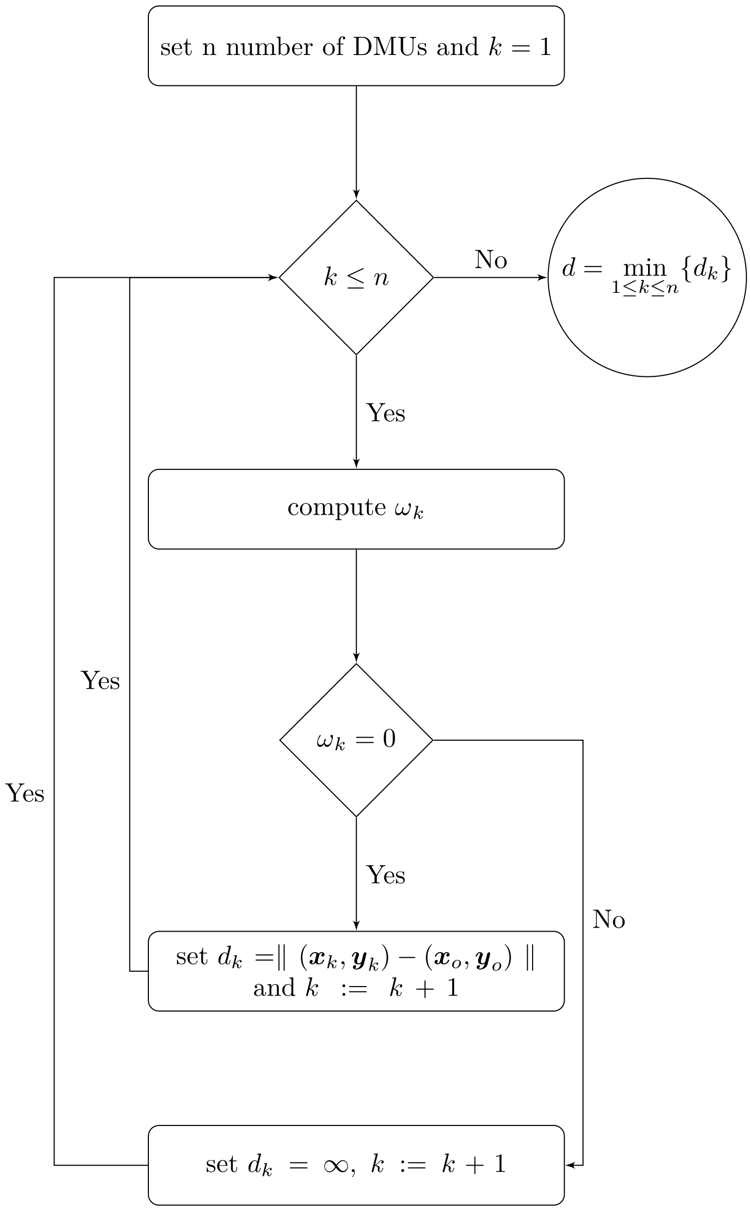
   and go to Step 3. Otherwise, go to Step 6.

3. If $\omega_k = 0$, then $(\boldsymbol{x}_k, \boldsymbol{y}_k)$ is a strong efficient DMU and go to Step 4. Otherwise, go to Step 5.

4. Compute $d_k = \| (\boldsymbol{x}_k, \boldsymbol{y}_k) - (\boldsymbol{x}_o, \boldsymbol{y}_o) \|$ and set $k := k + 1$, and return to Step 2.

5. Set $d_k = +\infty$ and $k := k + 1$ and return to Step 2.

6. Compute $d_{\|.\|}((\boldsymbol{x}_o, \boldsymbol{y}_o), \partial^{\text{S}}(T_{\text{FDH-VRS}})) = \min_{k=1,2,\ldots,n}\{d_k\}$ and stop.



4.2. Distance to the strong efficient frontier of $T_{\text{FDH-CRS}}$

This subsection discusses the distance of the input-output points to the strong efficient frontier of the production possibility set $T_{\text{FDH-CRS}}$. In order to do this, if $T_{\text{FDH-CRS}}$ is considered as the PPS in Problem (4.2), we have the following problem.

$$d_{\|.\|}((\boldsymbol{x}_o, \boldsymbol{y}_o), \partial^{\text{S}}(T_{\text{FDH-CRS}})) = \begin{array}{ll} \min\limits_{(\boldsymbol{x},\boldsymbol{y})} & \|(\boldsymbol{x}, \boldsymbol{y}) - (\boldsymbol{x}_o, \boldsymbol{y}_o)\| \\ \text{s.t.} & (\boldsymbol{x}, \boldsymbol{y}) \in \partial^{\text{S}}(T_{\text{FDH-CRS}}). \end{array} \tag{4.9}$$

Now to simplify Problem (4.9), first, the following theorem which presents a characterization for the strong efficient points of $T_{FDH-CRS}$ is expressed.

Theorem 4.2. *$(\boldsymbol{x}, \boldsymbol{y}) \in T_{FDH-CRS}$ is a strong efficient point if and only if there exist $j \in J$ and $\lambda \geq 0$ such that $(\boldsymbol{x}, \boldsymbol{y}) = \lambda(\boldsymbol{x}_j, \boldsymbol{y}_j)$, where $(\boldsymbol{x}_j, \boldsymbol{y}_j)$ is an observed efficient DMU.*

Proof: Suppose that $(\boldsymbol{x},\boldsymbol{y}) \in T_{FDH-CRS}$ is a strong efficient point. Therefore, there exists $j \in J$ and $\lambda_j \geq 0$ such that $\boldsymbol{x} \geq \lambda_j \boldsymbol{x}_j,\ \boldsymbol{y} \leq \lambda_j \boldsymbol{y}_j$. Since $(\boldsymbol{x},\boldsymbol{y})$ is a strong efficient point, so it is clear that $\boldsymbol{x} = \lambda_j \boldsymbol{x}_j, \boldsymbol{y} = \lambda_j \boldsymbol{y}_j$. That is $(\boldsymbol{x},\boldsymbol{y})$ locates on the halfline $\{\lambda(\boldsymbol{x}_j,\boldsymbol{y}_j)|\lambda \geq 0\}$. Furthermore, owing to the efficiency of $(\boldsymbol{x},\boldsymbol{y})$, it is clear that the observed DMU $(\boldsymbol{x}_j,\boldsymbol{y}_j)$ is a strong efficient DMU too.
Conversely, suppose that $(\boldsymbol{x},\boldsymbol{y}) = \lambda(\boldsymbol{x}_j,\boldsymbol{y}_j)$ for some $\lambda \geq 0$, where $(\boldsymbol{x}_j,\boldsymbol{y}_j)$ is a strong efficient DMU. Now, to prove the strong efficiency of $(\boldsymbol{x},\boldsymbol{y})$, by contradiction assume that $(\boldsymbol{x},\boldsymbol{y})$ is not a strong efficient point. Then there exists $(\overline{\boldsymbol{x}},\overline{\boldsymbol{y}}) \in T_{FDH-CRS}$ such that $(\overline{\boldsymbol{x}},-\overline{\boldsymbol{y}}) \lneqq (\boldsymbol{x},-\boldsymbol{y}) = \lambda(\boldsymbol{x}_j,-\boldsymbol{y}_j)$. This is a contradiction with the strong efficiency of $(\boldsymbol{x}_j,\boldsymbol{y}_j)$. Therefore, $(\boldsymbol{x},\boldsymbol{y})$ is a strong efficient DMU and the proof is completed. □

Theorem 4.2 implies that:

$$\partial^S(T_{FDH-CRS}) = \{\lambda(\boldsymbol{x}_j,\boldsymbol{y}_j)|\lambda \geq 0, (\boldsymbol{x}_j,\boldsymbol{y}_j) \text{ is a strong efficient DMU}\}.$$

Now, let's consider the bilevel form (4.10) for obtaining the minimum distance of $(\boldsymbol{x}_o,\boldsymbol{y}_o)$ from the strong efficient frontier of $T_{FDH-CRS}$.

$$\begin{array}{l} d_{\|.\|}((\boldsymbol{x}_o,\boldsymbol{y}_o),\partial^{\mathrm{S}}(T_{\text{FDH-CRS}})) = \min\limits_{(\boldsymbol{x},\boldsymbol{y},\boldsymbol{s}^-,\boldsymbol{s}^+)} \|\ (\boldsymbol{x},\boldsymbol{y}) - (\boldsymbol{x}_o,\boldsymbol{y}_o)\ \| + M(\mathbf{1}_m^t \boldsymbol{s}^- + \mathbf{1}_s^t \boldsymbol{s}^+) \\ \text{s.t.} (\boldsymbol{x},\boldsymbol{y}) \in T_{\text{FDH-CRS}} \\ \max\limits_{(\boldsymbol{s}^-,\boldsymbol{s}^+)} \mathbf{1}_m^t \boldsymbol{s}^- + \mathbf{1}_s^t \boldsymbol{s}^+ \\ \text{s.t. } (\boldsymbol{x}-\boldsymbol{s}^-, \boldsymbol{y}+\boldsymbol{s}^+) \in \cup_{j=1}^n \{(\boldsymbol{x},\boldsymbol{y})|\boldsymbol{x} \geq \lambda_j \boldsymbol{x}_j, \boldsymbol{y} \leq \lambda_j \boldsymbol{y}_j, \lambda_j \geq 0\} \\ \boldsymbol{s}^- \geq \mathbf{0}, \boldsymbol{s}^+ \geq \mathbf{0}. \end{array} \tag{4.10}$$

Similar to the discussion of the VRS case (subsection 2.1), problem (4.10) is equivalent to:

$$\begin{array}{ll} \min\limits_{(\boldsymbol{x},\boldsymbol{y})} & \|\ (\boldsymbol{x},\boldsymbol{y}) - (\boldsymbol{x}_o,\boldsymbol{y}_o)\ \| + M(\max\limits_{j\in J}\{\sum\limits_{i=1}^m (x_i - \lambda_j x_{ij}) + \sum\limits_{r=1}^s (\lambda_j y_{rj} - y_r)\}) \\ \text{s.t.} & (\boldsymbol{x},\boldsymbol{y}) \in T_{FDH-CRS} \\ & \max\limits_{j\in J} (\sum\limits_{i=1}^m (x_i - \lambda_j x_{ij}) + \sum\limits_{r=1}^s (\lambda_j y_{rj} - y_r)) \\ & \text{s.t. } x_i \geq \lambda_j x_{ij}, y_r \leq \lambda_j y_{rj}, \lambda_j \geq 0, \forall i, r. \end{array} \tag{4.11}$$

Since the optimal value of the lower level problem must be zero in optimality, therefore, Problem (4.11) is equivalent to the following problem:

$$\begin{array}{ll} \min\limits_{(\boldsymbol{x},\boldsymbol{y})} & \|\ (\boldsymbol{x},\boldsymbol{y}) - (\boldsymbol{x}_o,\boldsymbol{y}_o)\ \| \\ \text{s.t.} & (\boldsymbol{x},\boldsymbol{y}) \in T_{FDH-CRS} \\ & \max\limits_{j\in J}\{\sum\limits_{i=1}^m (x_i - \lambda_j x_{ij}) + \sum\limits_{r=1}^s (\lambda_j y_{rj} - y_r)|x_i \geq \lambda_j x_{ij}, y_r \leq \lambda_j y_{rj}, \lambda_j \geq 0, \forall i, r\} = 0. \end{array} \tag{4.12}$$

Theorem 4.3. $(\boldsymbol{x},\boldsymbol{y}) \in T_{FDH-CRS}$ *is not strong efficient unit if and only if for some* $j \in J$

$$r = 1,2,...,s,\ \{\frac{y_r}{y_{rj}}\} \leq \{\frac{x_i}{x_{ij}}\},\ i = 1,2,...,m$$

and $\{\frac{y_r}{y_{rj}}\} < \{\frac{x_i}{x_{ij}}\}$, *at least for one* i, r.

Proof: By attention to that

$$\max_j\{\sum_{i=1}^m (x_i - \lambda_j x_{ij}) + \sum_{r=1}^s (\lambda_j y_{rj} - y_r)|x_i \geq \lambda_j x_{ij}, y_r \leq \lambda_j y_{rj}, \lambda_j \geq 0, \forall i, r\} > 0$$

if and only if for some $j \in J$

$$r = 1, 2, ..., s,\ \{\frac{y_r}{y_{rj}}\} \leq \{\frac{x_i}{x_{ij}}\},\ i = 1, 2, ..., m,$$

and $\{\frac{y_r}{y_{rj}}\} < \{\frac{x_i}{x_{ij}}\}$, at least for one i, r, the proof is clear. □

Algorithm 2.

1. Set $k = 1$ and go to Step 2.

2. If $k \leq n$, go to Step 3; Otherwise go to Step 6.

3. If

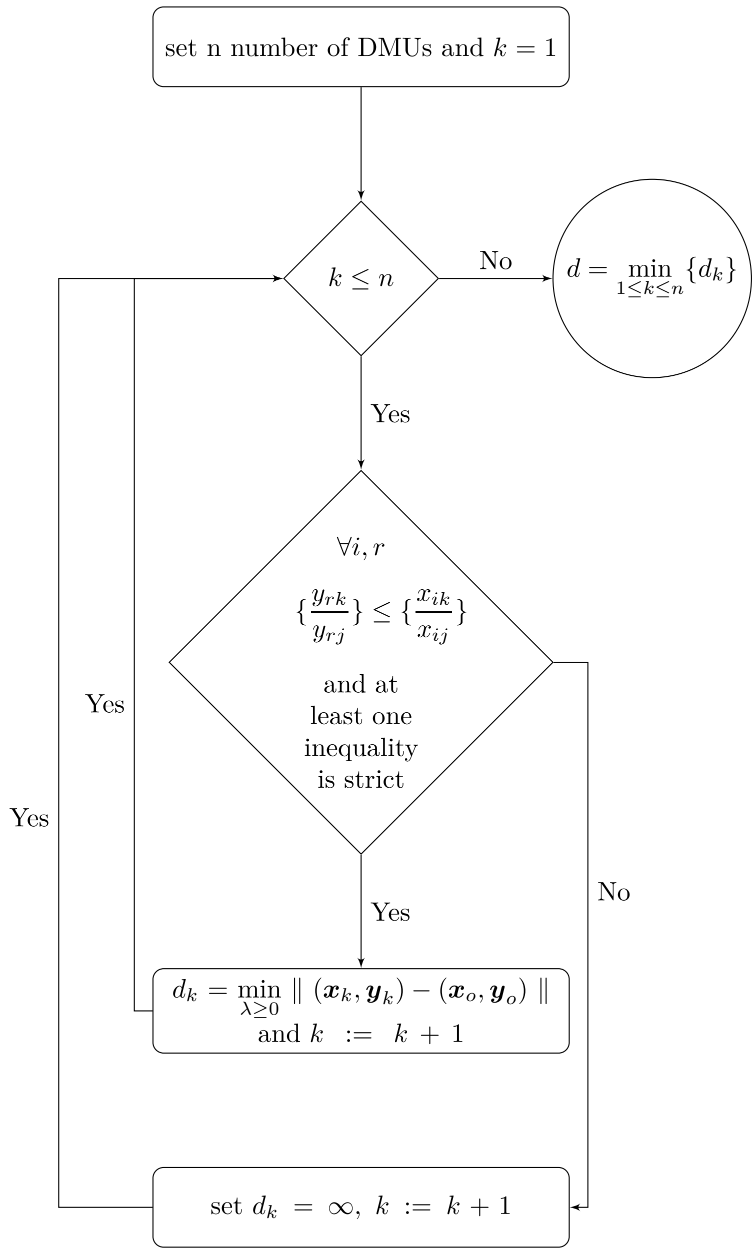
$$\{\frac{y_{rk}}{y_{rj}}\} \leq \{\frac{x_{ik}}{x_{ij}}\}, \forall i, r$$

   and at least one inequality is strict, then $(\boldsymbol{x}_k, \boldsymbol{y}_k)$ is not strong efficient and go to Step 5; Otherwise, go to Step 4.

4. Compute $d_k = \min\limits_{\lambda \geq 0} \parallel \lambda(\boldsymbol{x}_k, \boldsymbol{y}_k) - (\boldsymbol{x}_o, \boldsymbol{y}_o) \parallel$ and set $k := k + 1$ and return to Step 2.

5. Set $d_k = +\infty$ and $k := k + 1$ and return to Step 2.

6. Calculate $d_{\|.\|}((\boldsymbol{x}_o, \boldsymbol{y}_o), \partial^{\mathrm{S}}(T_{\text{FDH-CRS}})) = \min\limits_{k=1,2,...,n} \{d_k\}$ and stop.



Now, we explain two processes through a simple example.

Example 4.1. *Consider a simple case with six observed DMUs: a, b, c, d, e and f producing single-output y by single-input x. The data are listed in Table 1. Dashed line shows $T_{FDH-CRS}$ frontier and straight line shows $T_{FDH-VRS}$ frontier.*

For point M we use alghorithm1 to find closest strong efficient point. First we specify strong efficient points. In step2, for $k = 1, 2, 3$ and 4 we have $\omega_k = 0$. So the points a, b, c and d are strong efficient. For $k = 5$ we have $\omega_5 = max\{(2-1)+(2-.5)\} = 2.5$ and for $k = 6$ we have $\omega_6 = max\{(5.5-4)+(3.5-4)\} = 2$. So, e and f are inefficient. In step4 we use norm2 to calculate the distance from strong efficient frontier. Results are $d_1 = \| (5, 1) - (1, 2) \| = 4.12, d_2 = \| (5, 1) - (3, 3) \| = 2.83, d_3 = \| (5, 1) - (4, 4) \| = 3.16$ and $d_4 = \| (5, 1) - (8, 6) \| = 5.83$. We have $\min\{d_1, d_2, d_3, d_4\} = d_2$. So, b is the closest strong efficient point

DMUs	x	y
$a = (x_1, y_1)$	1	2
$b = (x_2, y_2)$	3	3
$c = (x_3, y_3)$	4	4
$d = (x_4, y_4)$	8	6
$e = (x_5, y_5)$	2	0.5
$f = (x_6, y_6)$	5.5	3.5

Table 1: Data of observed DMUs.

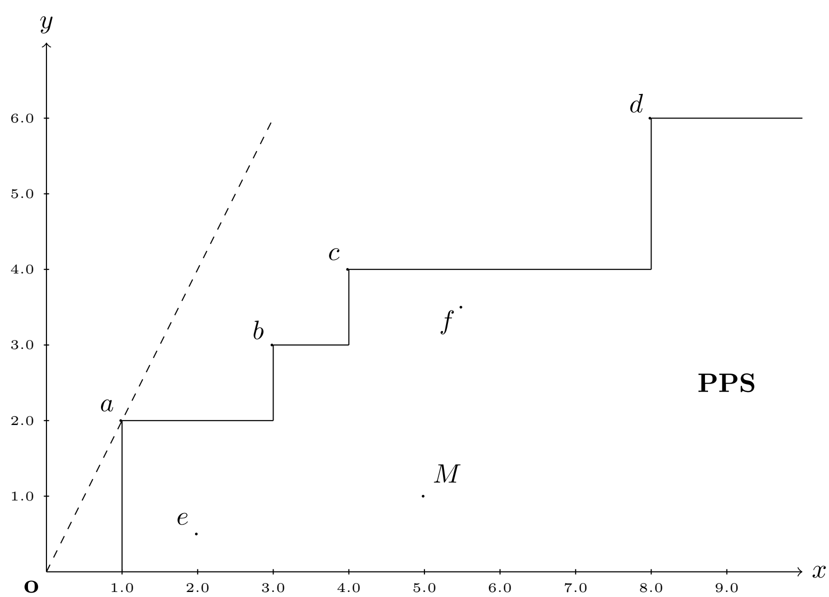


Figure 1: The production possibility set

to M.
Now, we use alghorithm2 to find closest strong efficient point to M. In step3 we see that only a is strong efficient point. For example for $k = 2$ and $j = 1$ we have $\frac{y_2}{y_1} < \frac{x_2}{x_1}$. For $k = 3, ..., 6$ same result holds. Now, according to step4 we have

$$d_1 = \min_{\lambda \geq 0} \parallel \lambda(1, 2) - (5, 1) \parallel = 4.0249$$

The closest strong efficient point is on line $\lambda(1, 2)$ that point is $(\frac{7}{5}, \frac{14}{5})$.

5. Illustrative Application to Bank Branches

In this section, the proposed approach is applied to a data set that has been analysed by Silva portela et al. [14]. The data correspond to 24 bank branches with two inputs (staff costs and other operating costs) and three outputs (value of current accounts, value of credit and interest revenues).
For an inefficient DMU, common methods (as much as possible) by reducing inputs and increasing outputs perform efficiently and with this method in mind, by less demanding levels of operation for the inputs and outputs define closest efficient target. Sometimes, it is reasonable that, instead of reducing input, by a small increase in input get significant increase in outputs or by a small decreasing in output get significant saving in inputs. While, this is in contrast with traditional DEA methods, we show that, this leads to a better results.

For instance, consider B19 in $T_{FDH-VRS}$. As it is shown, the closest strong efficient target for unit B19, obtained by our method, is B58. This point do not dominate B19 and is closer than B20 that Silva portela et al. [14] suggested. Also, in $T_{FDH-CRS}$ the closest efficient target for B19 is [8.482814.5313503.91015847.138]. As we see this point do not dominate B19, too.

The data and results are shown in Tables 1 and 2, respectively. The closest target of each unit has been recalculated by the proposed method. Tables 3 and 4 contents the results for the VRS and CRS, respectively.

6. Conclusion

In the context of data envelopment analysis, many efficiency improving projection models have been proposed; however, their drawbacks deemed researchers seek recent better models. In this article, we have proposed a numeration method to compute the minimum distance of units from the strong efficient frontier of the FDH non-convex production possibility set without using LP or MILP regular solving methods. Moreover, for an inefficient DMU, common methods consider the dominant efficient DMU. then, define closest efficient target. But, this method leads a closest efficient point that not necessarily dominate assessed inefficient DMU. This is the remarkable advantage of our proposed method. The applicability of the method is illustrated through analysing a real-world problem regarding 24 Portuguese bank branches.

7. Figures and Tables

Unit	Staff Costs	Other operating Costs	Current Accounts	Credit	Interest Revenue	FDH BCC-Eff.	FDH CCR-Eff.
B3	16.819	24.471	4892.629	10238.760	52.234		
B5	11.243	23.558	4777.107	8756.227	52.449		
B9	18.441	35.090	6450.385	12479.115	64.644		
B10	10.106	23.104	5223.611	12572.231	61.332	%100	%100
B11	15.129	32.781	7666.449	10221.426	67.682	%100	%100
B13	12.979	23.658	4991.984	10194.377	48.583		
B15	11.717	29314	4070.630	6418.995	40.328		
B16	18.306	31.359	7561.477	21922.138	101.725	%100	%100
B17	16.505	31.574	6322.393	1323.595	81.404	%100	
B19	12.211	24.411	3663.067	10103.516	49.062		
B20	11.981	17.857	3899.831	10658.024	51.052	%100	
B21	12.689	25.489	4797.797	10281.063	48.822		
B22	16.166	26.062	3946.813	7358.401	46.214		
B26	12.041	19.688	5524.905	7393.716	48.912	%100	%100
B27	10.021	16.780	3394.509	8269.236	39.565	%100	
B29	12.739	18.505	5635.758	6667.397	63.048	%100	%100
B50	12.505	17.508	4745.698	9603.156	48.199	%100	%100
B51	15.178	21.418	5758.861	6007.936	64.210	%100	
B52	14.146	22.291	4391.541	8259.170	50.503	%100	
B53	12.959	20.117	5372.053	7323.490	64.076	%100	%100
B56	9.073	19.259	2888.434	8694.691	39.974	%100	
B57	9.747	13.004	2107.062	5012.420	24.202	%100	
B58	10.639	22.566	3344.774	10293.887	43.311	%100	
B59	13.338	24.820	4354.301	10889.840	57.033		

Table 2: Data of bank branches.

Closest Target FDH Procedure		
Assessed DMU	Distance	Closest Efficient Point
Unit B19	602.98	B20
Unit B22	1004.59	B52

Table 3: Results from Silva Portela

Closest Target (VRS)		
Assessed DMU	Distance	Closest Efficient Point
Unit B19	370.93	B58
Unit B22	1004.59	B52

Table 4: Results for VRS.

Closest Target (CRS)				
Assessed DMU	Reference DMU	Lambda	Distance	Closest Efficient Point
$B19$	B16	0.46339	168.73	[8.4828 14.531 3503.9 10158 47.138]
$B22$	B50	0.77906	278.79	[9.7421 13.64 3697.2 7481.4 37.55]

Table 5: Results for CRS.

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