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Radial Positive Solutions for (p(x),q(x))-Laplacian Systems

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abstract: In this paper, we study the existence of radial positive solutions for nonvariational elliptic systems involving the $p(x)-$Laplacian operator, we show the existence of solutions using Leray-Schauder topological degree theory, sustained by Gidas-Spruck Blow-up technique.

Key Words: p(x)-Laplacian operator, elliptic systems, blow up technique, Leray-Schauder topological degree.

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1. Introduction

In this paper we study the existence of positive radial solutions of asymptotically homogeneous systems involving $p(x)-$Laplacian operator defined in $\mathbb{R}^N$, of the form

$$\begin{aligned} -\Delta_{p(x)}u &= a_{11}(|x|)f_{11}(u) + a_{12}(|x|)f_{12}(v) \qquad \text{in } \mathbb{R}^N, \\ -\Delta_{q(x)}v &= a_{21}(|x|)f_{21}(u) + a_{22}(|x|)f_{22}(v) \qquad \text{in } \mathbb{R}^N. \end{aligned} \tag{1.1}$$

Here $\Delta_{p(x)}$ is the so-called $p(x)-$Laplacian operator; namely $:= \Delta_{p(x)}u = div(|\nabla u|^{p(x)-2}\nabla u)$, with p and q are continuous real-valued functions such that $1 < p(x), q(x) < N$ $(N \geq 2)$ for all $x \in \mathbb{R}^N$. The coefficients $a_{ij}, i, j = 1, 2$, are positive continuous real-valued functions. The non linearities $f_{ij}, i, j = 1, 2$, belong to asymptotically homogeneous class of functions.

In recent years, several authors have used different methods to solve equations or quasi-linear systems defined in bounded or unbounded domains. Usually, we use critical points theory to show existence of weak solutions. There is a lot of work on this subject (see [8], [11], and therein..). This variational approach is used in particular to deal with systems derived from a potential, that is, the nonlinearities on the right-hand side correspond to the gradient of certain functional. Several articles were written about the homogeneous $p-$Laplacian operator. The reader can easily refer to the following list of work [6], [9], [10], [12], To examine system (1.1), we first exhibit a priori estimates using Gidas-Spruck "Blow-up" technique (see [4]). The main tool stay Leray-Schauder topological degree to establish the existence of fundamental states. This contribution is an extension to the work Djellit and Tas [7]. These authers consider the systems of the form

$$\begin{aligned} -\Delta_p u &= \lambda f(x, u, v) \qquad \text{in } \mathbb{R}^N, \\ -\Delta_q v &= \mu g(x, u, v) \qquad \text{in } \mathbb{R}^N. \end{aligned} \tag{1.2}$$

Where the nonlinearities f and g, satisfy polynomial growth conditions. Existence results are proved using fixed point theorems

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2. Preliminaries

First, we introduce definitions and notation utilized in this note. Let the Banach space

$$X=\left\{(u,v)\in C^0([0,+\infty[)\times C^0([0,+\infty[),\ \lim_{r\to+\infty}u(r)=\lim_{r\to+\infty}v(r)=0\right\}$$

be equipped with the norm

$$\|(u,v)\|_X=\|u\|_\infty+\|v\|_\infty,\qquad \|u\|_\infty=\sup_{r\in[0,+\infty[}|u(r)|.$$

Let $K=\{(u,v)\in X,\ u\geq 0,v\geq 0\}$ a positive cone of X. For $h\geq 0$ and $\lambda\in[0,1]$, we define two families of operators T_h and S_λ form X to itself by $T_h(u,v)=(w,z)$ such that (w,z) satisfies the system

$$\begin{aligned}
-\left(r^{N-1}|w'(r)|^{p(r)-2}w'(r)\right)' &= r^{N-1}a_{11}(r)f_{11}(|u(r)|)+r^{N-1}a_{12}(r)\left[f_{12}(|v(r)|)+h\right]\\
&\text{in}\quad [0,+\infty[,\\
-\left(r^{N-1}|z'(r)|^{q(r)-2}z'(r)\right)' &= r^{N-1}a_{21}(r)f_{21}(|u(r)|)+r^{N-1}a_{22}(r)f_{22}(|v(r)|)\\
&\text{in}\quad [0,+\infty[,\\
w'(0)=z'(0)=0, &\qquad \lim_{r\to+\infty}w(r)=\lim_{r\to+\infty}z(r)=0,
\end{aligned}\tag{2.1}$$

and $S_\lambda(u,v)=(w,z)$ such that (w,z) satisfies the system

$$\begin{aligned}
-\left(r^{N-1}|w'(r)|^{p(r)-2}w'(r)\right)' &= \lambda r^{N-1}a_{11}(r)f_{11}(|u(r)|)+\lambda r^{N-1}a_{12}(r)f_{12}(|v(r)|)\\
&\text{in}\quad [0,+\infty[,\\
-\left(r^{N-1}|z'(r)|^{q(r)-2}z'(r)\right)' &= \lambda r^{N-1}a_{21}(r)f_{21}(|u(r)|)+\lambda r^{N-1}a_{22}(r)f_{22}(|v(r)|)\\
&\text{in}\quad [0,+\infty[,\\
w'(0)=z'(0)=0, &\qquad \lim_{r\to+\infty}w(r)=\lim_{r\to+\infty}z(r)=0.
\end{aligned}\tag{2.2}$$

Let us recall the notion of "asymptotically homogeneous" functions and some of their properties. A function $\varphi:\mathbb{R}\to\mathbb{R}$ defined in a neighborhood at the infinity (respect. at the origin) is said asymptotically homogeneous at the infinity (respect. at the origin) of order $\rho>0$ if for all $\sigma>0$, we have $\lim_{s\to+\infty}\dfrac{\varphi(\sigma s)}{\varphi(s)}=\sigma^\rho$ (respect. $\lim_{s\to 0}\dfrac{\varphi(\sigma s)}{\varphi(s)}=\sigma^\rho$).

As an example, we have the function $\varphi(s)=|s|^{\alpha-2}s(\ln(1+|s|))^\beta$ with $\alpha>1$ and $\beta>1-\alpha$, It is asymptotically homogeneous at infinity of order $\alpha-1$ and at the origin of order $\alpha+\beta-1$.

Proposition 2.1. *[1] Let $\varphi:\mathbb{R}\to\mathbb{R}$ be a continuous, odd, asymptotically homogeneous at infinity (respect. at the origin) of order ρ such that $t\varphi(t)>0$ for all $t\neq 0$ and $\varphi(t)\to\infty$ as $t\to\infty$, then*

(i) For all $\varepsilon\in\,]0,\rho[$, there exists $t_0>0$ such that $\forall t\geq t_0$(respect. $0\leq t\leq t_0$), $c_1t^{\rho-\varepsilon}\leq\varphi(t)\leq c_2t^{\rho+\varepsilon}$; c_1,c_2 are positive constants. Moreover $\forall s\in[t_0,t]:(\rho+1-\varepsilon)\varphi(s)\leq(\rho+1+\varepsilon)\varphi(t)$.

(ii) If $(w_n),(t_n)$ are real sequences such that $w_n\to w$ and $t_n\to+\infty$ (respect. $t_n\to 0$) then $\lim_{n\to+\infty}\dfrac{\varphi(t_nw_n)}{\varphi(t_n)}=w^\rho$.

We assume that both the coefficients a_{ij} and the functions f_{ij} verify smooth conditions; explicitly:

(H1) For all $i, j = 1, 2,\ k = \pm$, the coefficient $a_{ij} : [0, +\infty[\to [0, +\infty[$ is continuous and satisfies $\exists\ \theta_{11}, \theta_{12} > p^k;\ \exists\ \theta_{21}, \theta_{22} > q^k$; there exists $R > 0$ such that $a_{ij}(\xi) = O(\xi^{-\theta_{ij}})$ for all $\xi > R$ and $\widetilde{a}_i = \min_{r\in[0,R]} a_{ij}(r) > 0; \quad i, j = 1, 2;\ i \neq j.$

(H2) For all $i, j = 1, 2$, the function $f_{ij} : \mathbb{R} \to \mathbb{R}$ is continuous, odd such that $sf_{ij}(s) > 0$ for all $s \neq 0$ and $\lim_{s\to+\infty} f_{ij}(s) = +\infty$.

(H3) For all $i, j = 1, 2$ and $k = \pm$, f_{ij} is asymptotically homogeneous at the infinity of order δ_{ij} satisfying $\dfrac{\delta_{12}\delta_{21}}{(p^k-1)(q^k-1)} > 1,\ \alpha_1\delta_{11} - \alpha_1(p^k-1) - p^k < 0,\ \alpha_2\delta_{22} - \alpha_2(q^k-1) - q^k < 0$ and $\max(\beta_1, \beta_2) \geq 0$ where $\alpha_1 = \dfrac{p^k(q^k-1) + \delta_{12}q^k}{\delta_{12}\delta_{21} - (p^k-1)(q^k-1)}, \alpha_2 = \dfrac{q^k(p^k-1) + \delta_{21}p^k}{\delta_{12}\delta_{21} - (p^k-1)(q^k-1)},\ \beta_1 = \alpha_1 - \dfrac{N-p^k}{p^k-1},\ \beta_2 = \alpha_2 - \dfrac{N-q^k}{q^k-1}$.

(H4) For all $i, j = 1, 2,\ k = \pm,\ f_{ij}$ is asymptotically homogeneous at the origin of order $\overline{\delta}_{ij}$ with $\overline{\delta}_{11}, \overline{\delta}_{12} > p^k - 1, \overline{\delta}_{21}, \overline{\delta}_{22} > q^k - 1$.

A nontrivial positive radial solution (u, v) to system $(T_0) \equiv (S_1)$ is also a solution to the following differential system:

$$\begin{aligned} -\left(r^{N-1}|u'(r)|^{p(r)-2}u'(r)\right)' &= r^{N-1}a_{11}(r)f_{11}(|u(r)|) + r^{N-1}a_{12}(r)f_{12}(|v(r)|) \\ &\text{in} \quad [0, +\infty[, \\ -\left(r^{N-1}|v'(r)|^{q(r)-2}v'(r)\right)' &= r^{N-1}a_{21}(r)f_{21}(|u(r)|) + r^{N-1}a_{22}(r)f_{22}(|v(r)|) \\ &\text{in} \quad [0, +\infty[, \\ u'(0) = v'(0) = 0, &\qquad \lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} v(r) = 0. \end{aligned} \tag{2.3}$$

To this end, we define the operator $L : K \to K$ by $L(u, v) = (w, z)$ such that

$$\begin{aligned} w(r) &= \int_r^{+\infty} \left(\eta^{1-N} \int_0^\eta \xi^{N-1}(a_{11}(\xi)f_{11}(u(\xi)) + a_{12}(\xi)f_{12}(v(\xi)))d\xi\right)^{\frac{1}{p(\eta)-1}} d\eta, \\ z(r) &= \int_r^{+\infty} (\eta^{1-N} \int_0^\eta \xi^{N-1}(a_{21}(\xi)f_{21}(u(\xi)) + a_{22}(\xi)f_{22}(v(\xi)))d\xi)^{\frac{1}{q(\eta)-1}} d\eta. \end{aligned}$$

3. Existence of solutions

To show the existence result, it is necessary to state some lemmas.

Lemma 3.1. *Under hypothesis* $(H1)$, *we have*

$\int_0^{+\infty} \left(\eta^{1-N} \int_0^\eta \xi^{N-1}a_{ij}(\xi)d\xi\right)^{\frac{1}{p(\eta)-1}} d\eta$
$\leq \int_0^{+\infty} \left(\eta^{1-N} \int_0^\eta \xi^{N-1}a_{ij}(\xi)d\xi\right)^{\frac{1}{p^k-1}} d\eta < +\infty$ *for* $i = 1,\ j = 1, 2$ *and* $k = \pm$.

$$\int_0^{+\infty}\left(\eta^{1-N}\int_0^\eta \xi^{N-1}a_{ij}(\xi)d\xi\right)^{\frac{1}{q(\eta)-1}}d\eta$$
$$\leq \int_0^{+\infty}\left(\eta^{1-N}\int_0^\eta \xi^{N-1}a_{ij}(\xi)d\xi\right)^{\frac{1}{q^k-1}}d\eta<+\infty \qquad \textit{for } i=2,\ j=1,2 \textit{ and } k=\pm.$$

Proof.
$$\int_0^{+\infty}\left(\eta^{1-N}\int_0^\eta \xi^{N-1}a_{ij}(\xi)d\xi\right)^{\frac{1}{p(\eta)-1}}d\eta$$
$$\leq \int_0^{+\infty}\left(\eta^{1-N}\int_0^\eta \xi^{N-1}a_{ij}(\xi)d\xi\right)^{\frac{1}{p^k-1}}d\eta$$
$$=\int_0^{R}\left(\eta^{1-N}\int_0^\eta \xi^{N-1}a_{ij}(\xi)d\xi\right)^{\frac{1}{p^k-1}}d\eta+\int_R^{+\infty}\left(\eta^{1-N}\int_0^\eta \xi^{N-1}a_{ij}(\xi)d\xi\right)^{\frac{1}{p^k-1}}d\eta.$$

The first integral in the right-hand side is finite since a_{ij} is continuous. The second one is also finite. Indeed, by virtue of *(H1)*, we have
$$\int_R^{+\infty}\left(\eta^{1-N}\int_0^\eta \xi^{N-1}a_{ij}(\xi)d\xi\right)^{\frac{1}{p^k-1}}d\eta$$
$$\leq \int_R^{+\infty}\left(\eta^{1-N}\int_0^\eta \xi^{N-1}c_{ij}(\xi)\xi^{-\theta_{ij}}d\xi\right)^{\frac{1}{p^k-1}}d\eta\leq c_{ij}R^{\frac{p^k-\theta_{ij}}{p^k-1}}$$

for $i=1,\ j=1,2$ and $k=\pm$.

This last term vanishes for sufficiently large R. Similary, we get the same achievement for $i=2,\ j=1,\ 2$ and $k=\pm$. □

Lemma 3.2. *If $u\in C^1([0,+\infty[)\cap C^2([0,+\infty[)$ is a nontrivial positive radial solution of the problem*
$$-\left(r^{N-1}|u'(r)|^{p(r)-2}u'(r)\right)'\geq 0 \qquad \textit{in } [0,+\infty[$$
such that $u(0)>0$ and $u'(0)\leq 0$, then
$$u(r)>0 \textit{ and } u'(r)\leq 0 \textit{ for all } r>0.$$

Proof. Let u be a nontrivial positive radial solution of the problem
$$-\left(r^{N-1}|u'(r)|^{p(r)-2}u'(r)\right)'\geq 0 \qquad \text{in } [0,+\infty[.$$

Suppose that $0<s<r$. Integrating from s to r, we obtain
$$r^{N-1}|u'(r)|^{p(r)-2}u'(r)\leq s^{N-1}|u'(s)|^{p(s)-2}u'(s).$$

Letting $s\to 0$, we get $u'(r)\leq 0$.
If $u'(r)=0$ then $u'(s)=0$ for all $0\leq s\leq r$. This means that u is either constant in $[0,+\infty[$ or there exists $r_0\geq 0$ such that $u'(r)<0$ for $r>r_0$ and $u'(r)=0$, $u(r)=u(0)$ for $0\leq r\leq r_0$. So u is non increasing and $u(0)>0$. □

Lemma 3.3. *Let $u\in C^1([0,+\infty[)\cap C^2([0,+\infty[)$ be a positive solution of the problem*
$$-\left(r^{N-1}|u'(r)|^{p-2}u'(r)\right)'\geq 0 \qquad \textit{in } [0,+\infty[$$
such that $u(0)>0$ and $u'(0)\leq 0$, then
The function M_p defined by $M_p(r)=ru'(r)+\frac{N-p}{p-1}u(r)$, $r\geq 0$, is nonnegative and nonincreasing. In particular, the function $r\to r^{\frac{N-p}{p-1}}u(r)$ is nondecreasing in $[0,+\infty[$.

Proof. Since u is a positive solution of the problem
$$-\left(r^{N-1}|u'(r)|^{p-2}u'(r)\right)'\geq 0 \qquad \text{in } [0,+\infty[\,.$$

we have $-r^{N-1}(p-1)|u'(r)|^{p-2}u''(r)-(N-1)r^{N-2}|u'(r)|^{p-2}u'(r)\geq 0$. In other words $ru''(r)+\frac{N-1}{p-1}u'(r)\leq 0$, or $(ru'(r))'+\frac{N-p}{p-1}u'(r)\leq 0$. This yields that M_p is nonincreasing. To show that $M_p(r)\geq 0$ for all $r\geq 0$, we use a contradiction argument. Indeed, assume that there exists $r_1>0$ such that $M_p(r_1)<0$. Since M_p is nonincreasing, for all $r>r_1, M_p(r)\leq M_p(r_1)$ or $u'(r)+\frac{N-p}{p-1}\frac{u(r)}{r}\leq\frac{M_p(r_1)}{r}$.

On the other hand $u(r)>0, \frac{N-p}{p-1}>0$, hence $u'(r)\leq\frac{M_p(r_1)}{r}$. Consequently, $u(r)-u(r_1)\leq M_p(r_1)\ln(\frac{r}{r_1})$, $r>r_1$. It follows immediately that $\lim\limits_{r\to+\infty}u(r)=-\infty$. This contradicts u begin positive. In particular

$$\frac{M_p(r)}{ru(r)}\geq 0 \qquad \forall r>0.$$

Finally, we obtain $\dfrac{u'(r)}{u(r)}+\dfrac{N-p}{p-1}\dfrac{1}{r}\geq 0$. In other words,

$$\left(\ln r^{\frac{N-p}{p-1}}u(r)\right)'\geq 0.$$

This implies that the function $r\to r^{\frac{N-p}{p-1}}u(r)$ is nondecreasing. □

The study of the function M_p is essential and help us to estimate $u(r)$

Lemma 3.4. *If (H1) is satisfied, then the operator L is compact.*

Proof. L is well defined. Indeed

$$\begin{aligned}w(r)&\leq c_{11}\int_r^{+\infty}\left(\eta^{1-N}\int_0^\eta \xi^{N-1}a_{11}(\xi)(u(\xi))^{\delta_{11}+\varepsilon}d\xi\right)^{\frac{1}{p^k-1}}d\eta\\&+c_{12}\int_r^{+\infty}\left(\eta^{1-N}\int_0^\eta \xi^{N-1}a_{12}(\xi)(u(\xi))^{\delta_{12}+\varepsilon}d\xi\right)^{\frac{1}{p^k-1}}d\eta\\&\leq c_{11}c_1\left(\|u\|_\infty\right)^{\frac{\delta_{11}+\varepsilon}{p^k-1}}+c_{12}c_2\left(\|v\|_\infty\right)^{\frac{\delta_{12}+\varepsilon}{p^k-1}}<+\infty.\end{aligned}$$

By Lemma 3.1,

$$c_j=\int_r^{+\infty}\left(\eta^{1-N}\int_0^\eta \xi^{N-1}a_{ij}(\xi)d\xi\right)^{\frac{1}{p^k-1}}d\eta<+\infty,$$

for $i=1, j=1,2$ and $k=\pm$
Similarly,

$$z(r)\leq c_{21}b_1\left(\|u\|_\infty\right)^{\frac{\delta_{21}+\varepsilon}{q^k-1}}+c_{22}b_2\left(\|v\|_\infty\right)^{\frac{\delta_{22}+\varepsilon}{q^k-1}}$$

$b_j=\int_r^{+\infty}(\eta^{1-N}\int_0^\eta \xi^{N-1}a_{ij}(\xi)d\xi)^{\frac{1}{q^k-1}}d\eta<+\infty$ for $i=2, j=1,2$ and $k=\pm$.
Obviously, $\sup\limits_{r\in[0,+\infty[}|w(r)|<+\infty$ and $\sup\limits_{r\in[0,+\infty[}|z(r)|<+\infty$.

Morever, we have $w\geq 0$, $z\geq 0$ and $\lim\limits_{r\to+\infty}w(r)=\lim\limits_{r\to+\infty}z(r)=0$.

Now, we show that L is compact. Indeed, let (u_n,v_n) be a bounded sequence of X. From the relation

$$L(u_n,v_n)=(w_n,z_n),$$

we can write

$$\begin{aligned} -\left(r^{N-1}|w_n'(r)|^{p(r)-2}w_n'(r)\right)' &= r^{N-1}a_{11}(r)f_{11}(u_n(r)) + r^{N-1}a_{12}(r)f_{12}(v_n(r)) \\ &\text{in} \quad [0,+\infty[\,, \\ -\left(r^{N-1}|z_n'(r)|^{q(r)-2}z_n'(r)\right)' &= r^{N-1}a_{21}(r)f_{21}(u_n(r)) + r^{N-1}a_{22}(r)f_{22}(v_n(r)) \\ &\text{in} \quad [0,+\infty[\,, \\ w'(0) = z'(0) = 0, &\qquad \lim_{r\to+\infty} w(r) = \lim_{r\to+\infty} z(r) = 0, \end{aligned} \tag{3.1}$$

For fixed $R > 0$, let $r \in [0, R]$ and put $\varphi(t) = |t|^{p(r)-1}$.
From the first equation of the above system, we obtain

$$\tfrac{d}{dr}\varphi(w_n'(r)) + \frac{N-1}{r}|w_n'(r)|^{p(r)-1} - a_{11}(r)f_{11}(u_n(r)) - a_{12}(r)f_{12}(v_n(r)) = 0,$$

Therefore

$$\tfrac{d}{dr}\varphi(w_n'(r)) - a_{11}(r)f_{11}(u_n(r)) - a_{12}(r)f_{12}(v_n(r)) \le 0,$$

in view of the part (i) of Proposition 2.1, we have

$$\tfrac{d}{dr}\varphi(w_n'(r)) \le a_{11}(r)(u_n(r))^{\delta_{11}+\varepsilon} + a_{12}(r)(v_n(r))^{\delta_{12}+\varepsilon}.$$

Since u_n and v_n are bounded, we get

$$\frac{d}{dr}\varphi(w_n'(r)) \le c_1 a_{11}(r) + c_2 a_{12}(r).$$

Integrating from 0 to R both last inqualities, we obtain

$$\varphi(w_n'(R)) \le c,$$

or

$$|(w_n'(R))|^{p(R)-1} \le c. \tag{3.2}$$

This means that at finite distance, w_n' is bounded.
In the same way, substuting q to p, we show that again $z_n'(r)$ is bounded on $[0, +\infty[$.

This yields $|w_n'(r)| \le c;\ |z_n'(r)| \le c\ \forall\, r \in [0, R]\,,\ \forall\, n \in \mathbb{N}$. Consequently, (w_n) and (z_n) are equicontinuous. According to Arzelà-Ascoli theorem, there exist two subsequences, denoted again as (w_n) and (z_n), such that $w_n \to w;\ z_n \to z$ in $C^0([0, R]);\ \forall\, R > 0$.

Let us prove now that (w_n, z_n) is a cauchy sequence in X. Indeed,

$$\begin{aligned} \sup_{r\in[0,+\infty[} |w_n(r) - w_m(r)| &\le \sup_{r\in[0,R[} |w_n(r) - w_m(r)| + \sup_{r\in[R,+\infty[} |w_n(r) - w_m(r)| \\ \sup_{r\in[R,+\infty[} |w_n(r) - w_m(r)| &\le \sup_{r\in[R,+\infty[} |w_n(r)| + \sup_{r\in[R,+\infty[} |w_m(r)| \\ &\le c_{11}c_1\left(\|u_n\|_\infty\right)^{\frac{\delta_{11}+\varepsilon}{p^k-1}} + c_{12}c_2\left(\|v_n\|_\infty\right)^{\frac{\delta_{12}+\varepsilon}{p^k-1}} \\ &\quad + c_{11}c_1\left(\|u_m\|_\infty\right)^{\frac{\delta_{11}+\varepsilon}{p^k-1}} + c_{12}c_2\left(\|v_m\|_\infty\right)^{\frac{\delta_{12}+\varepsilon}{p^k-1}} \end{aligned}$$

We have $c_1 + c_2 < \varepsilon$ as R sufficiently large. On the other hand (w_n) converges in $C^0([0, R])$.

It follows that (w_n) is a cauchy sequence in $C^0([0, +\infty[)$. In a similar manner, (z_n) is also a cauchy sequence in $C^0([0, +\infty[)$. Consequently (u_n, v_n) is a cauchy sequence in X. Hence L is compact. □

Theorem 3.5. *If hypotheses (H1)-(H3), are satisfies the system*

$$\begin{aligned} -\Delta_p u &= a_{12}(|x|)\,|v|^{\delta_{12}-1}\,v \qquad \text{in } \mathbb{R}^N, \\ -\Delta_q v &= a_{21}(|x|)\,|u|^{\delta_{21}-1}\,u \qquad \text{in } \mathbb{R}^N, \end{aligned} \tag{3.3}$$

has no non-trivial radial positive solutions; in particular (3.3) has no ground state.

Proof. Let us argue by contradiction. Let (u,v) be a radial positive solution of system (3.3). Then (u,v) satisfies the differential system

$$\begin{aligned} -\left(r^{N-1}|u'(r)|^{p-2}u'(r)\right)' &= r^{N-1}a_{12}(r)(v(r))^{\delta_{12}} \qquad \text{in } [0,+\infty[\\ -\left(r^{N-1}|v'(r)|^{q-2}v'(r)\right)' &= r^{N-1}a_{21}(r)(u(r))^{\delta_{21}} \qquad \text{in } [0,+\infty[\\ u'(0) = \ v'(0) &= 0. \end{aligned} \tag{3.4}$$

Hence,

$$-\left(r^{N-1}|u'(r)|^{p-2}u'(r)\right)' \geq r^{N-1}\widetilde{a}_1 v^{\delta_{12}} \tag{3.5}$$

$$-\left(r^{N-1}|u'(r)|^{q-2}u'(r)\right)' \geq r^{N-1}\widetilde{a}_2 v^{\delta_{21}} \tag{3.6}$$

with $v^{\delta_{ij}} = \min\limits_{[0,r]} v(r)^{\delta_{ij}} \qquad$ for $i \neq j$.

First, consider the case $\beta_1 > 0$ or $\beta_2 > 0$. Integrating both (3.5) and (3.6) from 0 to r and taking into account that $u'(r) < 0,\ v'(r) < 0$ for all $r > 0$, we obtain

$$\begin{aligned} -u'(r) &\geq \left(\frac{\widetilde{a}_1}{N}\right)^{\frac{1}{p-1}} r^{\frac{1}{p-1}} v^{\frac{\delta_{12}}{p-1}}, \\ -v'(r) &\geq \left(\frac{\widetilde{a}_2}{N}\right)^{\frac{1}{q-1}} r^{\frac{1}{q-1}} u^{\frac{\delta_{21}}{q-1}}. \end{aligned}$$

By Lemma 3.3, we have $M_p \geq 0,\ M_q \geq 0$, thus

$$\begin{aligned} 0 \geq -ru'(r) - \frac{N-p}{p-1}u(r) &\geq \left(\frac{\widetilde{a}_1}{N}\right)^{\frac{1}{p-1}} r^{\frac{p}{p-1}} v^{\frac{\delta_{12}}{p-1}} - \frac{N-p}{p-1}u(r), \\ 0 \geq -rv'(r) - \frac{N-q}{q-1}v(r) &\geq \left(\frac{\widetilde{a}_2}{N}\right)^{\frac{1}{q-1}} r^{\frac{q}{q-1}} u^{\frac{\delta_{21}}{q-1}} - \frac{N-q}{q-1}v(r). \end{aligned}$$

This yields

$$u(r) \geq Cr^{\frac{p}{p-1}} v^{\frac{\delta_{12}}{p-1}}, \tag{3.7}$$

$$v(r) \geq Cr^{\frac{q}{q-1}} u^{\frac{\delta_{21}}{q-1}}. \tag{3.8}$$

Combining these two inequalities, we have

$$u(r) \leq Cr^{-\alpha_1}, \tag{3.9}$$

$$v(r) \leq Cr^{-\alpha_2}. \tag{3.10}$$

Since $r^{\frac{N-p}{p-1}}u(r)$ and $r^{\frac{N-q}{q-1}}v(r)$ are nondecreasing, for all $r > r_0 > 0$,

$$u(r) \geq Cr^{-\frac{N-p}{p-1}}, \tag{3.11}$$

$$v(r) \geq Cr^{-\frac{N-q}{q-1}}. \tag{3.12}$$

Inequalities (3.9)−(3.12) imply either $r^{\beta_1} \leq C$ or $r^{\beta_2} \leq C$. This yields a contradiction. Suppose with out loss of generality now that $\beta_1 = 0$. Integrating with respect to r the first equation of System (3.4) from $r_0 > 0$ to r, we obtain

$$r^{N-1}|u'(r)|^{p-1}-r_0^{N-1}|u'(r_0)|^{p-1}\geq \widetilde{a}_1\int_{r_0}^{r}s^{N-1}v^{\delta_{12}}(s)ds.$$

On the other hand, by (3.8)

$$v^{\delta_{12}}(s)\geq Cs^{\frac{\delta_{12}q}{q-1}}u^{\frac{\delta_{12}\delta_{21}}{q-1}}(s).$$

Consequently,

$$r^{N-1}|u'(r)|^{p-1}\geq C\int_{r_0}^{r}s^{N-1+\frac{\delta_{12}q}{q-1}}u^{\frac{\delta_{12}\delta_{21}}{q-1}}(s)ds.$$

Taking into account inequality (3.11) and the fact that $\beta_1=0$, we have

$$r^{N-1}|u'(r)|^{p-1}\geq C\int_{r_0}^{r}s^{N-1+\frac{\delta_{12}q}{q-1}-\frac{N-p}{p-1}\frac{\delta_{12}\delta_{21}}{q-1}}ds=C\int_{r_0}^{r}s^{-1}ds=C\ln\tfrac{r}{r_0}.$$

On the other hand, $M_p(r)\geq 0$ for $r>0$ implies $\left(\frac{N-p}{p-1}\right)^{p-1}u^{p-1}(r)\geq r^{p-1}|u'(r)|^{p-1}$. Hence

$$u^{p-1}(r)\geq Cr^{p-1}|u'(r)|^{p-1}\geq Cr^{p-N}\ln\tfrac{r}{r_0}.$$

Then we write

$$r^{\frac{N-p}{p-1}}u(r)\geq C(\ln\tfrac{r}{r_0})^{\frac{1}{p-1}}.$$

This together with (3.9) yields a contradiction. □

We now show that the eventual radial positive solutions of System (2.1) are bounded.

Theorem 3.6. *Assume (H1)-(H4). If (u,v) is a ground state of(2.1). then there exists a constant $C>0$ (independent of u and v) such that $\|(u,v)\|_X\leq C$.*

Proof. Let (u,v) be a ground state of (2.1) for $h=0$, then (u,v) satisfies the system

$$\begin{aligned}-\left(r^{N-1}|u'(r)|^{p(r)-2}u'(r)\right)'&=r^{N-1}a_{11}(r)f_{11}(u(r))+r^{N-1}a_{12}(r)f_{12}(v(r))\\ &\text{in}\quad [0,+\infty[,\\ -\left(r^{N-1}|v'(r)|^{q(r)-2}v'(r)\right)'&=r^{N-1}a_{21}(r)f_{21}(u(r))+r^{N-1}a_{22}(r)f_{22}(v(r))\\ &\text{in}\quad [0,+\infty[,\\ u'(0)=v'(0)=0,&\qquad \lim_{r\to+\infty}u(r)=\lim_{r\to+\infty}v(r)=0,\end{aligned}\tag{3.13}$$

Assume now that there exists a sequence (u_n,v_n) of positive solutions of (3.13) such that $\|u_n\|_\infty\to\infty$ as $n\to\infty$ or $\|v_n\|_\infty\to\infty$ as $n\to\infty$. Taking $\gamma_n=\|u_n\|_\infty^{\frac{1}{\alpha_1}}+\|v_n\|_\infty^{\frac{1}{\alpha_2}}$, and using *(H3)*, we have $\alpha_1>0$ and $\alpha_2>0$. So $\gamma_n\to+\infty$ as $n\to\infty$.

Now we introduce the transformations

$$y=\gamma_n r,\qquad w_n(y)=\frac{u_n(r)}{\gamma_n^{\alpha_1}},\qquad z_n(y)=\frac{v_n(r)}{\gamma_n^{\alpha_2}}.$$

Observe that for all $y \in [0, +\infty[\,,\ 0 \le w_n(y) \le 1,\ 0 \le z_n(y) \le 1$. Furthermore it is easy to see that for any n the pair (w_n, z_n) is a solution of the system

$$
\begin{aligned}
&-(\gamma_n^{\alpha_1(p(\frac{y}{\gamma_n})-1)+p(\frac{y}{\gamma_n})} y^{N-1}|w_n'(y)|^{p(\frac{y}{\gamma_n})-2}w_n'(y))' \\
&= y^{N-1}a_{11}(\frac{y}{\gamma_n})f_{11}(\gamma_n^{\alpha_1}w_n(y)) + y^{N-1}a_{12}(\frac{y}{\gamma_n})f_{12}(\gamma_n^{\alpha_2}z_n(y)) \qquad \text{in} \quad [0,+\infty[\,, \\
&-(\gamma_n^{\alpha_2(q(\frac{y}{\gamma_n})-1)+q(\frac{y}{\gamma_n})} y^{N-1}|z_n'(y)|^{q(\frac{y}{\gamma_n})-2}z_n'(y))' \\
&= y^{N-1}a_{21}(\frac{y}{\gamma_n})f_{21}(\gamma_n^{\alpha_1}w_n(y)) + y^{N-1}a_{22}(\frac{y}{\gamma_n})f_{22}(\gamma_n^{\alpha_2}z_n(y)) \qquad \text{in} \quad [0,+\infty[\,, \\
&\qquad w_n'(0) = z_n'(0) = 0, \qquad \lim_{r\to+\infty} w_n(r) = \lim_{r\to+\infty} z_n(r) = 0
\end{aligned} \tag{3.14}
$$

Let $R > 0$ be fixed. We claim that (w_n') and (z_n') are bounded in $C([0,R])$. Indeed passing to a subsequence of (w_n') (denoted again (w_n')) assume that $\|w_n'\|_{([0,R])} \to +\infty$ as $n \to +\infty$. Hence there exists a sequence (y_n) in $[0,R]$ such that for all $A > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$, $|w_n'(y_n)| > A$.

Integrating with respect to y the first equation of System (3.14) we obtain

$$
|w_n'(y_n)|^{p(\frac{y_n}{\gamma_n})-1} = \frac{1}{y_n^{N-1}\gamma_n^{\alpha_1(p(\frac{y_n}{\gamma_n})-1)+p(\frac{y_n}{\gamma_n})}} \int_0^{y_n} \left(y^{N-1}a_{11}(\frac{y}{\gamma_n})f_{11}(\gamma_n^{\alpha_1}w_n(y)) + y^{N-1}a_{12}(\frac{y}{\gamma_n})f_{12}(\gamma_n^{\alpha_2}z_n(y))\right) dy.
$$

$$
|w_n'(y_n)|^{p(\frac{y_n}{\gamma_n})-1} \le \frac{1}{y_n^{N-1}} \int_0^{y_n} \left(y^{N-1}a_{11}(\frac{y}{\gamma_n})\frac{f_{11}(\gamma_n^{\alpha_1}w_n(y))}{\gamma_n^{\alpha_1(p^k-1)+p^k}} + y^{N-1}a_{12}(\frac{y}{\gamma_n})\frac{f_{12}(\gamma_n^{\alpha_2}z_n(y))}{\gamma_n^{\alpha_1(p^k-1)+p^k}}\right) dy.
$$

From the fact that $f_{1j},\ j = 1,2$, are asymptotically homogeneous at the infinity together with part (i) of Proposition 2.1, we arrive to the statement: for all $\varepsilon \in [0, \delta_{1j}[\,$, there exists $c_{1j}^1, c_{1j}^2 > 0,\ s_0 > 0$ such that for all $s \ge s_0$

$$
c_{1j}^1 s^{\delta_{1j}-\varepsilon} \le f_{1j}(s) \le c_{1j}^2 s^{\delta_{1j}+\varepsilon}.
$$

Since (w_n) and (z_n) are bounded, we conclude that

$$
\begin{aligned}
c_{11}^1\gamma_n^{\alpha_1(\delta_{11}-\varepsilon)-\alpha_1(p^k-1)-p^k} &\le \frac{f_{11}(\gamma_n^{\alpha_1}w_n(y))}{\gamma_n^{\alpha_1(p^k-1)+p^k}} \le c_{11}^2\gamma_n^{\alpha_1(\delta_{11}+\varepsilon)-\alpha_1(p^k-1)-p^k}, \\
c_{12}^1\gamma_n^{\alpha_2(\delta_{12}-\varepsilon)-\alpha_1(p^k-1)-p^k} &\le \frac{f_{12}(\gamma_n^{\alpha_2}z_n(y))}{\gamma_n^{\alpha_1(p^k-1)+p^k}} \le c_{12}^2\gamma_n^{\alpha_2(\delta_{12}+\varepsilon)-\alpha_1(p^k-1)-p^k}.
\end{aligned}
$$

By choosing ε sufficiently small, the assumption *(H3)* yields

$$
\frac{f_{11}(\gamma_n^{\alpha_1}w_n(y))}{\gamma_n^{\alpha_1(p^k-1)+p^k}} \to 0 \qquad \text{and} \qquad \frac{f_{12}(\gamma_n^{\alpha_2}z_n(y))}{\gamma_n^{\alpha_1(p^k-1)+p^k}} \to c_1 \qquad \text{as } n \to +\infty,
$$

where c_1 is positive constant. So there exists $n_1 \in \mathbb{N}$ such that for any $n \ge n_1$, we have

$$
|w_n'(y_n)|^{p(\frac{y_n}{\gamma_n})-1} \le \frac{a_{12}(0)}{y_n^{N-1}} c_1 \int_0^{y_n} y^{N-1}dy = \frac{c_1}{N}a_{12}(0)y_n \le \frac{Rc_1}{N}a_{12}(0) \equiv c.
$$

Setting $n \ge \max(n_0, n_1)$, we have $A < |w_n'(y_n)| \le c$. This contradicts the fact that A may be infinitely large. Similarly we prove that (z_n') is bounded in $(C[0,R])$. Consequently (w_n) and (z_n) are equicontinuous in $C([0,R])$. By Arzelà-Ascoli theorem, there exists a subsequence of (w_n) denoted again (w_n) (respect. (z_n)) such that $w_n \to w$ (respect. $z_n \to z$) in $C([0,R])$.

On the other hand,

$$
\|w_n\|_\infty^{\frac{1}{\alpha_1}} + \|z_n\|_\infty^{\frac{1}{\alpha_2}} = 1,
$$

this implies that the real-valued sequences $(\|w_n\|_\infty)$ and $(\|z_n\|_\infty)$ are bounded. Hence there exist subsequences denoted again $(\|w_n\|_\infty)$ and $(\|z_n\|_\infty)$ such that $\|w_n\|_\infty \to w_0, \|z_n\|_\infty \to z_0$ and $w_0^{\frac{1}{\alpha 1}} + z_0^{\frac{1}{\alpha 2}} = 1$. In view of the uniqueness of the limit in $C([0,R])$, we get $\|w\|_\infty^{\frac{1}{\alpha 1}} + \|z\|_\infty^{\frac{1}{\alpha 2}} = 1$. This implies that (w,z) is not identically null. Integrating from 0 to $y \in [0,R]$, the first and the second equation of System (3.14), we obtain

$$w_n(0) - w_n(y) = \int_0^y (g_n(t))^{\frac{1}{p(\frac{t}{\gamma_n})-1}}\,dt, \tag{3.15}$$

$$z_n(0) - z_n(y) = \int_0^y (h_n(t))^{\frac{1}{q(\frac{t}{\gamma_n})-1}}\,dt, \tag{3.16}$$

Clearly $g_n(y)$ and $h_n(y)$ are defined by

$$g_n(y) = \frac{1}{y^{N-1}\gamma_n^{\alpha_1(p(\frac{y}{\gamma_n})-1)+p(\frac{y}{\gamma_n})}} \int_0^y \left(t^{N-1}a_{11}(\frac{t}{\gamma_n})f_{11}(\gamma_n^{\alpha_1}w_n(t)) + t^{N-1}a_{12}(\frac{t}{\gamma_n})f_{12}(\gamma_n^{\alpha_2}z_n(t)) \right) dt.$$

$$g_n(y) \le \frac{1}{y^{N-1}} \int_0^y \left(t^{N-1}a_{11}(\frac{t}{\gamma_n})\frac{f_{11}(\gamma_n^{\alpha_1}w_n(t))}{\gamma_n^{\alpha_1(p^k-1)+p^k}} + t^{N-1}a_{12}(\frac{t}{\gamma_n})\frac{f_{12}(\gamma_n^{\alpha_2}z_n(t))}{\gamma_n^{\alpha_1(p^k-1)+p^k}} \right) dt.$$

$$h_n(y) = \frac{1}{y^{N-1}\gamma_n^{\alpha_2(q(\frac{y}{\gamma_n})-1)+q(\frac{y}{\gamma_n})}} \int_0^y \left(t^{N-1}a_{21}(\frac{t}{\gamma_n})f_{21}(\gamma_n^{\alpha_1}w_n(t)) + t^{N-1}a_{22}(\frac{t}{\gamma_n})f_{22}(\gamma_n^{\alpha_2}z_n(t)) \right) dt.$$

$$h_n(y) \le \frac{1}{y^{N-1}} \int_0^y \left(t^{N-1}a_{21}(\frac{t}{\gamma_n})\frac{f_{21}(\gamma_n^{\alpha_1}w_n(t))}{\gamma_n^{\alpha_2(q^k-1)+q^k}} + t^{N-1}a_{22}(\frac{t}{\gamma_n})\frac{f_{22}(\gamma_n^{\alpha_2}z_n(t))}{\gamma_n^{\alpha_2(q^k-1)+q^k}} \right) dt.$$

Compiling Proposition 2.1 and *(H3)*, we obtain

$$\frac{f_{11}(\gamma_n^{\alpha_1}w_n(t))}{\gamma_n^{\alpha_1(p^k-1)+p^k}} \to 0, \qquad \frac{f_{22}(\gamma_n^{\alpha_2}z_n(t))}{\gamma_n^{\alpha_2(q^k-1)+q^k}} \to 0,$$
$$\frac{f_{12}(\gamma_n^{\alpha_2}z_n(t))}{\gamma_n^{\alpha_1(p^k-1)+p^k}} = \frac{f_{12}(\gamma_n^{\alpha_2})}{\gamma_n^{\alpha_1(p^k-1)+p^k}}\frac{f_{12}(\gamma_n^{\alpha_2}z_n(t))}{f_{12}(\gamma_n^{\alpha_2})} \to cz^{\delta_{12}}(t),$$
$$\frac{f_{21}(\gamma_n^{\alpha_1}w_n(t))}{\gamma_n^{\alpha_2(q^k-1)+q^k}} = \frac{f_{21}(\gamma_n^{\alpha_1})}{\gamma_n^{\alpha_2(q^k-1)+q^k}}\frac{f_{21}(\gamma_n^{\alpha_1}w_n(t))}{f_{21}(\gamma_n^{\alpha_1})} \to cw^{\delta_{21}}(t),$$

as $n \to \infty$. By the Lebesgue theorem on dominated convergence, it follows that

$$g_n(y) \to \frac{c}{y^{N-1}} \int_0^y t^{N-1}a_{12}(0)z^{\delta_{12}}(t)dt,$$
$$h_n(y) \to \frac{c}{y^{N-1}} \int_0^y t^{N-1}a_{21}(0)w^{\delta_{21}}(t)dt,$$

as $n \to \infty$. Passing to the limit in (3.15) and (3.16), we arrive to

$$w(0) - w(y) = c\int_0^y \left(\frac{1}{\xi^{N-1}}\int_0^\xi t^{N-1}a_{12}(0)z^{\delta_{12}}(t)dt\right)^{\frac{1}{p(0)-1}} d\xi,$$
$$z(0) - z(y) = c\int_0^y \left(\frac{1}{\xi^{N-1}}\int_0^\xi t^{N-1}a_{21}(0)w^{\delta_{21}}(t)dt\right)^{\frac{1}{q(0)-1}} d\xi.$$

In this way, $w \ge 0, z \ge 0, w,\ z \in C^1([0,R]) \cap C^2([0,R])$ and satisfy the system

$$\begin{array}{ll} -(y^{N-1}|w'(y)|^{p-2}w'(y))' = ca_{12}(0)y^{N-1}(z(y))^{\delta_{12}} & \text{in } [0,R] \\ -(y^{N-1}|z'(y)|^{q-2}z'(y))' = ca_{21}(0)y^{N-1}(w(y))^{\delta_{21}} & \text{in } [0,R] \\ w'(0) = z'(0) = 0. & \end{array} \tag{3.17}$$

If we use the same arguments on $[0,R^*]$ where $R^* > R$, we obtain a solution (w^*,z^*) of System (3.17) with R^* instead of R, which coincide with (w,z) in $[0,R]$ To this end, we indefinitely extend (w,z) to $[0,+\infty[$. By Lemma 3.2 we have $w(y) > 0, z(y) > 0$, for all $y \ge 0$. The pair (w,z) also satisfies System (3.17). In other words (w,z) is a radial positive solution of (3.4). This contradicts Theorem 3.5. □

Lemma 3.7. *Under assumptions (H1)-(H4), there exists $h_0 > 0$ such that the problem $(u,v) = T_h(u,v)$ has no solution for $h \geq h_0$.*

Proof. Suppose by contradiction that there is a solution $(u,v) \in X$ of the above problem. Then (u,v) satisfies system

$$\begin{aligned} -\left(r^{N-1}|u'(r)|^{p(r)-2}u'(r)\right)' &= r^{N-1}a_{11}(r)f_{11}(|u(r)|) + r^{N-1}a_{12}(r)\left[f_{12}(|v(r)|) + h\right] \\ &\text{in} \quad [0,+\infty[, \\ -\left(r^{N-1}|v'(r)|^{q(r)-2}v'(r)\right)' &= r^{N-1}a_{21}(r)f_{21}(|u(r)|) + r^{N-1}a_{22}(r)f_{22}(|v(r)|) \\ &\text{in} \quad [0,+\infty[, \\ u'(0) = v'(0) = 0, &\qquad \lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} v(r) = 0, \end{aligned} \tag{3.18}$$

Assume that there exists a sequence (h_n) $h_n \to +\infty$ as $n \to +\infty$, such that (3.18) admits a pair of solutions (u_n, v_n). In accordance with Lemma 3.2, we have $u_n(r) > 0$, $v_n(r) > 0$, $u_n'(r) \leq 0$, and $v_n'(r) \leq 0$, for all $n \in \mathbb{N}$. Integrating the first equation of System (3.18), from R to $2R$, $R > 0$, we obtain

$$u_n(R) \geq \int_R^{2R} \left(\eta^{1-N}\int_0^\eta \xi^{N-1}a_{12}\left(\xi\right)h_n d\xi\right)^{\frac{1}{p^k-1}} d\eta \geq cRh_n^{\frac{1}{p^k-1}}.$$

Here

$$c = \left(\frac{1}{(2R)^{N-1}}\int_0^R \xi^{N-1}a_{12}(\xi)d\xi\right)^{\frac{1}{p^k-1}}.$$

Consequently $u_n(R) \geq cRh_n^{\frac{1}{p^k-1}}$. Passing to the limit we get $u_n(R) \to +\infty$. On the other hand, integrating the second equation of (3.18), from R to $2R$, we obtain

$$v_n(R) \geq \int_R^{2R} \left(\eta^{1-N}\int_0^\eta \xi^{N-1}a_{21}(\xi)f_{21}(u_n(\xi))d\xi\right)^{\frac{1}{q^k-1}} d\eta \geq cR(f_{21}(u_n(R)))^{\frac{1}{q^k-1}}.$$

By hypothesis *(H3)* and Proposition 2.1, we have $v_n(R) \geq c(u_n(R))^{\frac{\delta_{21}-\varepsilon}{q^k-1}}$. Operating similarly, we obtain $u_n(R) \geq c(v_n(R))^{\frac{\delta_{12}-\varepsilon}{p^k-1}}$. It follows from the last two inequalities, that

$$(u_n(R))^{\frac{(\delta_{12}-\varepsilon)(\delta_{21}-\varepsilon)-(p^k-1)\left(q^k-1\right)}{(p^k-1)(q^k-1)}} \leq \frac{1}{c}.$$

This is the desired contradiction since $u_n(R)$ increases to infinitely. □

Lemma 3.8. *There exists $\overline{\rho} > 0$ such that for all $\rho \in]0, \overline{\rho}[$ and all $(u,v) \in X$ satisfying $\|(u,v)\| = \rho$, the equation $(u,v) = S_\lambda(u,v)$ has no solution.*

Proof. Assume that there exist $(\rho_n) \in \mathbb{R}_+$, $\rho_n \to 0$; $(\lambda_n) \subset [0,1]$ and $(u_n, v_n) \in X$ such that $(u_n, v_n) = S_{\lambda_n}(u_n, v_n)$ with $\|(u_n, v_n)\| = \rho_n$. Taking *(H4)* into account,

$$\begin{aligned} \|u_n\|_\infty &\leq c\lambda_n^{\frac{1}{p^k-1}}\left(\|u_n\|_\infty^{\frac{\overline{\delta}_{11}+\varepsilon}{p^k-1}} + \|v_n\|_\infty^{\frac{\overline{\delta}_{12}+\varepsilon}{p^k-1}}\right), \\ \|v_n\|_\infty &\leq c\lambda_n^{\frac{1}{q^k-1}}\left(\|u_n\|_\infty^{\frac{\overline{\delta}_{21}+\varepsilon}{q^k-1}} + \|v_n\|_\infty^{\frac{\overline{\delta}_{22}+\varepsilon}{q^k-1}}\right). \end{aligned}$$

Adding term by term, we obtain

$$\|(u_n,v_n)\| \leq C\left(\|(u_n,v_n)\|^{\frac{\overline{\delta}_{11}+\varepsilon}{p^k-1}} + \|(u_n,v_n)\|^{\frac{\overline{\delta}_{12}+\varepsilon}{p^k-1}} + \|(u_n,v_n)\|^{\frac{\overline{\delta}_{21}+\varepsilon}{q^k-1}} + \|(u_n,v_n)\|^{\frac{\overline{\delta}_{22}+\varepsilon}{q^k-1}}\right).$$

This implies

$$1 \leq C\left(\|(u_n,v_n)\|^{\frac{\overline{\delta}_{11}+\varepsilon}{p^k-1}-1} + \|(u_n,v_n)\|^{\frac{\overline{\delta}_{12}+\varepsilon}{p^k-1}-1} + \|(u_n,v_n)\|^{\frac{\overline{\delta}_{21}+\varepsilon}{q^k-1}-1} + \|(u_n,v_n)\|^{\frac{\overline{\delta}_{22}+\varepsilon}{q^k-1}-1}\right).$$

The above inequality contradicts the fact that $\|(u_n,v_n)\| = \rho_n \to 0$ as $n \to +\infty$ □

Theorem 3.9. *Under hypotheses (H1)-(H4), System (1.1) has positive radial solution.*

Proof. To show the existence of ground states for (1.1) (or (2.1) with $h=0$), it is sufficient to prove that the compact operator T_0 admits a fixed point. In view of Theorem 3.6, the eventual fixed point (u,v) of T_0 are bounded; explicitly there exists $C>0$ such that $\|(u,v)\|_X \leq C$. Let us chose $R_1 > C$ and let us designate by B_{R_1} the ball of X, centered at the origin with radius R_1. To this end, the Leray-Schauder degree $\deg_{LS}(I-T_h, B_{R_1}, 0)$ is well defined. It being understood that I denote the identical operator in X. Moreover, by Lemma 3.7, we have $\deg_{LS}(I-T_h, B_{R_1}, 0) = 0$ for all $h \geq h_0$. It follows from the homotopy invariance of the Leray-Schauder degree that

$$\deg_{LS}(I-T_0, B_{R_1}, 0) = \deg_{LS}(I-T_h, B_{R_1}, 0) = 0$$

On the other hand, by Lemma 3.8, there exists $0 < \rho < \overline{\rho} < R_1$ such that $\deg_{LS}(I-S_\lambda, B_\rho, 0)$ is well defined. Once again, the homotopy invariance of the Leray-Schauder degree yields

$$\begin{aligned} 1 = & \quad \deg_{LS}(I, B_\rho, 0) \\ = & \quad \deg_{LS}(I-S_\lambda, B_\rho, 0) \\ = & \quad \deg_{LS}(I-S_1, B_\rho, 0) \\ = & \quad \deg_{LS}(I-T_0, B_\rho, 0). \end{aligned}$$

Using the additivity of the Leray-Schauder degree,

$$\deg_{LS}(I-T_0, B_{R_1}\setminus B_\rho, 0) = \deg_{LS}(I-T_0, B_{R_1}, 0) - \deg_{LS}(I-T_0, B_\rho, 0) = -1.$$

This implies that T_0 has fixed point in $B_{R_1}\setminus B_\rho$. Consequently, there exists a nontrivial ground state.
□

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