Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) v. 2023 (41) : 1–11.
ISSN-0037-8712 in press
doi:10.5269/bspm.48439

The Existence of One Solution for Impulsive Differential Equations via Variational Methods

Zhaleh Rouhani and Ghasem A. Afrouzi

abstract: We prove the existence of at least one non-trivial weak solution for a nonlinear Dirichlet boundary value problem subject to perturbations of impulsive terms via employing a critical point theorem for differentiable functionals.

Key Words: Existence result, boundary value problem, impulsive condition, variational methods, critical point theory.

Contents

1. Introduction

In this paper, we study the following nonlinear impulsive differential problem:

$$\begin{cases} -u''(t)+a(t)u'(t)+b(t)u(t)=\lambda g(t,u(t)), & t\in[0,T], t\neq t_j,\\ u(0)=u(T)=0, & \\ \Delta u'(t_j)=\lambda I_j(u(t_j)), \quad j=1,2,\dots,n, & \end{cases} \tag{1.1}$$

where $\lambda\geq 0, g:[0,T]\times\mathbb{R}\to\mathbb{R}, a,b\in L^\infty([0,T])$ with $ess\inf_{t\in[0,T]}a(t)\geq 0$ and $ess\inf_{t\in[0,T]}b(t)\geq 0, 0=t_0<t_1<t_2<\cdots<t_n<t_{n+1}=T, \Delta u'(t_j)=u'(t_j^+)-u'(t_j^-)=\lim_{t\to t_j^+}u'(t)-\lim_{t\to t_j^-}u'(t)$ and $I_j:\mathbb{R}\to\mathbb{R}$ are continuous for every $j=1,2,\dots,n$.

Impulsive differential equations are considered by many authors and one of the reasons of getting this attention can be the main role they play in many real world phenomena such as medicine, biology, mechanics, engineering, etc. One of the most important application of the impulsive differential equation is that it is the main tool to study the dynamics of that process which are subject to sudden changes in their state. The existence and multiplicity of solutions for impulsive differential equations have been examined in many works, and for an overview on this subject, we refer the reader to the papers [2,4,5,6,7,8,10,11,12,16,18,19,20,21,22,23,25,26,27,28,29]. For instance, in paper [6] the authors studied the existence of n distinct pairs of nontrivial solutions for the following impulsive differential equations with Dirichlet boundary conditions by using variational methods and critical point theory,

$$u''(t)+\lambda h(t,u(t))=0, \qquad t\neq t_j \quad a.e. t\in[0,T]$$

$$-\Delta u'(t_j)=I_j(u(t_j)), \qquad j=1,2,\dots p,$$

$$u(0)=u(T)=0,$$

where $0=t_0<t_1<\cdots<t_p<t_{p+1}=T, \lambda>0, h\in C([0,T]\times\mathbb{R},\mathbb{R}), I_j\in C(\mathbb{R},\mathbb{R}), j=1,2,\dots,p$, $\Delta(u'(t_j))=u'(t_j^+)-u'(t_j^-)$, $u'(t_j^+)$ and $u'(t_j^-)$ denote the right and the left limits, respectively, of $u'(t_j)$ at $t=t_j, j=1,2,\dots,p$. In [28], Zhang and Yuan dealt with the existence and multiplicity of solutions for the nonlinear Dirichlet value problem with impulses. Using the variational methods and critical points theory, they gave some new criteria to guarantee that the impulsive problem has at least one nontrivial

2010 *Mathematics Subject Classification:* 34C25, 58E30, 47H04.
Submitted June 22, 2019. Published September 16, 2020

solution, assuming that the nonlinearity is superquadratic at infinity, subquadratic at the origin, and the impulsive functions have sublinear growth. Moreover, if the nonlinearity and the impulsive functions are odd, then the impulsive problem has infinitely many distinct solutions. More precisely in [19] the authors studied the existence of solutions for following second-order impulsive differential equation by using the critical point theorem of Y.Jabri and an even functional theorem.

$$\begin{cases} -u''(t)+g(t)u(t)=f(t,u(t)), & a.e. t\in[0,T],\\ \Delta u'(t_j)=I_ju(t_j), & j=1,2,\dots,p\\ u(0)=u(T)=0, \end{cases}$$

where $g \in L^\infty[0,T]$, T is a real positive number, $\Delta(u'(t_j)) = u'(t_j^+) - u'(t_j^-) = \lim_{s\to t_j^+} u'(s) - \lim_{s\to t_j^-} u'(s), f : [0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous, $t_j, j=1,2,\dots,p$ are the instants where the impulses occur and $0=t_0<t_1<t_2<\dots<t_p<t_{p+1}=T, I_j:\mathbb{R}\to\mathbb{R}(j=1,2,\dots,p)$ are continuous. Also they gave some criteria to guarantee that the impulsive differential equation has at least one solution, infinitely many solutions under the assumption that a nonlinear term satisfies sublinear, superlinear, asymptotically linear, respectively. Some results are extended and conditions of assumptions are simplified. In [27] the authors considered the existence of solutions for following nonlinear impulsive problem with periodic boundary conditions, by using critical point theory,

$$\begin{cases} -u''(t)+cu(t)=\lambda f(t,u(t)), & t\neq t_j \quad a.e. t\in[0,T],\\ \Delta u'(t_j)=I_ju(t_j), & j=1,2,\dots,p-1\\ u(0)=u(T), \qquad u'(0^+)=u'(T^-), \end{cases}$$

where $c\in\mathbb{R}, \lambda\in\mathbb{R}\setminus\{0\}$ are two parameters, $T>0, f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous, $0=t_0<t_1<t_2<\dots<t_p=T, \Delta u'(t_j)=u'(t_j^+)-u'(t_j^-)=\lim_{t\to t_j^+}u'(t)-\lim_{t\to t_j^-}u'(t), u'(0^+)=\lim_{t\to 0^+}u'(t)$ and $u'(T^-)=\lim_{t\to T^-}u'(t), I_j:\mathbb{R}\to\mathbb{R}, j=1,2,...,p-1$ are continuous. Also they obtained some existence theorems of infinitely many solutions for the problem when the impulsive functions are super linear then extend and improve some results. In [7] the authors obtained some new existence results of solutions for some Dirichlet impulsive differential problems using critical point theory, they gave some new criteria to guarantee that the impulsive problem has at least one nontrivial solution or infinitely many solutions, assuming that the nonlinearity is superquadratic and has a sublinear growth. In the paper [4] the authors using variational methods studied second-order impulsive differential equations with Dirichlet boundary conditions, depending on two real parameters, and showed that an appropriate growth condition of the nonlinear term, under small perturbations of impulsive terms, ensures the existence of three solutions, while in the paper [5] they established multiplicity results for the same equations, and the have ensured the existence of infinitely many solutions using variational methods. Recently Graef et al. in [11] investigated the existence of infinitely many periodic solutions to a class of perturbed second-order impulsive Hamiltonian systems while the existence of nontrivial classical solutions for a class of Dirichlet boundary value problems with impulsive effects via variational methods and critical point theory is established in [10].

In the present paper, we are interested in ensuring the existence of at least one non-trivial solution for the nonlinear Dirichlet boundary value problem (1.1).

2. Preliminaries

A classical solution of (1.1) is a function u such that:

$$u\in\{w\in C([0,T]):w_{|[t_j,t_{j+1}]}\in H^2([t_j,t_{j+1}])\},$$

that satisfies the equation in (1.1) a.e. on $[0,T]\setminus\{t_1,\dots,t_n\}$, the limits $u'(t_j^+), u'(t_j^-), j=1,2,\dots,n$, exist, that satisfies the impulsive conditions $\Delta u'(t_j)=\lambda I_ju(t_j)$and the boundary conditions $u(0)=u(T)=0$. If a,b and g are continuous, then a classical solution satisfies the equation in (1.1) for all $t\in[0,T]\setminus\{t_1,t_2,\dots,t_n\}$. We consider the following slightly different form of problem (1.1):

$$\begin{cases} -(p(t)u'(t))'+q(t)u(t)=\lambda f(t,u(t)), & t\in[0,T], t\neq t_j,\\ u(0)=u(T)=0,\\ \Delta u'(t_j)=\lambda I_ju(t_j), \quad j=1,2,\dots,n, \end{cases} \tag{2.1}$$

where $p \in C^1([0,T],]0,+\infty[)$ and $q \in L^\infty([0,T])$ with $ess\inf_{t\in[0,T]} q(t) \geq 0$. By choosing $p(t), q(t)$ and $f(t,u)$ as follows, it is easy to see that the solutions of (2.1) are solutions of (1.1).

$$p(t) = e^{-\int_0^t a(\tau)d\tau}, q(t) = b(t)e^{-\int_0^t a(\tau)d\tau}, f(t,u) = g(t,u)e^{-\int_0^t a(\tau)d\tau}.$$

Let $E = H_0^1(0,T)$, and consider the inner product

$$\prec u, v \succ = \int_0^T p(t)u'(t)v'(t)dt + \int_0^T q(t)u(t)v(t)dt$$

which its corresponding norm is

$$\|u\| = \Big(\int_0^T p(t)(u'(t))^2dt + \int_0^T q(t)(u(t))^2dt\Big)^{\frac{1}{2}}.$$

These following Lemmas will be helpful in the proving main results.

Lemma 2.1. *([4, Proposition 2.1]) Let $u \in E$. Then*

$$\|u\|_\infty \leq \frac{1}{2}\sqrt{\frac{T}{p^*}}\|u\|, \tag{2.2}$$

where $p^* := \min_{t\in[0,T]} p(t)$

Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function.

Definition 2.2. *A function $u \in E$ is said to be a weak solution of (2.1) if u satisfies*

$$\int_0^T p(t)u'(t)v'(t)dt + \int_0^T q(t)u(t)v(t)dt - \lambda\int_0^T f(t,u(t))v(t)dt + \mu\sum_{j=1}^n p(t_j)I_j(u(t_j))v(t_j) = 0$$

for any $v \in E$.

Lemma 2.3. *([4, Lemma 2.1]) $u \in E$ is a weak solution of (1.1) if and only if u is a classical solution of (2.1).*

We establish our main results by applying the following version of Ricceri's variational principle [24, Theorem 2.1].

Theorem 2.4. *Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is sequentially weakly lower semicontinuous, strongly continuous, and coercive and Ψ is sequentially weakly upper semicontinuous. For every $r > \inf_X \Phi$, let us put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\sup_{v\in\Phi^{-1}(]-\infty,r])}\Psi(v) - \Psi(u)}{r - \Phi(u)}$$

Then, for any $r > \inf_X \Phi$ and every $\lambda \in]0, \frac{1}{\varphi(r)}[$, the restriction of the functional $I_\lambda = \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of I_λ in X.

We refer the interested reader to the papers [1,9,13,14,15,17] in which Theorem 2.4 has been successfully employed to the existence of at least one non-trivial solution for boundary value problems.

3. Main results

In this section we illustrate our main result to prove existence of solution for the problem.

Put $F(t,\zeta) = \int_0^\zeta f(t,x)dx$ for each $(t,\zeta) \in [0,T] \times \mathbb{R}$.

Theorem 3.1. *Assume that*

$$\sup_{\gamma>0} \frac{\gamma^2}{\int_0^T \max_{|\xi|\leq\gamma} F(t,\xi)dt} > \frac{T}{2p^*}, \tag{3.1}$$

$$\limsup_{\gamma\to+\infty} \frac{\Sigma_{j=1}^n \max_{|\xi|\leq\gamma} -\mathcal{I}j(\xi)}{\gamma^2} < +\infty \tag{3.2}$$

whose potential $\mathcal{I}_j(\xi) := \int_0^\xi I_j(x)dx, \xi \in \mathbb{R}$, *and there are non-empty open sets* $D \subsetneq (0,T)$ *and* $B \subset D$ *of positive Lebesgue measures such that*

$$\limsup_{\xi\to0^+} \frac{ess\inf_{t\in B} F(t,\xi) - \sum_{j=1,t_j\in B}^n p(t_j)\mathcal{I}_j(\xi)}{\xi^2} = +\infty$$

and

$$\liminf_{\xi\to0^+} \frac{ess\inf_{t\in D} F(t,\xi) - \sum_{j=1,t_j\in D}^n p(t_j)\mathcal{I}_j(\xi)}{\xi^2} > -\infty.$$

Then, for each

$$\lambda \in \Lambda = (0, \frac{2p^*}{T} \sup_{\gamma>0} \frac{\gamma^2}{\int_0^T \max_{|\xi|\leq\gamma} F(t,\xi)dt}),$$

the problem (1.1) *admits at least one non-trivial weak solution* $u_\lambda \in E$. *Moreover, we have*

$$\lim_{\lambda\to0^+} \|u_\lambda\| = 0$$

and the real function

$$\lambda \to \frac{1}{2}\Big(\int_0^T p(t)(u'(t))^2dt + \int_0^T q(t)(u(t))^2dt\Big)$$
$$-\lambda(\int_0^T F(t,u(t))dt - \sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)dx)$$

is negative and strictly decreasing in the open interval Λ.

Proof. We apply Theorem 2.4 to the problem (2.1). To this end, for each $u \in E$, set

$$\Phi(u) = \frac{1}{2}\|u\|^2, \qquad \Psi(u) = \int_0^T F(t,u(t))dt - \sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)dx,$$

and

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u).$$

Clearly, the functionals Φ and Ψ satisfy the required conditions in Theorem 2.4. The functional Ψ is Gâteaux differentiable and sequentially weakly upper semicontinuous whose Gâteaux derivative at the point $u \in E$ is the functional $\Psi'(u) \in E^*$, given by

$$\Psi'(u)(v) = \int_0^t F(t,u(t))v(t)dt - \sum_{j=1}^n p(t_j)I_j(u(t_j))v(t_j)dx$$

for every $v \in E$. Moreover, we observe that the functional Φ is Gâteaux differentiable whose Gâteaux derivative at the point $u \in E$ is the functional $\Phi'(u) \in E^*$, given by

$$\Phi'(u)(v) = \int_0^T p(t)u'(t)v'(t)dt + \int_0^T q(t)u(t)v(t)dt$$

for every $v \in E$. From the condition (3.1), there exists $\bar{\gamma} > 0$ such that

$$\frac{\bar{\gamma}^2}{\int_0^T \max_{|\xi|\leq\bar{\gamma}} F(t,\xi)dt} > \frac{T}{2p^*}$$

Put

$$r = \frac{2p^*}{T}\bar{\gamma}^2.$$

Also, we have $\Phi^{-1}(-\infty, r) = \{u : \Phi(u) < r\} \subseteq \{u : |u(t)| \leq \bar{\gamma},\ \forall\, t \in [0,T]\}$. Hence

$$\Psi(u) = \int_0^T F(t,u(t))dt - \sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)dx \leq \int_0^T \max_{|\xi|\leq\bar{\gamma}} F(t,\xi)dt$$

for every $u \in E$ such that $\Phi(u) < r$. Then

$$\sup_{\Phi(u)<r} \Psi(u) \leq \int_0^T \max_{|\xi|\leq\bar{\gamma}} F(t,\xi)dt + \sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)dx.$$

By considering the above computations and the definition of $\varphi(r)$, since $0 \in \Phi^{-1}(-\infty, r)$ and $\Phi(0) = \Psi(0) = 0$, one has

$$\begin{aligned}\varphi(r) &= \inf_{u\in\Phi^{-1}(-\infty,r)} \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v) - \Psi(u)}{r - \Phi(u)}\\ &\leq \frac{\sup_{v\in\Phi^{-1}(-\infty,r)} \Psi(v)}{r}\\ &\leq \frac{T}{2p^*}\frac{\int_0^T \max_{|\xi|\leq\bar{\gamma}} F(t,\xi)dt}{\bar{\gamma}^2} + \frac{T\|p\|_\infty}{2p^*}\sum_{j=1}^n \frac{max_{|\xi|\leq\bar{\gamma}}(-I_j)(\xi)}{\bar{\gamma}^2}\end{aligned}$$

so we have

$$\varphi(r) \leq \frac{T}{2p^*}\frac{\int_0^T \max_{|\xi|\leq\bar{\gamma}} F(t,\xi)dt}{\bar{\gamma}^2} + \frac{T\|p\|_\infty}{2p^*}\sum_{j=1}^n \frac{max_{|\xi|\leq\bar{\gamma}}(-I_j)(\xi)}{\bar{\gamma}^2} \tag{3.3}$$

Hence, putting $\lambda^* = \dfrac{1}{\frac{T}{2p^*}\frac{\int_0^T \max_{|\xi|\leq\bar{\gamma}} F(t,\xi)dt}{\bar{\gamma}^2} + \frac{T\|p\|_\infty}{2p^*}\sum_{j=1}^n \frac{max_{|\xi|\leq\bar{\gamma}}(-I_j)(\xi)}{\bar{\gamma}^2}}$ Theorem 2.4 ensures that for every $\lambda \in (0, \lambda^*) \subseteq (0, \dfrac{1}{\varphi(r)})$ the functional I_λ admits at least one critical point (local minima) $u_\lambda \in \Phi^{-1}(-\infty, r)$. For every fixed $\lambda \in (0, \lambda^*)$ we show that $u_\lambda \neq 0$ and the map

$$(0, \lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda)$$

is negative. Let us verify that

$$\limsup_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty$$

Due to our assumptions at zero, we can fix a sequence $\{\xi_n\} \subset \mathbb{R}^+$ converging to zero and two constants σ, κ (with $\sigma > 0$) such that

$$\lim_{n\to\infty} \frac{ess\inf_{t\in B} F(t,\xi_n) - \sum_{j=1,t_j\in B}^n p(t_j)\int_0^{u(t_j)} I_j(x)dx}{\xi_n^2} = +\infty$$

and

$$ess\inf_{t\in D} F(t,\xi_n) - \sum_{j=1,t_j\in D}^n p(t_j)\int_0^{\xi_n} I_j(x)dx \geq \kappa\xi_n^2$$

for every $\xi \in [0,\sigma]$. Now, fix a set $C \subset B$ of positive measure and a function $\nu \in E$ such that:
(i) $v(t) \in [0,T]$, for every $t \in [0,T]$
(ii) $v(t) = 1$, for every $t \in C$
(iii) $v(t) = 0$, for every $t \in [0,T]\backslash D$
Hence, fix $M > 0$ and consider a real positive number η with

$$M < \frac{\eta\ meas(C) + \kappa \int_{D\backslash C} |v(t)|^2 dt}{\frac{\|v\|^2}{2}}.$$

Then, there is $\nu \in \mathbb{N}$ such that $\xi_n < \sigma$ and

$$ess \inf_{t\in B} F(t,\xi_n) - \sum_{j=1, t_j\in B}^{n} p(t_j) \int_0^{\xi_n} I_j(x)dx \geq \eta \xi_n^2$$

for every $n > \nu$. Now, for every $n > \nu$, bearing in mind the properties of the function v (that is $0 \leq \xi_n v(t) < \sigma$ for n sufficiently large), one has

$$\begin{aligned}
\frac{\Psi(\xi_n v)}{\Phi(\xi_n v)} =& \frac{\int_C F(t,\xi_n)dt + \int_{D\backslash C} F(t,\xi_n v(t))dt}{\Phi(\xi_n v)} \\
&- \frac{\sum_{j=1,t_j\in C}^{n} p(t_j) \int_0^{\xi_n} I_j(x)dx + \sum_{j=1,t_j\in D\backslash C}^{n} p(t_j) \int_0^{\xi_n v(t_j)} I_j(x)dx}{\Phi(\xi_n v)} \\
&\geq \frac{\eta\ meas(C) + \kappa \int_{D\backslash C} |v(t)|^2 dt}{\frac{\|v\|^2}{2}} > M.
\end{aligned}$$

Since M could be arbitrarily large it concludes that

$$\limsup_{\|u\|\to 0^+} \frac{\Psi(u)}{\Phi(u)} = +\infty$$

Hence, there exists a sequence $\omega_n \subset E$ strongly converging to zero such that, for n sufficiently large, $\omega_n \in \Phi^{-1}(-\infty, r)$ and $I_\lambda(\omega_n) := \Phi(\omega_n) - \lambda\Psi(\omega_n) < 0$, since u_λ is a global minimum of the restriction of I_λ to $\Phi^{-1}(-\infty, r)$, we conclude that

$$I_\lambda(u_\lambda) < 0 \tag{3.4}$$

so that u_λ is not trivial. From (3.4) we easily see that the map

$$(0,\lambda^*) \ni \lambda \mapsto I_\lambda(u_\lambda) \tag{3.5}$$

is negative. Now we show that

$$\lim_{\lambda\to 0^+} \|u_\lambda\| = 0.$$

By considering that Φ is coercive and that for $\lambda \in (0,\lambda^*)$ the solution $u_\lambda \in \Phi^{-1}(-\infty, r)$, one has that there exists a positive constant L such that $\|u_\lambda\| \leq L$ for every $\lambda \in (0,\lambda^*)$. Clearly, there exists a positive constant M such that

$$\left| \int_0^T F(t,u_\lambda(t))u_\lambda(t)dt + \sum_{j=1}^{n} p(t_j) \int_0^{u(t_j)} I_j(x)dx \right| \leq M\|u_\lambda\| \leq ML. \tag{3.6}$$

for every $\lambda \in (0,\lambda^*)$. Since u_λ is a critical point of I_λ, we have $I'_\lambda(u_\lambda)(v) = 0$ for any $v \in E$ and every $\lambda \in (0,\lambda^*)$. In particular $I'_\lambda(u_\lambda)(u_\lambda) = 0$, that is,

$$\Phi'(u_\lambda)(u_\lambda) = \lambda \int_0^T f(t,u_\lambda(t))u_\lambda(t)dt \tag{3.7}$$

for every $\lambda \in (0, \lambda^*)$. By (3.7) it follows that

$$0 \leq \|u_\lambda\|^2 \leq \Phi'(u_\lambda)(u_\lambda) = \lambda \int_0^T f(t, u_\lambda(t))dt$$

for any $\lambda \in (0, \lambda^*)$.. Letting $\lambda \to 0^+$, by (3.6) we get

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0,$$

as claimed. Finally, we show that the map

$$\lambda \mapsto I_\lambda(u_\lambda)$$

is strictly decreasing in $(0, \lambda^*)$. For our goal we see that for any $u \in E$, one has

$$I_\lambda(u) = \lambda\big(\frac{\Phi(u)}{\lambda} - \Psi(u)\big). \tag{3.8}$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \lambda^*$ and let u_{λ_i} be the global minimum of the functional I_{λ_i} restricted to $\Phi(-\infty, r)$ for $i = 1, 2$. Also, let

$$m_{\lambda_i} = \frac{\Phi(u_{\lambda_i})}{\lambda_i} - \Psi(u_{\lambda_i}) = \inf_{\nu \in \Phi^{-1}(-\infty, r)} \big(\frac{\Phi(\nu)}{\lambda_i} - \Psi(\nu)\big),$$

for every $i = 1, 2$. Clearly, (3.5) together with (3.8) and the positivity of λ imply that

$$m_{\lambda_i} < 0, \quad for\ i = 1, 2. \tag{3.9}$$

Moreover, since $0 < \lambda_1 < \lambda_2$, we have

$$m_{\lambda_2} < m_{\lambda_1}. \tag{3.10}$$

Then, by (3.8)-(3.10) and the fact $0 < \lambda_1 < \lambda_2$, we obtain

$$I_{\lambda_2}(u_{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = I_{\lambda_1}(u_{\lambda_1}),$$

so that the map $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $\lambda \in (0, \lambda^*)$. Since $\lambda < \lambda^*$ is arbitrary, we see that $\lambda \mapsto I_\lambda(u_\lambda)$ is strictly decreasing in $(0, \lambda^*)$. The proof is complete. □

We here present the following example in which the hypotheses of Theorem 3.1 are fulfilled.

Example 3.2. *Consider the following problem*

$$\begin{cases} -(\frac{\sqrt{t+2}}{t+1}u'(t))' + u'(t) + u(t) = \lambda f(t, u(t)), & t \in [0,1], t \neq \frac{1}{4}, \frac{1}{5} \\ u(0) = u(1) = 0, \\ \Delta u'(\frac{1}{4}) = \lambda(-e^{u(\frac{1}{4})}(u^2(\frac{1}{4}) + 2u(\frac{1}{4}))), \\ \Delta u'(\frac{1}{5}) = \lambda(-e^{u(\frac{1}{5})}(u^2(\frac{1}{5}) + 2u(\frac{1}{5}))), \end{cases} \tag{3.11}$$

where $f(t, x) = 2 + t \sin x$ for all $(t, x) \in [0, 1] \times \mathbb{R}$. By the expression of f we have $F(t, x) = 2x - t\cos x + t$ for all $(t, x) \in [0, 1] \times \mathbb{R}$. Then, for each

$$\lambda \in \Lambda = (0, +\infty),$$

the problem (3.11) admits at least one non-trivial weak solution $u_\lambda \in E$. Moreover, we have

$$\lim_{\lambda \to 0^+} \|u_\lambda\| = 0$$

and the real function

$$\lambda \to \frac{1}{2}\Big(\int_0^1 \frac{\sqrt{t+2}}{t+1}(u'(t))^2dt + \int_0^1 (u(t))^2dt\Big) - \lambda\Big(\int_0^1 (2u(t) - t\cos(u(t)) + t)dt$$
$$- \sum_{j=1}^{n}\Big(\frac{\sqrt{t_j+2}}{t_j+1}\Big)\int_0^{u(t_j)} I_j(x)dx\Big)$$

is negative and strictly decreasing in the open interval Λ.

We give some remarks of our results as follows.

Remark 3.3. *Here employing Ricceri's variational principle we are looking for the existence of critical points of the functional* I_λ *naturally associated with the problem* (2.1). *We emphasize that by direct minimization, we can not argue,in general for finding the critical points of* I_λ. *Because, in general,* I_λ *can be unbounded from the following in* E. *Indeed, for example, in the case when* $f(x) = 1 + |x|^{\tau-2}x$ *for every* $x \in \mathbb{R}$, *for any fixed* $u \in E \setminus \{0\}$ *and* $\iota \in \mathbb{R}$, *we obtain*

$$\begin{aligned} I_\lambda(\iota u) =& \Phi(\iota u) - \lambda\int_0^t f(\iota u(t))dt - \lambda\sum_{j=1}^{n} p(t_j)\int_0^{\iota u(t_j)} I_j(x)dx \\ \leq& \frac{1}{2}\iota^2\|u\|^2 - \lambda\iota\|u\| - \lambda\iota^{\tau-1}\|u\|^\tau - \lambda p^*\frac{L}{2}\|u\|^2 \to -\infty \end{aligned}$$

as $\iota \to +\infty$.

Remark 3.4. *We want to point out that the energy functional* I_λ *associated with the problem* (2.1) *is not coercive. Indeed, when* $f(x) = 1 + |x|^{\tau-2}x$ *we have*

$$\begin{aligned} I_\lambda(\iota u) =& \Phi(\iota u) - \lambda\int_0^t f(\iota u(t))dt - \lambda\sum_{j=1}^{n} p(t_j)\int_0^{\iota u(t_j)} I_j(x)dx \\ \leq& \frac{1}{2}\iota^2\|u\|^2 - \lambda\iota\|u\| - \lambda\iota^{\tau-1}\|u\|^\tau - \lambda p^*\frac{L}{2}\|u\|^2 \to -\infty \end{aligned}$$

as $\iota \to +\infty$

Remark 3.5. *If in Theorem 3.1 the function* $f(t,x) \geq 0$ *for a. e.* $(t,x) \in [0,T] \times \mathbb{R}$, *the condition* (3.1) *becomes to the more simple form*

$$\sup_{\gamma>0} \frac{\gamma^2}{\int_0^T F(t,\xi)dt} > \frac{T}{2p^*}. \tag{3.12}$$

Moreover, if the following assumption is verified

$$\limsup_{\gamma\to+\infty} \frac{\gamma^2}{\int_0^T F(t,\xi)dt} > \frac{T}{2p^*},$$

then the condition (3.12) *automatically holds.*

Remark 3.6. *In Theorem 3.1 if* f *is nonnegative, then the solution* u_λ *is positive. Indeed, arguing by a contradiction, assume that the set* $A = \{t \in]0,1] : u_\lambda(t) < 0\}$ *is non-empty and of positive measure. Put* $\bar{u}(t) = \min\{0, u_\lambda(t)\}$ *for all* $t \in [0,1]$. *Clearly,* $\bar{u} \in E$ *and one has*

Remark 3.7. *Now, let us prove that the critical points of the energy functional* I_λ *are nonnegative. Arguing by a contradiction, assume that* u *is a critical point of* I_λ *and the open set*

$$A := \big\{t \in [0,T] : u(t) < 0\big\}$$

is of positive Lebesgue measure. Put $\bar{u} := \min\{0.u\}$. Clearly, $\bar{u} \in X$ and, taking into account that u is a critical point, one has

$$
\begin{aligned}
0 =&\Phi'(u)(\bar{u}) - \lambda\Psi'(u)(\bar{u})\\
&= \int_0^T p(t)u'(t)\bar{u}'(t)dt + \int_0^T q(t)u(t)\bar{u}(t)dt - \lambda\int_0^T f(t,u(t))\bar{u}(t)dt\bar{u}(t_j)\\
&+\Sigma_{j=1}^n p(t_j)I_j(u(t_j)) = 0
\end{aligned}
$$

Thus, from our sign assumption on the data we have

$$
\begin{aligned}
0 \leq&\Big(\int_0^T p(t)(u'(t))^2dt + \int_0^T q(t)(u(t))^2dt\Big)^{\frac{1}{2}}\\
\leq& \int_0^T p(t)(u'(t))^2dt + \int_0^T q(t)(u(t))^2dt\\
&-\lambda(\int_0^t F(t,u(t))dt - \sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)dx) \leq 0
\end{aligned}
$$

so it is absurd.

Now, we present a special case of Theorem 3.1 as follows.

Theorem 3.8. *Let $\alpha \in L^\infty([0,T])$ such that $ess\inf_{t\in[0,T]}\alpha(t) > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) = 0$, and put $F(\xi) = \int_0^\xi f(t)dt$ for all $\xi \in \mathbb{R}$. Assume that*

$$
\sup_{\gamma>0}\frac{\gamma^2}{\max_{|\xi|\leq\gamma}F(\xi)} > \frac{T\int_0^T\alpha(t)dt}{2p^*}
$$

and condition (3.2) holds, there are non-empty open sets $D \subseteq (0,T)$ and $B \subset D$ of positive Lebesgue measures such that

$$
\limsup_{\xi\to0^+}\frac{F(\xi)ess\inf_{t\in B}\alpha(t) - \sum_{j=1,t_j\in B}^n p(t_j)\mathfrak{I}_j(\xi)}{\xi^2} = +\infty
$$

and

$$
\liminf_{\xi\to0^+}\frac{F(\xi)ess\inf_{t\in D}\alpha(t) - \sum_{j=1,t_j\in D}^n p(t_j)\mathfrak{I}_j(\xi)}{\xi^2} > -\infty.
$$

Then, for each $\lambda \in \Lambda = \left(0, \dfrac{2p^*}{T\int_0^T\alpha(t)dt}\sup_{\gamma>0}\dfrac{\gamma^2}{\max_{|\xi|\leq\gamma}F(\xi)}\right)$, the problem

$$
\begin{cases}
-(p(t)u'(t))' + q(t)u(t) = \lambda\alpha(t)f(u), & t\in[0,T], t\neq t_j,\\
u(0) = u(T) = 0,\\
\Delta u'(t_j) = \lambda I_j u(t_j), \quad j = 1,2,\ldots,n,
\end{cases}
\tag{$S_{\lambda,\alpha}$}
$$

admits at least one non-trivial weak solution $u_\lambda \in E$ such that

$$
\lim_{\lambda\to0^+}\|u_{\lambda\|=0}
$$

and the real function

$$
\lambda \to \frac{1}{2}\Big(\int_0^T p(t)(u'(t))^2dt + \int_0^T q(t)(u(t))^2dt\Big) - \lambda(\int_0^T \alpha(t)F(u(t))dt
$$
$$
-\sum_{j=1}^n p(t_j)\int_0^{u(t_j)} I_j(x)dx)
$$

is negative and strictly decreasing in Λ.

At the end, we present the following example to illustrate previous theorem.

Example 3.9. *Consider the following problem*

$$\begin{cases} -(u'(t))' + u(t) = \lambda\alpha(t)f(u), & t \in [0,1], t \neq t_j, \\ u(0) = u(1) = 0, \\ \Delta u'(t_j) = \lambda I_j u(t_j), \quad j = 1, 2, \ldots, n, \end{cases}$$

where $f(x) = xe^x, \alpha(t) = \sqrt{t+1}$ *for all* $x \in \mathbb{R}, t \in [0,1]$ *and* $I_j(\xi) = -\xi^2$ *for all* $\xi \in \mathbb{R}$, $j = 1, 2$. *By the expression of* f *we have* $F(x) = xe^x - e^x + 1$ *for all* $x \in \mathbb{R}$. *Obviously are conditions of previous theorem are fulfilled. So, for each* $\lambda \in \Lambda = \left(0, \frac{4}{3}(2\sqrt{2} - 1)\right)$, *the above problem admits at least one non-trivial weak solution* $u_\lambda \in E$ *such that*

$$\lim_{\lambda \to 0^+} \|u_{\lambda\|=0}$$

and the real function

$$\lambda \to \frac{1}{2}\Big(\int_0^1 (u'(t))^2 dt + \int_0^1 (u(t))^2 dt\Big) - \lambda(\int_0^1 \alpha(t)F(u(t))dt$$

$$-\sum_{j=1}^{n} \int_0^{u(t_j)} I_j(x)dx)$$

is negative and strictly decreasing in Λ.

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Zhaleh Rouhani,
Department of Mathematics, Faculty of Mathematical Sciences,
University of Mazandaran,
Babolsar, Iran.
E-mail address: zh.rouhani@yahoo.com

and

Ghasem A. Afrouzi (corresponding author),
Department of Mathematics, Faculty of Mathematical Sciences,
University of Mazandaran,
Babolsar, Iran.
E-mail address: afrouzi@umz.ac.ir