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Existence Results to Steklov System Involving the (p, q)-Laplacian

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ABSTRACT: In this paper, a quasilinear elliptic system involving a pair of (p,q)-Laplacian operators with Steklov boundary value conditions is studied. Using the Mountain Pass Geometry, we prove the existence of at least one weak solution. For the infinitely many weak solutions, we based on Bratsch's Fountain Theorem [9].

Key Words: Quasilinear elliptic equations with p-Laplacian, weak solution, mountain pass geometry.

Contents

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, with a smooth boundary $\partial\Omega$ and $1 < p < \infty,\ 1 < q < \infty$. We consider the system

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + |u|^{p-2}u = \frac{\partial F}{\partial u}(x,u,v) & \text{on } \partial\Omega, \\ \Delta_q v = 0 & \text{in } \Omega, \\ |\nabla v|^{q-2}\frac{\partial v}{\partial \nu} + |v|^{q-2}v = \frac{\partial F}{\partial v}(x,u,v) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $\frac{\partial}{\partial \nu}$ is the outer normal derivative. We denote

$$p^{\partial} = \begin{cases} \frac{(N-1)p}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$$

The function $F : \partial\Omega \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is assumed to be continuous in $x \in \partial\Omega$, of class C^1 in $u, v \in \mathbb{R}$ and satisfying the following hypotheses:
$(\mathbf{H_1})$ $F(x, s_1, s_2) \leq c_1 + c_2|s_1|^{p_1} + c_3|s_2|^{q_1} + c_4|s_1|^{\alpha}|s_2|^{\beta},\ \forall(x, s_1, s_2) \in \partial\Omega \times \mathbb{R}^2$, where p_1, q_1, α, β denote positive constants such that: $p_1 < p^{\partial},\ q_1 < q^{\partial},\ \frac{\alpha}{p} + \frac{\beta}{q} = 1,\ p_1 > p,\ q_1 > q$.
$(\mathbf{H_2})$ There exist $M > 0,\ \eta_1 > p,\ \eta_2 > q$: for all $x \in \partial\Omega$, for all $(s_1, s_2) \in \mathbb{R}^2$:

$$|s_1|^{\eta_1} + |s_2|^{\eta_2} \geq 2M \text{ such that} : 0 < F(x, s_1, s_2) \leq \frac{s_1}{\eta_1}\frac{\partial F}{\partial s_1}(x, s_1, s_2) + \frac{s_2}{\eta_2}\frac{\partial F}{\partial s_2}(x, s_1, s_2),$$

and

$$F(x, s_1, s_2) \geq c_5(|s_1|^{\eta_1} + |s_2|^{\eta_2} - 1).$$

$(\mathbf{H_3})$ $\lim\limits_{(s_1,s_2)\to(0,0)} \frac{F(x,s_1,s_2)}{|s_1|^p+|s_2|^q} = 0$, uniformly with respect to $x \in \partial\Omega$.

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Where $c_i, i = 1, 2...$, denote positive constants. These hypotheses are only needed to insure the Mountain Pass Geometry and the Palais-Smale condition for the Euler-Lagrange functional associated to the system (1.1).

Many publication has been studying existence results for nonlinear elliptic systems with Dirichlet, Neumann or Steklov conditions, for example we cite [1,2,11].
The quasilinear elliptic systems involving a general (p,q)-Laplacian operator has been received considerable attention in recent years. This is partly due to their frequent appearance in applications such as; the reaction-diffusion problems, the non-Newtonien fluids, astronomy, etc. (see [4]). Also these problems are very interesting from a purely mathematical point of view as well.
Various authors discuss this kind of problems such as [5,8,16], existence and nonexistence theorems were obtained.
In this paper, we will generalize the results of [12] to the (p,q)-Laplacian with Steklov boundary conditions, the first result is based on the Mountain Pass Theorem of Ambrosetti and Rabinowitz which was proposed in 1973 (see, [3]), it is a result of great intuitive appeal as well as practical importance in the determination of critical points of functionals, which has became one of the main tools for finding solutions to elliptic problems of variational type. For the second one we shall give a variant of Bratsch's Fountain Theorem [9].

Our main results are the proofs of the following theorems.

Theorem 1.1. *If the hypotheses $(H_1), (H_2)$ and (H_3) hold true, then the problem (1.1) has at least one weak solution.*

Theorem 1.2. *If the functional $F(x, u, v)$ is even in (u,v) and the hypotheses (H_1) and (H_2) hold true, then the problem (1.1) has infinitely many (pairs) weak solutions.*

This paper is organized as follows, section 1 contains an introduction and the main results. In section 2, which has a preliminary character, we will give some assumptions and facts that will be needed in the paper, in section 3, we will give the proofs of our main results.

2. Preliminaries

In this section, we introduce some notations used below and recall some background facts concerning the generalized Lebesgue-Sobolev space.
For every $1 < p < \infty$ and bounded domain $\Omega \subset \mathbb{R}$ and measurable $u : \Omega \to \mathbb{R}$ we denote the norm of $L^p(\Omega)$ by

$$\|u\|_{L^p(\Omega)} = \Big(\int_\Omega |u|^p dx \Big)^{\frac{1}{p}}.$$

The dual space of $L^p(\Omega)$ is $L^{p'}(\Omega)$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, we have the following Hölder type inequality:

$$\Big| \int_\Omega uvdx \Big| \leq 2\|u\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)}. \tag{2.1}$$

The separable and reflexive Banach space $W^{1,p}(\Omega)$ is endowed with its natural norm

$$\|u\|_{1,p} = \left(\int_\Omega |u|^p dx + \int_\Omega |\nabla u|^p dx \right)^{\frac{1}{p}}.$$

Consider the space $\mathrm{W} = \mathrm{W}^{1,\mathrm{p}}(\Omega) \times \mathrm{W}^{1,\mathrm{q}}(\Omega)$ equipped with the norm

$$\|(u, v)\|_W = \|u\|_{1,p} + \|v\|_{1,q}, \ \textit{for all}\ (u, v) \in \mathrm{W}.$$

We introduce the norm $\|.\|_{p,q}$ which is equivalent to $\|.\|_W$ and will be used later in this paper, where

$$\|w\|_{p,q} = \|u\|_p + \|v\|_q,$$

where

$$\|u\|_p = \Big(\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} |u|^p d\sigma\Big)^{\frac{1}{p}},$$

and

$$\|v\|_q = \Big(\int_\Omega |\nabla v|^q dx + \int_{\partial\Omega} |v|^q d\sigma\Big)^{\frac{1}{q}}.$$

$\|.\|_p$ is also a norm on $\mathrm{W}^{1,\mathrm{p}}(\Omega)$ which is equivalent to $\|u\|_{1,p}$, the same for $\|.\|_q$ is a norm on $\mathrm{W}^{1,\mathrm{q}}(\Omega)$. Then $\|.\|_{p,q}$ is a norm on W which is equivalent to $\|.\|_W$(see [Theorem 2.1] [10]).

Definition 2.1. *We say that (u, v) is a weak solution of (1.1) if :*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx + \int_{\partial\Omega} |u|^{p-2}u\varphi d\sigma = \int_{\partial\Omega} \frac{\partial F}{\partial u}(x, u, v)\varphi d\sigma,$$

and

$$\int_\Omega |\nabla v|^{q-2}\nabla v \nabla\psi dx + \int_{\partial\Omega} |v|^{q-2}v\psi d\sigma = \int_{\partial\Omega} \frac{\partial F}{\partial v}(x, u, v)\psi d\sigma,$$

for every (u, v) and $(\varphi, \psi) \in W = W^{1,p}(\Omega) \times W^{1,q}(\Omega)$.

Let

$$\mathcal{F}(u, v) = \int_{\partial\Omega} F(x, u, v)d\sigma, \tag{2.2}$$

then

$$\mathcal{F}'(u, v)(\varphi, \psi) = \mathcal{F}_u(u, v)(\varphi) + \mathcal{F}_v(u, v)(\psi),$$

where

$$\mathcal{F}_u(u, v)\varphi = \int_{\partial\Omega} \frac{\partial F}{\partial u}(x, u, v)\varphi d\sigma \text{ and } \mathcal{F}_v(u, v)\psi = \int_{\partial\Omega} \frac{\partial F}{\partial v}(x, u, v)\psi d\sigma.$$

The Euler-Lagrange functional associated to (1.1) is given by

$$\Phi(u, v) = \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q - \mathcal{F}(u, v), \tag{2.3}$$

it is clear that $\Phi \in C^1(W, \mathbb{R})$ and

$$(\Phi'(u, v)(\varphi, \psi)) = (\Phi_u(u, v)(\varphi)) + (\Phi_v(u, v)(\psi)),$$

where

$$(\Phi_u(u, v)(\varphi)) = \int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx + \int_{\partial\Omega} |u|^{p-2}u\varphi d\sigma - \mathcal{F}_u(u, v)\varphi$$

and

$$(\Phi_v(u, v)(\psi)) = \int_\Omega |\nabla v|^{q-2}\nabla v \nabla\psi dx + \int_{\partial\Omega} |v|^{q-2}v\psi d\sigma - \mathcal{F}_v(u, v)\psi.$$

Hence it is easy to see that the critical points of Φ are weak solutions to problem (1.1) and we use this fact in the search concerning weak solutions from the next sections.
Let W^* is the dual space of W endowed with the norm $\|.\|_{W^*}$, therefore

$$\|\Phi'(u, v)\|_{W^*} = \|\Phi_u(u, v)\|_{W^*,p} + \|\Phi_v(u, v)\|_{W^*,q},$$

where $\|.\|_{W^*,p}$ (respectively $\|.\|_{W^*,q}$) is the norm of the dual space of $W^{1,p}(\Omega)$ (respectively $W^{1,q}(\Omega)$).

3. Proof of main results

3.1. The existence result

To prove our Theorem 1.1, we shall give a variant of the Mountain Pass Theorem of Ambrosetti and Rabinowitz (see [14,17]) as follows.

Theorem 3.1. *Let X be a Banach space endowed with the norm $\|.\|_X$. Assume that $I \in C^1(X,\mathbb{R})$ satisfies the Palais-Smale condition, also I has a Mountain Pass Geometry, that is,*
1) Any sequence $(u_n)_n \subset X$ such that $(I(u_n))_n$ is bounded and $I'(u_n) \to 0$ in X as $n \to \infty$, contains a subsequence converging to a critical point of I.
2) There exist $r, c' > 0$ such that $I(u) \geq c'$ if $\|u\|_X = r$.
3) There exist $\check{u} \in X$ such that $\|\check{u}\| > r$ and $I(\check{u}) < c'$.
Then I has a nontrivial critical point $u_0 \in X \setminus \{0, \check{u}\}$ with critical value

$$I(u_0) = \inf_{\gamma\in\Gamma} \sup_{u\in\gamma} I(u) \geq c' > 0,$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = \check{u}\}$.

To this aim, we prove three auxiliary lemmas.
We give the first one.

Lemma 3.2. *Assume (H_1) and (H_2) hold true, then the functional Φ introduced by (2.3) satisfies the Palais-Smale condition.*

Proof. Let (u_n, v_n) a Palais-Smale sequence for the functional Φ, then there exist $c > 0$ such that $\Phi(u_n, v_n) \leq c$ and $\|\Phi'(u_n, v_n)\|_{W^*} \to 0$ as $n \to +\infty$.
We first show that (u_n, v_n) is bounded. To do so, we argue by contradiction and we assume that, up to a subsequence (u_n, v_n) is not bounded.

We have $\Phi(u_n, v_n) = \frac{1}{p}\|u_n\|_p^p + \frac{1}{q}\|v_n\|_q^q - \mathcal{F}(u_n, v_n)$. From (H_2), we have

$$\begin{aligned}\Phi(u_n, v_n) &\geq \frac{1}{p}\|u_n\|_p^p + \frac{1}{q}\|v_n\|_q^q - \int_{\partial\Omega} \frac{u_n}{\eta_1}\frac{\partial F}{\partial u_n}(x, u_n, v_n)d\sigma - \int_{\partial\Omega} \frac{v_n}{\eta_2}\frac{\partial F}{\partial v_n}(x, u_n, v_n)d\sigma \\ &\geq (\frac{1}{p} - \frac{1}{\eta_1})\|u_n\|_p^p + (\frac{1}{q} - \frac{1}{\eta_2})\|v_n\|_q^q + \frac{1}{\eta_1}\Big(\|u_n\|_p^p - \int_{\partial\Omega} u_n\frac{\partial F}{\partial u_n}(x, u_n, v_n)d\sigma\Big) \\ &\quad + \frac{1}{\eta_2}\Big(\|v_n\|_q^q - \int_{\partial\Omega} v_n\frac{\partial F}{\partial u_n}(x, u_n, v_n)d\sigma\Big),\end{aligned}$$

it follows

$$\Phi(u_n, v_n) \geq (\frac{1}{p} - \frac{1}{\eta_1})\|u_n\|_p^p + (\frac{1}{q} - \frac{1}{\eta_2})\|v_n\|_q^q - \frac{1}{\eta_1}\|\Phi_u(u_n, v_n)\|_{W^*,p}\|u_n\|_p - \frac{1}{\eta_2}\|\Phi_v(u_n, v_n)\|_{W^*,q}\|v_n\|_q.$$

Without loss of generality we may take $\|u_n\|_p \geq \|v_n\|_q$ and for $\eta_1 \leq \eta_2$, we obtain

$$\Phi(u_n, v_n) \geq (\frac{1}{p} - \frac{1}{\eta_1})\|u_n\|_p^p + (\frac{1}{q} - \frac{1}{\eta_2})\|v_n\|_q^q - \frac{1}{\eta_1}(\|\Phi_u(u_n, v_n)\|_{W^*,p} + \|\Phi_v(u_n, v_n)\|_{W^*,q})\|u_n\|_p,$$

$$\Phi(u_n, v_n) \geq (\frac{1}{p} - \frac{1}{\eta_1})\|u_n\|_p^p + (\frac{1}{q} - \frac{1}{\eta_2})\|v_n\|_q^q - \frac{1}{\eta_1}\|\Phi'(u_n, v_n)\|_{W^*}\|u_n\|_p.$$

Since $\Phi(u_n, v_n) \leq c$ and $\|\Phi'(u_n, v_n)\|_{W^*} \to 0$ as $n \to +\infty$, therefore

$$(\frac{1}{p} - \frac{1}{\eta_1})\|u_n\|_p^p + (\frac{1}{q} - \frac{1}{\eta_2})\|v_n\|_q^q \leq c.$$

Since $\eta_1 > p$ and $\eta_2 > q$, this can not hold true. Hence (u_n, v_n) is bounded.
Taking into account the fact that W is a reflexive Banach space we infer that, up to a subsequence, $(u_n, v_n) \rightharpoonup (u, v)$ weakly in W.
Moreover, $|(\Phi'(u_n, v_n), (u_n - u, v_n - v))| \leq \|\Phi'(u_n, v_n)\|_{W^*} \|(u_n - u, v_n - v)\|_{p,q}$. Since $\|\Phi'(u_n, v_n)\|_{W^*} \to 0$ as $n \to +\infty$. It follows that $(\Phi'(u_n, v_n), (u_n - u, v_n - v)) \to 0$ as $n \to +\infty$. It means that $(\Phi_u(u_n, v_n)(u_n - u)) + (\Phi_v(u_n, v_n)(v_n - v)) \to 0$ as $n \to +\infty$.

So we have
$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n (\nabla u_n - \nabla u) dx + \int_{\partial\Omega} |u_n|^{p-2} u_n (u_n - u) d\sigma - \mathcal{F}_u(u_n, v_n)(u_n - u) \to 0$ as $n \to +\infty$,
and
$\int_\Omega |\nabla v_n|^{q-2} \nabla v_n (\nabla v_n - \nabla v) dx + \int_{\partial\Omega} |v_n|^{q-2} v_n (v_n - v) d\sigma - \mathcal{F}_v(u_n, v_n)(v_n - v) \to 0$ as $n \to +\infty$.

By Hölder type inequality (2.1), we have

$$\left| \int_{\partial\Omega} |u_n|^{p-2} u_n (u_n - u) \right| \leq 2 \||u_n|^{p-2} u\|_{L^{p'}(\partial\Omega)} \|u_n - u\|_{L^p(\partial\Omega)},$$

$$\left| \mathcal{F}_u(u_n, v_n)(u_n - u) \right| \leq 2 \|\tfrac{\partial F}{\partial u}(x, u_n, v_n)\|_{L^{p'}(\partial\Omega)} \|u_n - u\|_{L^p(\partial\Omega)}.$$

Now we use the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$, we obtain that $u_n \to u$ strongly in $L^p(\partial\Omega)$. Therefor we deduce that

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n (\nabla u_n - \nabla u) dx \to 0 \text{ as } n \to +\infty.$$

Using S^+ property, we conclude that $u_n \to u$ strongly in $W^{1,p}(\Omega)$. Similarly we show that $v_n \to v$ strongly in $W^{1,q}(\Omega)$, so we have $(u_n, v_n) \to (u, v)$ strongly in W and the proof of this lemma is complete.
□

We carry on to the next lemma.

Lemma 3.3. *If (H_1) and (H_3) hold true, then there exist $r > 0$ and $c' > 0$ such that $\Phi(u, v) \geq c'$ for every $(u, v) \in W$ satisfying $\|(u, v)\|_{p,q} = r$.*

Proof. We have $\Phi(u, v) = \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q - \mathcal{F}(u, v)$. Using $(H_1), (H_3)$ and (2.2), we obtain

$$\begin{aligned}\Phi(u, v) &\geq \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q - \int_{\partial\Omega} \Big(o(|u|^p + |v|^q) + c_2|u|^{p_1} + c_3|v|^{q_1} + c_4|u|^\alpha |v|^\beta d\sigma \Big),\\ &\geq \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q - o(\|u\|^p_{L^p(\partial\Omega)} + \|v\|^q_{L^p(\partial\Omega)}) - c_2\|u\|^{p_1}_{L^{p_1}(\partial\Omega)} - c_3\|v\|^{q_1}_{L^{q_1}(\partial\Omega)} - c_4 \int_{\partial\Omega} |u|^\alpha |v|^\beta d\sigma.\end{aligned}$$

Using young inequality and $\frac{\alpha}{p} + \frac{\beta}{q} = 1$, and the continuous embedding

$$W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega),$$
$$W^{1,p}(\Omega) \hookrightarrow L^{p_1}(\partial\Omega),$$
$$W^{1,q}(\Omega) \hookrightarrow L^q(\partial\Omega),$$
$$W^{1,q}(\Omega) \hookrightarrow L^{q_1}(\partial\Omega).$$

We conclude that
$\Phi(u, v) \geq \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q - o(c_1^p\|u\|_p^p + c_1^q\|v\|_q^q) - c_2'\|u\|_p^{p_1} - c_3'\|v\|_q^{q_1} - (c_4^p\|u\|_p^p + c_4^q\|v\|_q^q)$.
Set $o(\max\{c_1^p, c_1^q\}) \leq \frac{1}{2}\min\{\frac{1}{p}, \frac{1}{q}\}$, $\max\{c_4^p, c_4^q\} \leq -\frac{1}{2}\min\{\frac{1}{p}, \frac{1}{q}\}$ and $C \leq \max\{c_2', c_3'\}$. Then

$$\Phi(u, v) \geq \frac{1}{p}\|u\|_p^p + \frac{1}{q}\|v\|_q^q - C(\|u\|_p^{p_1} + \|v\|_q^{q_1}). \tag{3.1}$$

Since $p_1 > p$ and $q_1 > q$, then there exist $c' > 0$ such that $\Phi(u, v) \geq c' > 0$ for $\|(u, v)\|_{p,q} = r$, where r is chosen sufficiently small, thus the proof is complete. □

Finally we give the third lemma.

Lemma 3.4. *If (H_2) hold true, then there exist $(\check{u},\check{v}) \in W$ and $t > 1$ with $\|(\check{u},\check{v})\|_{p,q} > r$ such that $\Phi(t\check{u},t\check{v}) < c'$.*

Proof. Let $(\check{u},\check{v}) \in W$ and $t > 1$, we have

$$\begin{aligned}\Phi(t\check{u},t\check{v}) &= \frac{1}{p}\|t\check{u}\|_p^p + \frac{1}{q}\|t\check{v}\|_q^q - \mathcal{F}(t\check{u},t\check{v})\\ &= \frac{t^p}{p}\|\check{u}\|_p^p + \frac{t^q}{q}\|\check{v}\|_q^q - \int_{\partial\Omega} F(x,t\check{u},t\check{v})d\sigma.\end{aligned}$$

From (H_2), we have $F(x,s_1,s_2) \geq c_5(|s_1|^{\eta_1} + |s_2|^{\eta_2} - 1),\ \forall(x,s_1,s_2) \in \partial\Omega \times \mathbb{R}^2$,
then

$$\begin{aligned}\Phi(t\check{u},t\check{v}) &\leq \frac{t^p}{p}\|\check{u}\|_p^p + \frac{t^q}{q}\|\check{v}\|_q^q - c_5\int_{\partial\Omega}|t\check{u}|^{\eta_1}d\sigma - c_5\int_{\partial\Omega}|t\check{v}|^{\eta_2}d\sigma - c_5|\partial\Omega|\\ &\leq \frac{t^p}{p}\|\check{u}\|_p^p + \frac{t^q}{q}\|\check{v}\|_q^q - c_5t^{\eta_1}\int_{\partial\Omega}|\check{u}|^{\eta_1}d\sigma - c_5t^{\eta_2}\int_{\partial\Omega}|\check{v}|^{\eta_2}d\sigma - c_5|\partial\Omega|,\end{aligned}$$

where $|\partial\Omega|$ is s the Lebesgue measure of $\partial\Omega$.
Due to the fact that $\eta_1 > p$ and $\eta_2 > q$, we arrive at

$$\lim_{t\to+\infty}\Phi(t\check{u},t\check{v}) = -\infty.$$

□

Now we can give the proof of our Theorem1.1.

Proof of Theorem 1.1. From lemma 3.2, 3.3 and 3.4, we can apply the Mountain Pass Theorem 3.1, we deduce that there exist a nontrivial critical point for the the Euler-Largange functional Φ, thus the problem (1.1) has at least one nontrivial weak solution (u_0,v_0) with

$$\Phi(u_0,v_0) = \inf_{\gamma\in\Gamma}\sup_{(u,v)\in\gamma}\Phi(u,v) \geq c' > 0.$$

□

3.2. The multiplicity results

In this subsection we prove under some conditions on the function F that the problem (1.1) possesses infinitely many nontrivial weak solutions. The proof is based on Bartsch's Fountain Theorem [9]. Before giving the statement of this theorem, we introduce the general context. Since $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$ are reflexive and separable Banach space (and their dual), then W and W^* are too.
It was proved (see [21]) that for a reflexive and separable Banach space there exist $(e_n)_{n\in\mathbb{N}} \subset W$ and $(f_n)_{n\in\mathbb{N}} \subset W^*$ such that

$$f_n(e_m) = \delta_{n,m} = \begin{cases} 1 & \text{if } n = m,\\ 0 & \text{if } n \neq m,\end{cases}$$

and $W = \overline{span}\{e_n : \ n = 1,2...\}$ and $W^* = \overline{span}\{f_n : \ n = 1,2...\}$.
For $i = 1,2...$ we define $X_i = span\{e_i\}$, $Y_i = \bigoplus_{j=1}^{i} X_j$ $Z_i = \bigoplus_{j=i}^{\infty} X_j$.
Let us recall the version of the Fountain Theorem which will be used in the sequel.

Theorem 3.5. ***Fountain Theorem*** *(see [19]). Let $I \in C^1(X,\mathbb{R})$ be an even functional, where $(X,\|.\|)$ is a separable and reflexive Banach space, assume that for each $i = 1,2...$ there exist $\rho_i > \gamma_i > 0$ such that*
(i) $\displaystyle\inf_{u\in Z_i,\|u\|=\gamma_i} I(u) \to +\infty$ *as $i \to +\infty$.*

(ii) $\max_{u\in Y_i,\|u\|=\rho_i} I(u) \leq 0.$

(iii) I satisfies the $(PS)_c$ condition for every $c>0$, that is, any sequence $(u_n)_n \in X$ such that $I(u_n)\leq c$ and $\|I'(u_n)\|_{X^*}\to 0$ as $n\to+\infty$ contains a subsequence converging to a critical point of I.
Then I has an unbounded sequence of critical points.

For $W=W^{1,p}(\Omega)\times W^{1,q}(\Omega)$ which is a separable and reflexive Banach space and for X_i, Y_i, Z_i define as above and for Φ given by (2.3) we can use the previous theorem to prove the result of Theorem 1.2. It is clear that the idea is to show that Φ satisfies hypotheses (i)-(iii) of Theorem 3.5. By Lemma 3.2 the hypothesis (iii) is fulfilled, since Lemma 3.4 takes place and we get from above that

$$\lim_{t\to+\infty}\Phi(t\check{u},t\check{v})=-\infty,$$

this implies that $\max_{(u,v)\in Y_i,\|(u,v)\|=\rho_i}\Phi(u,v)\leq 0.$ for every ρ_i large enough.
So we only focus on (i).
For every $\theta>1$, $u\in L^\theta(\partial\Omega)$ and $v\in L^\theta(\partial\Omega)$, we define $|(u,v)|_\theta=\max\{\|u\|_{L^\theta(\partial\Omega)},\|v\|_{L^\theta(\partial\Omega)}\}$. Set $\theta=\max_{x\in\overline{\Omega}}\{\alpha,\beta,p_1,q_1\}$ and $\theta'=\inf_{x\in\overline{\Omega}}\{\alpha,\beta,p_1,q_1\}$.
An important result for the proof is represented by the next proposition.

Proposition 3.6. *([12], Lemma 7) Define*

$$\delta_i=\sup\{|(u,v)|_\theta:\|(u,v)\|_{p,q}=1,(u,v)\in Z_i\},\text{then}\lim_{i\to+\infty}\delta_i=0. \tag{3.2}$$

We present the lemma corresponding to hypothesis (i).

Lemma 3.7. *Assume (H_1) hold true, then for every $i=1,2...$ there exist $\gamma_i>0$ such that*

$$\inf_{(u,v)\in Z_i,\|(u,v)\|=\gamma_i}\Phi((u,v))\to+\infty \text{ as } i\to+\infty.$$

Proof. By (3.1), (3.2) and (H_1), we have

$$\begin{aligned}\Phi(u,v)\geq& min\{\frac{1}{p},\frac{1}{q}\}\|(u,v)\|^{\min(p,q)}-C(\delta_i\|(u,v)\|)^{p_1}-C(\delta_i\|(u,v)\|)^{q_1},\\ \geq& min\{\frac{1}{p},\frac{1}{q}\}\|(u,v)\|^{\min(p,q)}-C\delta_i^{\theta'}\|(u,v)\|^{\theta},\end{aligned}$$

where θ,θ' are defined as above. Choosing $\gamma_i=\left(\min\{\frac{1}{p},\frac{1}{q}\}\frac{1}{2C\delta_i^{\theta'}}\right)^{\frac{1}{\theta-\min\{p,q\}}}\to+\infty$ as $i\to+\infty$. Consequently, for $\|(u,v)\|=\gamma_i$, then $\Phi(u,v)\geq\frac{1}{2}\min\{\frac{1}{p},\frac{1}{q}\}\|(u,v)\|^{\min(p,q)}$. Then the proof is complete. □

Proof of Theorem 1.2. From lemma 3.7 and applying the Fountain Theorem 3.5, we achieve the proof. □

Remark 3.8. *consider the following system*

$$\begin{cases} -div(|\nabla u|^{p-2}\nabla u)+|u|^{p-2}u=0 & in\ \Omega,\\ |\nabla u|^{p-2}\frac{\partial u}{\partial\nu}=\frac{\partial F}{\partial u}(x,u,v) & on\ \partial\Omega,\\ -div(|\nabla v|^{q-2}\nabla v)+|v|^{q-2}v=0 & in\ \Omega,\\ |\nabla v|^{q-2}\frac{\partial v}{\partial\nu}=\frac{\partial F}{\partial v}(x,u,v) & on\ \partial\Omega.\end{cases} \tag{3.3}$$

The same results will be obtained, using the norm

$$\|(u,v)\|_W=\|u\|_{1,p}+\|v\|_{1,q},\ for\ all\ (u,v)\in \mathrm{W},$$

where

$$\|u\|_{1,p}=\Big(\int_\Omega|\nabla u|^p dx+\int_\Omega|u|^p d\sigma\Big)^{\frac{1}{p}},$$

and

$$\|v\|_{1,q}=\Big(\int_\Omega|\nabla v|^q dx+\int_\Omega|v|^q d\sigma\Big)^{\frac{1}{q}}.$$

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