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Asymptotically Lacunary μ-Statistical Equivalence of Generalized Difference Sequences in Probabilistic Normed Spaces*

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ABSTRACT: The current article introduces the notion of asymptotically lacunary (Δ^n, μ)-statistical equivalent sequence in the settings of a probabilistic norm N. Furthermore, the article presents the concepts of asymptotically (Δ^n, μ)-strongly Cesáro equivalent sequences and asymptotically (Δ^n, μ)-strongly Cesáro Orlicz equivalent sequences in the theory of probabilistic normed spaces and also investigates their various properties including some inclusion relations as well as some equivalent conditions in this new settings.

Key Words: Probabilistic normed space, μ-statistical convergence, difference sequence, lacunary sequence, asymptotically equivalent sequence, Orlicz function.

Contents

1. Introduction

Ever since the theory of metric spaces was introduced by Fréchet [10], it has been emerged as an important area of research in various field of mathematics such as geometry, analysis etc. In numerous branches of mathematics, it has been found much convenient to have a notion of distance which is applicable to the elements of abstract sets. In context to this, Fréchet [10] introduced the theory of metric spaces in 1906. In this theory, he defined the notion of distance between two elements of a set by associating a non-negative real number with each ordered pair of elements of the set satisfying certain properties. But it is not always possible for associating such a single number with a pair of elements. In such type of situations, it is better to look upon the distance concept as a statistical rather than a determinate one. In this context, generalizing the notion of metric space, Menger [17] introduced the notion of statistical metric space, now known as probabilistic metric space. Using the concept of statistical metric and generalizing the idea of ordinary normed linear space, Šerstnev [24] introduced the notion of probabilistic normed space (in short PN-space) in 1962, where the norms of the vectors are represented by distribution function rather than a positive number. Situations where crisp norm is unable to measure the length of a vector precisely, the notion of probabilistic norm happens to be useful, one may refer to Alsina et al. [1]. This theory is important as a generalization of deterministic results of normed linear spaces and furnishes us vital tools appropriate to the investigation of geometry of nuclear physics, topological spaces, convergence of random variables, continuity properties, linear operators, etc.

The concept of statistical convergence was introduced by Steinhaus [25] and Fast [9] independently and then studied by many researcher see for instance [4,7,12,13,18,19,20,22,26,27,31]. The idea of lacunary strong convergence was introduced by Freedman et al. [11] and investigated by many researcher

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from different aspects [2,3,7,19,22,26,27,32]. As an important generalization of statistical convergence Fridy and Orhan [12,13] studied the concept of lacunary statistical convergence.

The notion of difference sequence was first proposed by Kizmaz [15]. It was generalized by Et and Çolak [8] as generalized difference sequence. Then this concept has been investigated by various authors [2,3,22,26,27,28] from different aspects.

The concept of asymptotically equivalent sequences was first introduced by Marouf [16]. Patterson [18] investigated the statistical analogue of this notion by introducing asymptotically statistical equivalent sequences, which was further extended using a lacunary sequence by Patterson and Savas [19]. In that paper, they introduced the idea of asymptotically lacunary statistical equivalent sequences. This idea was further generalized using a generalized difference sequence and an Orlicz function in order to develop the concepts of asymptotically Δ^n-lacunary statistical equivalent sequences and Cesáro Orlicz asymptotically statistical equivalent sequences by Braha [3]. The work of Patterson and Savas [19] was generalized from the aspect of PN-spaces by Esi [7].

An interesting generalization of the theory of asymptotically statistical equivalent sequences is to study the concept of statistical convergence using a complete $\{0,1\}$ valued measure μ defined on an algebra of subsets of natural numbers has been introduced by Connor [4,5].

Motivated by the literature reviewed above, we thought of studying asymptotically lacunary statistical equivalence of generalized difference sequences in the theory of PN-spaces using the two valued measure μ in this article. In context to this, we introduce the notion of asymptotically lacunary (Δ^n,μ)-statistical equivalent sequences in PN-spaces and investigate some results. We introduce the concepts of asymptotically (Δ^n,μ)-statistical equivalent sequences, asymptotically (Δ^n,μ)-strongly Cesáro equivalent sequences and asymptotically (Δ^n,μ)-strongly Cesáro Orlicz equivalent sequences in the theory of PN-spaces and also investigate their various inclusion relations as well as some equivalent conditions in this new settings.

A short outline of the article is as follows: In Section 2 we procure some basic definitions and examples which are going to be used during this investigation. We have introduced asymptotically (Δ^n,μ)-statistical equivalent sequences in PN-spaces and discussed some of their properties in Section 3. In this section itself, we have further introduced the concepts of asymptotically lacunary (Δ^n,μ)-statistical equivalent sequences and asymptotically (Δ^n,μ)-strongly Cesáro equivalent sequences in PN-spaces and also investigates their various inclusion relations as well as some equivalent conditions in this new settings. In the Section 4, we are deal with the concept of asymptotically (Δ^n,μ)-strongly Cesáro Orlicz equivalent sequences in the theory of PN-spaces and investigate few inclusion properties. Finally, Section 5 summarizes the article with a conclusion.

2. Preliminaries

Throughout the paper, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{Z}^+$ denote the sets of natural, real, non-negative real numbers and non-negative integers, respectively.

Definition 2.1. *[21] A function $f:\mathbb{R}^+\to[0,1]$ is called a distribution function if it is non-decreasing, left-continuous with $\inf_{t\in\mathbb{R}^+} f(t)=0$ and $\sup_{t\in\mathbb{R}^+} f(t)=1$. Let D denotes the set of all distribution functions.*

Definition 2.2. *[21] A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is said to be a continuous t-norm if it satisfies the following conditions, for all $a,b,c,d\in[0,1]$:*

(i) $a*1=a$,

(ii) $a*b=b*a$,

(iii) $a*b\le c*d$, *whenever* $a\le c$ *and* $b\le d$,

(iv) $(a * b) * c = a * (b * c)$.

Definition 2.3. *[1] A triplet $(X, N, *)$ is called a probabilistic normed space (in short a PN-space) if X is a real vector space, N a mapping from X into D (for $x \in X$, the distribution function $N(x)$ is denoted by N_x and $N_x(t)$ is the value of N_x at $t \in \mathbb{R}^+$) and $*$ a t-norm satisfying the following conditions:*

(i) $N_x(0) = 0$,

(ii) $N_x(t) = 1$, *for all* $t > 0$ *if and only if* $x = 0$,

(iii) $N_{\alpha x}(t) = N_x\left(\frac{t}{|\alpha|}\right)$, *for all* $\alpha \in \mathbb{R} \setminus \{0\}$,

(iv) $N_{x+y}(s + t) \geq N_x(s) * N_y(t)$, *for all* $x, y \in X$ *and* $s, t \in \mathbb{R}^+$.

Example 2.4. *Let $(X, ||.||)$ be a normed linear space. Let $a * b = \min\{a, b\}$, for all $a, b \in [0, 1]$ and $N_x(t) = \dfrac{t}{t + ||x||}$, $x \in X$ and $t \geq 0$. Then $(X, N, *)$ is a PN-space.*

Definition 2.5. *[8] For an integer $n \in \mathbb{Z}^+$, the generalized difference operator $\Delta^n x_i$ is defined as $\Delta^n x_i = \Delta^{n-1} x_i - \Delta^{n-1} x_{i+1}$, where $\Delta^0 x_i = x_i$ and $\Delta x_i = x_i - x_{i+1}$, for all $i \in \mathbb{N}$.*

Definition 2.6. *[6] A sequence $x = (x_i)$ in a PN-space $(X, N, *)$ is said to be Δ^n-convergent to $x_0 \in X$ in terms of the probabilistic norm N, if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer i_0 such that $N_{\Delta^n x_i - x_0}(\varepsilon) > 1 - \lambda$, whenever $i \geq i_0$. In this case, we write $N\text{-}\lim \Delta^n x = x_0$.*

Definition 2.7. *[6] A sequence $x = (x_i)$ in a PN-space $(X, N, *)$ is said to be Δ^n-Cauchy sequence in terms of the probabilistic norm N, if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer i_0 such that $N_{\Delta^n x_i - \Delta^n x_j}(\varepsilon) > 1 - \lambda$, for all $i, j \geq i_0$.*

Before proceeding further, we discuss some basic notions as defined follows.

Definition 2.8. *[16] Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent of multiple L provided that $\lim\limits_k \frac{x_k}{y_k} = L$. It is denoted by $x \sim y$ and it is said to be simply asymptotically equivalent if $L = 1$.*

Definition 2.9. *A number sequence $x = (x_k)$ is said to be statistically convergent to a number L provided that, for every $\varepsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0,$$

where $|\cdot|$ denotes the cardinality of the enclosed set. In this case, we write $\operatorname{stat-lim} x = L$.

Definition 2.10. *[19] Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistical equivalent of multiple L provided that for every $\varepsilon > 0$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \leq n : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0.$$

It is denoted by $x \overset{S^L}{\sim} y$ and it is said to be simply asymptotically statistical equivalent if $L = 1$.

Let S^L denote the set of all sequences $x = (x_k)$ and $y = (y_k)$ such that $x \overset{S^L}{\sim} y$.

Definition 2.11. *[12] An increasing sequence $\theta = \{k_r\}$, $r = 0, 1, 2, \ldots$ with $k_0 = 0$ of non-negative integers is said to be a lacunary sequence such that $h_r = k_r - k_{r-1} \to \infty$ whenever $r \to \infty$. The intervals governed by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r.*

Definition 2.12. *[19] Let θ be a lacunary sequence. Then two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple L provided that for every $\varepsilon > 0$, we have*

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \frac{x_k}{y_k} - L \right| \geq \varepsilon \right\} \right| = 0.$$

It is denoted by $x \overset{S_\theta^L}{\sim} y$ and it is said to be simply asymptotically lacunary statistical equivalent if $L = 1$.

Let S_θ^L denote the set of all sequences $x = (x_k)$ and $y = (y_k)$ such that $x \overset{S_\theta^L}{\sim} y$.

Definition 2.13. *[7] Let $(X, N, *)$ be a PN-space and θ be a lacunary sequence. Then two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistical equivalent of multiple L in terms of the probabilistic norm N provided that for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, we have*

$$\lim_r \frac{1}{h_r} \left| \left\{ k \in I_r : N_{\frac{x_k}{y_k} - L}(\varepsilon) \leq 1 - \lambda \right\} \right| = 0.$$

It is denoted by $x \overset{S_\theta^L(N)}{\sim} y$ and it is said to be simply asymptotically lacunary statistical equivalent if $L = 1$.

Let $S_\theta^L(N)$ denote the set of all sequences $x = (x_k)$ and $y = (y_k)$ such that $x \overset{S_\theta^L(N)}{\sim} y$.

3. Asymptotically Lacunary (Δ^n, μ)-Statistical Equivalent Sequences in PN-spaces

Throughout the paper, μ will denote a complete $\{0, 1\}$-valued finitely additive measure defined on a field Γ of all finite subsets of $\mathbb{N}$ and suppose that $\mu(A) = 0$, if $|A| < \infty$; if $A \subset B$ and $\mu(B) = 0$, then $\mu(A) = 0$; and $\mu(\mathbb{N}) = 1$.

Using the above notion of μ, we introduce the following definition in the theory of PN-space keeping in mind that the notion is going to be useful in establishing the main results of the present work.

Definition 3.1. *A sequence $x = (x_i)$ in a PN-space $(X, N, *)$ is said to be μ-statistically convergent to x_0 in terms of the probabilistic norm N, if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$,*

$$\mu(\{i \in \mathbb{N} : N_{x_i - x_0}(\varepsilon) \leq 1 - \lambda\}) = 0.$$

We denote it by $\mu\text{-}\operatorname{stat}_N\text{-}\lim x = x_0$.

Definition 3.2. *Let $(X, N, *)$ be a PN-space. Then two non-negative sequence $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically (Δ^n, μ)-statistical equivalent of multiple L in terms of the probabilistic norm N, if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$,*

$$\mu\left(\left\{ k \in \mathbb{N} : N_{\frac{\Delta^n x_k}{\Delta^n y_k} - L}(\varepsilon) \leq 1 - \lambda \right\} \right) = 0.$$

It is denoted by $x \overset{\mu\text{-}S_N^L(\Delta^n)}{\sim} y$ or written as $\mu\text{-}\operatorname{stat}_N\text{-}\lim \frac{\Delta^n x_k}{\Delta^n y_k} = L$ and it is said to be simply asymptotically (Δ^n, μ)-statistical equivalent if $L = 1$.

Let ${}^\mu S_N^L(\Delta^n)$ denote the set of all sequences $x = (x_k)$ and $y = (y_k)$ such that $x \overset{\mu\text{-}S_N^L(\Delta^n)}{\sim} y$.

In view of the Definition 3.2 and other properties of measure, we state the following lemma without proof.

Lemma 3.3. *Let $(X, N, *)$ be a PN-space. Then for every $\varepsilon > 0$ and $\lambda \in (0, 1)$, the following statements are equivalent:*

(i) $\mu\text{-}\operatorname{stat}_N\text{-}\lim \frac{\Delta^n x_k}{\Delta^n y_k} = L$,

(ii) $\mu\left(\left\{k \in \mathbb{N} : N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}\right) = 0,$

(iii) $\mu\left(\left\{k \in \mathbb{N} : N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) > 1-\lambda\right\}\right) = 1,$

(iv) $\mu\text{-}\operatorname{stat-lim} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) = 1.$

Theorem 3.4. *Let $(X, N, *)$ be a PN-space. If two sequences (x_k) and (y_k) in X are asymptotically (Δ^n, μ)-statistical equivalent of multiple L in terms of the probabilistic norm N, then L is unique.*

Proof: We assume that $x \overset{\mu\text{-}S_N^{L_1}(\Delta^n)}{\sim} y$ and $x \overset{\mu\text{-}S_N^{L_2}(\Delta^n)}{\sim} y$ with $L_1 \neq L_2$. Now for a given $\lambda > 0$, we choose $r \in (0,1)$ such that $(1-r) * (1-r) > 1-\lambda$. Then for any $\varepsilon > 0$, we define the following sets:

$$A_1 = \left\{k \in \mathbb{N} : N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L_1}(\varepsilon) \leq 1-r\right\}$$
$$A_2 = \left\{k \in \mathbb{N} : N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L_2}(\varepsilon) \leq 1-r\right\}.$$

Since $x \overset{\mu\text{-}S_N^{L_1}(\Delta^n)}{\sim} y$, so $\mu(A_1) = 0$ and as $x \overset{\mu\text{-}S_N^{L_2}(\Delta^n)}{\sim} y$, so $\mu(A_2) = 0$ for all $\varepsilon > 0$ and $r \in (0,1)$. Now, let $A_1 \cap A_2 = A$. Then we observe that $\mu(A) = 0$ which implies that $\mu(\mathbb{N} \setminus A) = 1$. If $k \in \mathbb{N} \setminus A$, then we have

$$\begin{aligned}
N_{L_1-L_2}(\varepsilon) &= N_{\left(L_1-\frac{\Delta^n x_k}{\Delta^n y_k}\right)+\left(\frac{\Delta^n x_k}{\Delta^n y_k}-L_2\right)}(\varepsilon/2+\varepsilon/2)\\
&\geq N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L_1}(\varepsilon/2) * N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L_2}(\varepsilon/2)\\
&> (1-r) * (1-r)\\
&> 1-\lambda.
\end{aligned}$$

Since $\lambda > 0$ is arbitrary, we get $N_{L_1-L_2}(\varepsilon) = 1$, for all $\varepsilon > 0$ which gives $L_1 = L_2$. Hence the proof. $\square$

Definition 3.5. *Let $(X, N, *)$ be a PN-space. Then two non-negative sequence $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically (Δ^n, μ)-strongly Cesáro equivalent of multiple L in terms of the probabilistic norm N, if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, we have*

$$\mu\left(\left\{m \in \mathbb{N} : \frac{1}{m}\sum_{k=1}^{m} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}\right) = 0.$$

It is denoted by $x \overset{\mu|\sigma_1|_N^L(\Delta^n)}{\sim} y$ and it is said to be simply asymptotically (Δ^n, μ)-strongly Cesáro equivalent if $L = 1$.

Let ${}_\mu|\sigma_1|_N^L(\Delta^n)$ denote the set of all sequences $x = (x_k)$ and $y = (y_k)$ such that $x \overset{\mu|\sigma_1|_N^L(\Delta^n)}{\sim} y$.

Theorem 3.6. *Let $x = (x_k)$ and $y = (y_k)$ be two non-negative sequences in a PN-space $(X, N, *)$. If x and y are asymptotically (Δ^n, μ)-statistical equivalent of multiple L in terms of the probabilistic norm N, then they are asymptotically (Δ^n, μ)-strongly Cesáro equivalent of multiple L in terms of the probabilistic norm N.*

Proof: Let $x = (x_k)$ and $y = (y_k)$ be two non-negative sequences such that $x \overset{\mu\text{-}S_N^L(\Delta^n)}{\sim} y$. Then for every $\varepsilon > 0$ and $\lambda \in (0,1)$, we have

$$\mu\left(\left\{k \in \mathbb{N} : N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}\right) = 0.$$

Given $\varepsilon > 0$, we have

$$N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq \frac{1}{m}\sum_{k=1}^{m} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon),$$

for all $m \in \mathbb{N}$. Thus for every $\varepsilon > 0$ and $\lambda \in (0,1)$, we have

$$\left\{m \in \mathbb{N} : \frac{1}{m}\sum_{k=1}^{m} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\} \subseteq \left\{k \in \mathbb{N} : N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}$$

and hence

$$\mu\left(\left\{m \in \mathbb{N} : \frac{1}{m}\sum_{k=1}^{m} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}\right) \leq \mu\left(\left\{k \in \mathbb{N} : N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}\right) = 0.$$

Consequently $x \overset{\mu|\sigma_1|^L_N(\Delta^n)}{\sim} y$. □

The converse of the above result is not true, which has been illustrated with the help of the following counterexample.

Example 3.7. *Let us consider $n = 1, L = \frac{1}{2}, \varepsilon = 0.1$ and $\lambda = 0.9$. We define $x = (x_k)$ as follows:*

$$x_k = \begin{cases} 1, & k \text{ is odd} \\ 2, & k \text{ is even,} \end{cases}$$

and $\Delta y_k = 1$, for all k. Then it can be easily proved that $x \overset{\mu|\sigma_1|^L_N(\Delta^n)}{\sim} y$ but $x \overset{\mu\text{-}S^L_N(\Delta^n)}{\nsim} y$.

Definition 3.8. *Let $(X, N, *)$ be a PN-space and $\theta = (k_r)$ be a lacunary sequence. Then two non-negative sequence $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary (Δ^n, μ)-statistical equivalent of multiple L in terms of the probabilistic norm N, if for every $\varepsilon > 0$ and $\lambda \in (0,1)$*

$$\mu\left(\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k \in I_r} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}\right) = 0.$$

It is denoted by $x \overset{\mu_\theta\text{-}S^L_N(\Delta^n)}{\sim} y$ and it is said to be simply asymptotically lacunary (Δ^n, μ)-statistical equivalent if $L = 1$.

Let ${}^{\mu}_{\theta}S^L_N(\Delta^n)$ denote the set of all sequences $x = (x_k)$ and $y = (y_k)$ such that $x \overset{\mu_\theta\text{-}S^L_N(\Delta^n)}{\sim} y$.

Theorem 3.9. *Let $x = (x_k)$ and $y = (y_k)$ be two non-negative sequences in a PN-space $(X, N, *)$ and $\theta = (k_r)$ be a lacunary sequence such that $\liminf_r q_r > 1$. Then $x \overset{\mu\text{-}S^L_N(\Delta^n)}{\sim} y$ implies $x \overset{\mu_\theta\text{-}S^L_N(\Delta^n)}{\sim} y$.*

Proof: Suppose that $\liminf_r q_r > 1$. Then there exists $\delta > 0$ such that $q_r \geq 1 + \delta$, for all $r \in \mathbb{N}$, which yields

$$\frac{h_r}{k_r} \geq \frac{\delta}{1+\delta} \quad \text{and} \quad \frac{k_{r-1}}{h_r} \leq \frac{1}{\delta}.$$

Now if $x \overset{\mu\text{-}S^L_N(\Delta^n)}{\sim} y$, then for every $\varepsilon > 0$ and $\lambda \in (0,1)$, we have

$$\mu\left(\left\{k \in \mathbb{N} : N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}\right) = 0.$$

From the Theorem 3.6, we have if $x \overset{\mu\text{-}S_N^L(\Delta^n)}{\sim} y$, then $x \overset{\mu|\sigma_1|_N^L(\Delta^n)}{\sim} y$. That is, for every $\varepsilon > 0$ and $\lambda \in (0,1)$, we have

$$\mu\left(\left\{m \in \mathbb{N} : \frac{1}{m}\sum_{k=1}^{m} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}\right) = 0.$$

Now for every $\varepsilon > 0, \lambda \in (0,1)$ and $r \in \mathbb{N}$, we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) &= \frac{1}{h_r}\sum_{k=1}^{k_r} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) - \frac{1}{h_r}\sum_{k=1}^{k_{r-1}} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\\ &= \frac{k_r}{h_r}\left[\frac{1}{k_r}\sum_{k=1}^{k_r} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right] - \frac{k_{r-1}}{h_r}\left[\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right]\end{aligned}$$

But the terms $\frac{1}{k_r}\sum_{k=1}^{k_r} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)$ and $\frac{1}{k_{r-1}}\sum_{k=1}^{k_{r-1}} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)$ both converges to 1. Hence

$$\frac{1}{h_r}\sum_{k\in I_r} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) = 1 > 1-\lambda.$$

Therefore,

$$\mu\left(\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}\right) = 0.$$

Consequently $x \overset{\mu_\theta\text{-}S_N^L(\Delta^n)}{\sim} y$. □

Theorem 3.10. *Let $x = (x_k)$ and $y = (y_k)$ be two non-negative sequences in a PN-space $(X, N, *)$ and $\theta = (k_r)$ be a lacunary sequence such that $\limsup_r q_r < \infty$. Then $x \overset{\mu_\theta\text{-}S_N^L(\Delta^n)}{\sim} y$ implies $x \overset{\mu\text{-}S_N^L(\Delta^n)}{\sim} y$.*

Proof: If $\limsup_r q_r < \infty$, then there exists $B > 0$ such that $q_r < B$, for all $r \geq 1$. Now if $x \overset{\mu_\theta\text{-}S_N^L(\Delta^n)}{\sim} y$, then for every $\varepsilon > 0$ and $\lambda \in (0,1)$, we have

$$\mu\left(\left\{r \in \mathbb{N} : \frac{1}{h_r}\sum_{k\in I_r} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}\right) = 0.$$

We set

$$A = \left\{j \in \mathbb{N} : \frac{1}{h_j}\sum_{k\in I_j} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}.$$

Then we have $\mu(A) = 0$. Now let m be an integer with $k_{r-1} < m < k_r$ where $r > R, R > 0$. Then

$$\begin{aligned}\frac{1}{m}\left\{k \le m : N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \le 1-\lambda\right\} &\subseteq \frac{1}{k_{r-1}}\left\{k \le k_r : N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \le 1-\lambda\right\}\\ &\subseteq \frac{1}{k_{r-1}}\left\{k \le I_1 : \frac{1}{h_1}\sum_{k\in I_1} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \le 1-\lambda\right\}\\ &\cup \frac{1}{k_{r-1}}\left\{k \le I_2 : \frac{1}{h_2}\sum_{k\in I_2} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \le 1-\lambda\right\}\\ &\vdots\\ &\cup \frac{1}{k_{r-1}}\left\{k \le I_r : \frac{1}{h_r}\sum_{k\in I_r} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \le 1-\lambda\right\}\end{aligned}$$

Hence we have

$$\mu\left(\left\{k \in \mathbb{N} : N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \le 1-\lambda\right\}\right) = 0,$$

since $\mu(A) = 0$. Consequently $x \overset{\mu\text{-}S_N^L(\Delta^n)}{\sim} y$. □

Corollary 3.11. *Let $(X, N, *)$ be a PN-space and $\theta = (k_r)$ be a lacunary sequence. If $1 < \liminf_r q_r < \limsup_r q_r < \infty$, then $x \overset{\mu\text{-}S_N^L(\Delta^n)}{\sim} y$ if and only if $x \overset{\mu_\theta\text{-}S_N^L(\Delta^n)}{\sim} y$.*

4. Asymptotically (Δ^n, μ)-Strongly Cesáro Orlicz Equivalent Sequences in PN-spaces

In this segment, we introduce the notion of asymptotically (Δ^n, μ)-strongly Cesáro Orlicz equivalent sequences in the theory of PN-spaces and investigate few inclusion properties.

Let Q_t be the set of all subsets of $\mathbb{N}$, which does not contain more than t elements. Suppose that (ϕ_t) is a non decreasing sequence of positive real numbers with the property that $\phi_t \to \infty$ as $t \to \infty$ and $\phi_t \le t$ for every $t \in \mathbb{N}$. Furthermore, a function $M : (0, \infty] \to (0, \infty]$ that is non decreasing, continuous and convex with the the property that $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. Since the Orlicz function M is a convex function with $M(0) = 0$, hence it satisfies the property $M(\lambda y) \le \lambda M(y)$, for all $\lambda \in (0, 1]$.

Definition 4.1. *Let $(X, N, *)$ be a PN-space. Then two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically (Δ^n, μ)-strongly Cesáro Orlicz equivalent of multiple L in terms of the probabilistic norm N, if for every $\lambda \in (0, 1)$ and $\varepsilon > 0$, we have*

$$\mu\left(\left\{m \in \mathbb{N} : \frac{1}{m}\sum_{k=1}^{m} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) \le 1-\lambda\right\}\right) = 0,$$

denoted by $x \overset{\mu|\sigma_1|_N^L(M\text{-}\Delta^n)}{\sim} y$ and simply asymptotically (Δ^n, μ)-strongly Cesáro Orlicz equivalent if $L = 1$.

Definition 4.2. *Let $(X, N, *)$ be a PN-space. Then two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically (Δ^n, μ, ϕ_t)-strongly Cesáro Orlicz equivalent of multiple L in terms of the probabilistic norm N, if for every $\lambda \in (0, 1)$ and $\varepsilon > 0$, we have*

$$\mu\left(\left\{t \in \mathbb{N} : \frac{1}{\phi_t}\sum_{k\in\sigma, \sigma\in Q_t} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) \le 1-\lambda\right\}\right) = 0,$$

denoted by $x \overset{\mu_{\phi_t}\text{-}|\sigma_1|_N^L(M\text{-}\Delta^n)}{\sim} y$ and simply asymptotically (Δ^n, μ, ϕ_t)-strongly Cesáro Orlicz equivalent if $L = 1$.

Definition 4.3. *Let $(X, N, *)$ be a PN-space. Then two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be $x \overset{\mu_\phi\text{-}S_N^L(\Delta^n)}{\sim} y$, if for every $\lambda \in (0,1)$ and $\varepsilon > 0$, we have*

$$\mu\left(\left\{t \in \mathbb{N} : \frac{1}{\phi_t}\sum_{k\in\sigma,\sigma\in Q_t} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}\right) = 0.$$

Theorem 4.4. *For two non-negative sequences $x = (x_k)$ and $y = (y_k)$ in a PN-space $(X, N, *)$, $x \overset{\mu\text{-}S_N^L(\Delta^n)}{\sim} y$ implies $x \overset{\mu_\phi\text{-}S_N^L(\Delta^n)}{\sim} y$.*

Proof: Let $x = (x_k)$ and $y = (y_k)$ be two non-negative sequences in a PN-space $(X, N, *)$ such that $x \overset{\mu\text{-}S_N^L(\Delta^n)}{\sim} y$. Then for every $\lambda \in (0,1)$ and $\varepsilon > 0$, we have

$$\mu\left(\left\{k \in \mathbb{N} : N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}\right) = 0.$$

Given $\varepsilon > 0$, we have

$$N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq \frac{1}{\phi_t}\sum_{k\in\sigma,\sigma\in Q_t} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon),$$

for all $t \in \mathbb{N}$. Then for every $\lambda \in (0,1)$ and $\varepsilon > 0$, we have

$$\left\{t \in \mathbb{N} : \frac{1}{\phi_t}\sum_{k\in\sigma,\sigma\in Q_t} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\} \subseteq \left\{k \in \mathbb{N} : N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}$$

and hence

$$\mu\left(\left\{t \in \mathbb{N} : \frac{1}{\phi_t}\sum_{k\in\sigma,\sigma\in Q_t} N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}\right)$$
$$\leq \mu\left(\left\{k \in \mathbb{N} : N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon) \leq 1-\lambda\right\}\right) = 0.$$

Consequently, $x \overset{\mu_\phi\text{-}S_N^L(\Delta^n)}{\sim} y$. □

Theorem 4.5. *Let M be an Orlicz function. Then the following results hold:*

(i) $x \overset{\mu|\sigma_1|_N^L(M\text{-}\Delta^n)}{\sim} y$ *implies* $x \overset{\mu_{\phi_t}\text{-}|\sigma_1|_N^L(M\text{-}\Delta^n)}{\sim} y$.

(ii) If $\liminf_t \frac{\phi_t}{\phi_{t-1}} = 1$, *then* $x \overset{\mu_{\phi_t}\text{-}|\sigma_1|_N^L(M\text{-}\Delta^n)}{\sim} y$ *implies* $x \overset{\mu|\sigma_1|_N^L(M\text{-}\Delta^n)}{\sim} y$.

Proof: (i) From the definition of ϕ_t, it pursues that $\inf_t \frac{t}{t-\phi_t} \geq 1$. Then there exists $\delta > 0$ such that $\frac{t}{\phi_t} \leq \frac{1+\delta}{\delta}$. Let $x \overset{\mu|\sigma_1|_N^L(M\text{-}\Delta^n)}{\sim} y$. Then for every $\lambda \in (0,1)$ and $\varepsilon > 0$, we have

$$\mu\left(\left\{m \in \mathbb{N} : \frac{1}{m}\sum_{k=1}^{m} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) \leq 1-\lambda\right\}\right) = 0.$$

By the Lemma 3.3, we have

$$\mu\text{-}\operatorname{stat-lim}\frac{1}{m}\sum_{k=1}^{m} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) = 1.$$

Now for every $\lambda \in (0,1)$ and $\varepsilon > 0$, we have

$$\begin{aligned}
\frac{1}{\phi_t}\sum_{k\in\sigma,\sigma\in Q_t} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) - 1 &= \frac{1}{\phi_t}\sum_{k=1}^{t} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) \\
&\quad - \frac{1}{\phi_t}\sum_{k\in\{1,2,\ldots,t\}\setminus\sigma,\sigma\in Q_t} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) - 1 \\
&= \frac{t}{\phi_t}\left[\frac{1}{t}\sum_{k=1}^{t} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) - 1\right] + \frac{t}{\phi_t} \\
&\quad - \frac{1}{\phi_t}\sum_{k\in\{1,2,\ldots,t\}\setminus\sigma,\sigma\in Q_t} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) - 1 \\
&\leq \frac{t}{\phi_t}\left[\frac{1}{t}\sum_{k=1}^{t} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) - 1\right] + \frac{1}{\delta} \\
&\quad - \frac{1}{\phi_t} M\left(N_{\frac{\Delta^n x_{k_0}}{\Delta^n y_{k_0}}-L}(\varepsilon)\right),
\end{aligned}$$

where $k_0 \in \{1,2,\ldots,t\}\setminus\sigma$. Since μ-stat-lim $\frac{1}{t}\sum\limits_{k=1}^{t} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) = 1$ and M is continuous, so letting $t \to \infty$ on the last relation, we have

$$\frac{1}{\phi_t}\sum_{k\in\sigma,\sigma\in Q_t} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) - 1 < \frac{1}{\delta} = \gamma, \text{ (say } \delta = \tfrac{1}{\gamma}\text{, for } \gamma > 0).$$

Hence, we have

$$\mu\text{-}\operatorname{stat-lim} \frac{1}{\phi_t}\sum_{k\in\sigma,\sigma\in Q_t} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) = 1.$$

Consequently, $x \overset{\mu_{\phi_t}\text{-}|\sigma_1|_N^L(M\text{-}\Delta^n)}{\sim} y$.

(ii) Let $\liminf\limits_{t} \frac{\phi_t}{\phi_{t-1}} = 1$. Suppose that $x \overset{\mu_{\phi_t}\text{-}|\sigma_1|_N^L(M\text{-}\Delta^n)}{\sim} y$. Then for every $\lambda \in (0,1)$ and $\varepsilon > 0$, such that

$$\mu\left(\left\{t \in \mathbb{N} : \frac{1}{\phi_t}\sum_{k\in\sigma,\sigma\in Q_t} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) \leq 1-\lambda\right\}\right) = 0.$$

Then for $\varepsilon > 0$, we have

$$H_t = \frac{1}{\phi_t}\sum_{k\in\sigma,\sigma\in Q_t} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) \to 1.$$

Then for $\delta > 0$, there exists $t_0 \in \mathbb{N}$ such that $H_t < 1+\varepsilon$, for all $t \geq t_0$. Also we can find $T > 0$ such that

$H_t < T,\ t = 1, 2, \ldots$. Let m be an integer with $\phi_{t-1} < m \le [\phi_t]$. Then

$$\begin{aligned}
\frac{1}{m}\sum_{k=1}^{m} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right) &\le \frac{1}{\phi_{t-1}}\sum_{k=1}^{[\phi_t]} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right)\\
&= \frac{1}{\phi_{t-1}}\left[\sum_{k=1}^{[\phi_1]} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right)\right.\\
&\left.+\cdots+\sum_{k=[\phi_{t-1}]}^{[\phi_t]} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right)\right]\\
&\le \frac{\phi_1}{\phi_{t-1}}\cdot\frac{1}{\phi_1}\sum_{k\in\sigma,\sigma\in Q_1} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right)\\
&+\cdots+\frac{\phi_t}{\phi_{t-1}}\cdot\frac{1}{\phi_t}\sum_{k\in\sigma,\sigma\in Q_t} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right)\\
&\le \sup_{1\le t\le t_0} H_t\frac{\phi_{t_0}}{\phi_{t-1}} + \frac{\phi_{t_0+1}}{\phi_{t-1}}H_{t_0+1}+\cdots+\frac{\phi_t}{\phi_{t-1}}H_t\\
&< T\frac{\phi_{t_0}}{\phi_{t-1}} + (1+\varepsilon)\frac{\phi_{t_0+1}+\phi_{t_0+2}+\cdots+\phi_t}{\phi_{t-1}}.
\end{aligned}$$

Since $\phi_{t-1}\to\infty$ as $m\to\infty$, it follows that

$$\mu\text{-}\operatorname{stat}\text{-}\lim \frac{1}{m}\sum_{k=1}^{m} M\left(N_{\frac{\Delta^n x_k}{\Delta^n y_k}-L}(\varepsilon)\right)\to 1.$$

Consequently, $x \overset{\mu|\sigma_1|_N^L(M\text{-}\Delta^n)}{\sim} y$. $\square$

5. Conclusion

In this article, we have introduced asymptotically (Δ^n,μ)-statistical equivalent sequences in PN-spaces and discussed some of their properties. We have further introduced the concepts of asymptotically lacunary (Δ^n,μ)-statistical equivalent sequences and asymptotically (Δ^n,μ)-strongly Cesáro equivalent sequences in the theory of PN-spaces. Moreover, we have introduced the concept of asymptotically (Δ^n,μ)-strongly Cesáro Orlicz equivalent sequences and obtained some inclusion relations as well as some equivalent conditions in this new settings.

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