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The Continuous Generalized Wavelet Transform Associated with q-Bessel Operator

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ABSTRACT: The continuous generalized wavelet transform associated with q-Bessel operator is defined, which will invariably be called continuous q-Bessel wavelet transform . Certain and boundedness results and inversion formula for continuous q-Bessel wavelet transform are obtained. Discrete q-Bessel wavelet transform is defined and a reconstruction formula is derived for discrete q- Bessel wavelet.

Key Words: q-Bessel function, q- Bessel Fourier transform, wavelet transform.

Contents

1. Introduction

A complex-valued continuous function ϕ with the property

$$\int_0^{\infty} \phi(t)dt = 0, \tag{1.1}$$

is called a wavelet.The wavelet transform of a function $f \in L^2(\mathbf{R})$ with respect to the wavelet $\phi \in L^2(\mathbf{R})$ is defined by

$$(W_\phi)(b,a) = \int_{-\infty}^{+\infty} f(t)\overline{\phi_{b,a}(t)}dt,\ b \in \mathbf{R},\ a > 0, \tag{1.2}$$

where

$$\phi_{b,a}(t) = a^{-1/2}\phi((t-b)/a). \tag{1.3}$$

In terms of the translation T_b defined by

$$T_b\phi(t) = \phi(t-b),\ b \in \mathbf{R} \tag{1.4}$$

and dilation D_a defined by

$$D_a\phi(t) = |a|^{-1/2}\phi(t/a),\ a \neq 0, \tag{1.5}$$

we can write

$$\phi_{b,a}(t) = T_b D_a \phi(t). \tag{1.6}$$

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We can also express (1.2) as the convolution:

$$(W_\phi f)(b,a) = (f * g_{0,a})(b), \tag{1.7}$$

where

$$g(t) := \overline{\phi(-t)}\,. \tag{1.8}$$

2. The q-Bessel operator and q-Bessel function

The q-Bessel operator defined by

$$\Delta_{q,\alpha} f(x) = \frac{1}{x^{2\alpha+1}} D_q \left[x^{2\alpha+1} D_q f\right]\left(q^{-1}x\right), \tag{2.1}$$

where

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)\,x},\ x \neq 0,\ q \neq 1. \tag{2.2}$$

For $a, q \in \mathbf{C}$, the q-shift factorial $(a;q)_k$ is defined as a product of k factors

$$(a;q)_k = (1-a)(1-aq)\ldots\left(1-aq^{k-1}\right),\ k \in \mathbf{N}^*,\ (a;q)_0 = 1. \tag{2.3}$$

If $\mid q \mid< 1$, this definition remains meaningful for $k = +\infty$ as a convergent infinite product:

$$(a;q)_\infty = \prod_{k=0}^{\infty}\left(1 - aq^k\right). \tag{2.4}$$

We also write $(a_1,\ldots,a_r;q)_k$ for the product of rq-shifted factorials:

$$(a_1,\ldots,a_r;q)_k = (a_1;q)_k \ldots (a_r;q)_k\,,\ k \in \mathbf{N} \text{ or } k = \infty. \tag{2.5}$$

A q-hypergeometric series is a power series (for the moment still formal) in one complex variable z with power series coefficients which depend, apart from q, on r complex upper parameters $a_1,\ldots,a_r$ and s complex lower parameters $b_1,\ldots,b_s$ as follows:

$$r\phi_s(a_1,\ldots,a_r;b_1,\ldots,b_s;q,x) = \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k\,(q;q)_k}\left[(-1)^k q^{\frac{k(k-1)}{2}}\right]^{1+s-r} x^k, \text{ for } r,s \in \mathbf{N}. \tag{2.6}$$

The q-Bessel function is defined by

$$j_\alpha\left(x;q^2\right) = \Gamma_{q^2}(\alpha+1)\sum_{k=0}^{\infty}\frac{(-1)^k q^{k(k-1)}}{\Gamma_{q^2}(k+1)\,\Gamma_{q^2}(\alpha+k+1)}\left(\frac{x}{1+q}\right)^{2k}. \tag{2.7}$$

This function is bounded and for every $x \in \mathbf{R}_q$ and $\alpha > -\frac{1}{2}$, we have

$$\left|j_\alpha\left(x;q^2\right)\right| \leq \frac{1}{(q;q^2)_\infty^2}, \tag{2.8}$$

$$\left(\frac{1}{x}D_q\right) j_\alpha\left(.;q^2\right) = -\frac{(1-q)}{(1-q^{2\alpha+2})} j_{\alpha-1}\left(qx;q^2\right), \tag{2.9}$$

$$\left(\frac{1}{x}D_q\right)\left(x^{2\alpha} j_\alpha\left(x;q^2\right)\right) = \frac{\left(1-q^{2\alpha}\right)}{(1-q)} x^{2(\alpha-1)} j_{\alpha-1}\left(x;q^2\right), \tag{2.10}$$

$$\left|D_q j_\alpha\left(x;q^2\right)\right| \leq \frac{x\,(1-q)}{(1-q^{2\alpha+2})\,(q;q^2)_\infty^2}. \tag{2.11}$$

We remark that for $\lambda \in \mathbf{C}$, the function $j_\alpha(\lambda x, q^2)$ is the unique solution of the q-differential system

$$\begin{cases} \Delta_{q,\alpha} U(x,q) = -\lambda^2 U(x,q) \\ U(0,q) = 1 \; ; \; D_{q,x} U(x,q)\,|_{x=0} = 0, \end{cases} \tag{2.12}$$

where $\Delta_{q,\alpha}$ is the q-Bessel operator defined by

$$\Delta_{q,\alpha} f(x) = \frac{1}{x^{2\alpha+1}} D_q \left[x^{2\alpha+1} D_q f \right] \left(q^{-1} x \right) \tag{2.13}$$

$$= q^{2\alpha+1} \Delta_q f(x) + \frac{1-q^{2\alpha+1}}{(1-q)\, q^{-1} x} D_q f\left(q^{-1} x \right), \tag{2.14}$$

where

$$\Delta_q f(x) = \Lambda_q^{-1} D_q^2 f(x) = \left(D_q^2 f \right) \left(q^{-1} x \right) \tag{2.15}$$

and for $k \in \mathbf{N}$ and $\lambda \in \mathbf{R}_{q,+}$

$$\Delta_{q,x}^k j_\alpha \left(\lambda x; q^2 \right) = (-1)^k \lambda^{2k} j_\alpha \left(\lambda x; q^2 \right). \tag{2.16}$$

3. q-Functional spaces

We begin by putting

$$\mathbf{R}_{q,+} = \left\{ +q^k, \; k \in \mathbf{Z} \right\}, \; \tilde{\mathbf{R}}_{q,+} = \left\{ +q^k, \; k \in \mathbf{Z} \right\} \cup \{0\} \tag{3.1}$$

and we denote by $L^p_{\alpha,q}(\mathbf{R}_{q,+})$, $p \leq [0,\infty[$, $\left(\texttt{resp.} L^\infty_{\alpha,q}(\mathbf{R}_{q,+}) \right)$the space of functions f such that,

$$\|f\|_{p,\alpha,q} = \left(\int_0^\infty |f(x)|^p \, d_q\sigma(x) \right)^{\frac{1}{p}} < +\infty, \tag{3.2}$$

$$\texttt{resp.} \|f\|_{\infty,q} = \texttt{ess.} \sup_{x \in \mathbf{R}_q} |f(x)| < +\infty, \tag{3.3}$$

$$d_q\sigma(x) = \frac{(1+q)^{-\alpha}}{\Gamma_{q^2}(\alpha+1)} x^{2\alpha+1} d_q x = b_{\alpha,q} x^{2\alpha+1} d_q x. \tag{3.4}$$

4. q-Bessel translation operator

$T^\alpha_{q,x}$, $x \in \mathbf{R}_{q,+}$ is the q-generalized translation operator associated with the q-Bessel transform is introduced in [12], is defined as follows

$$\phi(x,y) = T^{\alpha,q}_y f(x) = \int_0^{+\infty} f(t)\, D_{\alpha,q}(x,y,t)\, d_q\sigma(t), \; \alpha > -1, \tag{4.1}$$

with

$$D_{\alpha,q}(x,y,z) = \int_0^{+\infty} j_\alpha \left(xt; q^2 \right) j_\alpha \left(yt; q^2 \right) j_\alpha \left(zt; q^2 \right) d_q\sigma(t) \tag{4.2}$$

and

$$\int_0^{+\infty} D_{\alpha,q}(x,y,z)\, d_q\sigma(z) = 1. \tag{4.3}$$

In particular the following product formula holds

$$T^\alpha_{q,x} j_\alpha \left(y; q^2 \right) = j_\alpha \left(x; q^2 \right) j_\alpha \left(y; q^2 \right). \tag{4.4}$$

It is shown in [12] that for $f \in L^p_{\alpha,q}(\mathbf{R}_{q,+})$

$$\left\| T^\alpha_{q,x} f \right\|_{p,\alpha,q} \leq \|f\|_{p,\alpha,q}, \tag{4.5}$$

and the map $y \to T^{\alpha,q}_y f$ is continuous from $(0,\infty)$ into $(0,\infty)$.

5. q-Convolution and q-Bessel Fourier transform

The q-Bessel Fourier transform $F_{\alpha,q}$ and the q -Bessel convolution product are defined for suitable functions f, g as follows

$$\hat{f}_{\alpha,q}(\lambda) = \int_0^\infty f(x)\, j_\alpha\left(\lambda x; q^2\right) d_q\sigma(x), \tag{5.1}$$

$$f *_{\alpha,q} g(x) = \int_0^{+\infty} T_{q,x}^\alpha f(y)\, g(y)\, d_q\sigma(y). \tag{5.2}$$

It is shown in [11], that the q-Bessel Fourier transform $F_{\alpha,q}$ satisfies the following properties:

Theorem 5.1. *If $f \in L^1_{\alpha,q}(\mathbf{R}_{q,+})$ then $F_{\alpha,q}(f) \in C_{q,*,0}(\mathbf{R}_{q,+})$ and*

$$\left\|\hat{f}_{\alpha,q}\right\| \leq B_{\alpha,q}\|f\|_{1,\alpha,q}, \tag{5.3}$$

where

$$B_{\alpha,q} = \frac{1}{(1-q)} \frac{\left(-q^2; q^2\right)_\infty \left(-q^{2\alpha+2}; q^2\right)_\infty}{\left(q^2; q^2\right)_\infty}. \tag{5.4}$$

Theorem 5.2. *Given two functions $f, g \in L^1_{\alpha,q}(\mathbf{R}_{q,+})$,then*

$$f *_{\alpha,q} g \in L^1_{\alpha,q}(\mathbf{R}_{q,+}) \tag{5.5}$$

and

$$F_{\alpha,q}(f *_{\alpha,q} g) = F_{\alpha,q}(f)\, F_{\alpha,q}(g). \tag{5.6}$$

Theorem 5.3. *(Inversion formula): If $f \in L^1_{\alpha,q}(\mathbf{R}_{q,+})$ such that $F_{\alpha,q}(f) \in L^1_{\alpha,q}(\mathbf{R}_{q,+})$,then for all $x \in \mathbf{R}_{q,+}$, we have*

$$f(x) = \int_0^\infty \hat{f}_{\alpha,q}(f)(y)\, j_\alpha\left(xy; q^2\right) d_q\sigma(y) \tag{5.7}$$

Theorem 5.4. *(q-Plancherel theorem) If $\hat{f}_{\alpha,q}$ is an isomorphisom of $L^2_{\alpha,q}(\mathbf{R}_{q,+})$,we have*

$$\left\|\hat{f}_{\alpha,q}(\lambda)\right\|_{2,\alpha,q} = \|f\|_{2,\alpha,q}, \text{ for } f \in L^2_{\alpha,q}(\mathbf{R}_{q,+}) \text{ and } F^{-1}_{\alpha,q}(f) = F_{\alpha,q}(f). \tag{5.8}$$

Theorem 5.5. *(i) For $f \in L^p_{\alpha,q}(\mathbf{R}_{q,+})$, $p \in [1, \infty[$, $g \in L^1_{\alpha,q}(\mathbf{R}_{q,+})$, we have*

$$f *_{\alpha,q} g \in L^p_{\alpha,q}(\mathbf{R}_{q,+}) \text{ and } \|f *_{\alpha,q} g\|_{p,\alpha,q} \leq \|f\|_{p,\alpha,q}\|g\|_{1,\alpha,q}.$$

(ii) $\displaystyle\int_0^\infty F_{\alpha,q}(f)(\xi)\, g(\xi)\, d_q\sigma(\xi) = \int_0^\infty f(\xi)\, F_{\alpha,q}(g)(\xi)\, d_q\sigma(\xi),\ f, g \in L^1_{\alpha,q}(\mathbf{R}_{q,+}).$

(iii) $F_{\alpha,q}\left(T^\alpha_{q,x} f\right)(\xi) = j_\alpha\left(\xi x; q^2\right) F_{\alpha,q}(f)(\xi),\ f \in L^1_{\alpha,q}(\mathbf{R}_{q,+}).$

6. The continuous generalized wavelet transform associated with q -Bessel operator

Let $\psi \in L^p_{\alpha,q}(\mathbf{R}_{q,+})$, $1 \leq p < \infty$ be given. For $b \geq 0$ and$a > 0$ define the q-Bessel wavelet

$$\psi^{\alpha,q}_{b,a}(x) := D_a T_b^{\alpha,q}\psi(x) = D_a\psi(b, x) = a^{-2\alpha-2}\psi\left(\frac{b}{a}, \frac{x}{a}\right) \tag{6.1}$$

$$= a^{-2\alpha-2}\int_0^\infty D_{\alpha,q}\left(\frac{b}{a}, \frac{x}{a}, z\right)\psi(z)\, d_q\sigma(z), \tag{6.2}$$

the integral being convergent by virtue to (4.5).
Using the wavelet $\psi^{\alpha,q}_{b,a}$, we now define the continuous q -Bessel wavelet transform which will send each L^p-function defined on the positive half line to a function $B_{\alpha,q}(b, a)$ on the first quadrant as follows.

$$B_{\alpha,q}(b, a) := \left(B^{\alpha,q}_\psi f\right)(b, a) := \left\langle f(t), \psi^{\alpha,q}_{b,a}(t)\right\rangle_{\alpha,q} = \int_0^\infty f(t)\, \overline{\psi^{\alpha,q}_{b,a}(t)}\, d_q\sigma(t) \tag{6.3}$$

$$= a^{-2\alpha-2}\int_0^\infty\int_0^\infty f(t)\,\overline{\psi(z)}D_{\alpha,q}\left(\frac{b}{a},\frac{t}{a},z\right)d_q\sigma(z)\,d_q\sigma(t), \tag{6.4}$$

provided the integral is convergent; see Theorem 5.3 for existence.

Theorem 6.1. *Let $\psi \in L^p_{\alpha,q}(\mathbf{R}_{q,+})$, $1 \le p < \infty$. Then for $y \ge 0$,*
(i) the map $y \to T_y^{\alpha,q}\psi$ is continuous from $L^p_{\alpha,q}(\mathbf{R}_{q,+})$ into $L^{p'}_{\alpha,q}(\mathbf{R}_{q,+})$.
(ii)the function $\psi^{\alpha,q}_{b,a}$ is defined almost everywhere on $[0,\infty)$, and

$$\left\|\psi^{\alpha,q}_{b,a}(x)\right\|_{p,\alpha,q} \le a^{(2\alpha+2)\left(\frac{1}{p}-1\right)}\|\psi\|_{p,\alpha,q}. \tag{6.5}$$

Proof. We can write, for $\frac{1}{p}+\frac{1}{p'}=1$,

$$|\psi(x,y)| = \left|T_y^{\alpha,q}\psi(x)\right| = \left|\int_0^\infty \psi(z)\,D^{1/p}_{\alpha,q}(x,y,z)\,D^{1/p'}_{\alpha,q}(x,y,z)\right| d_q\sigma(z)$$
$$\le \left(\int_0^\infty |\psi(z)|^p D_{\alpha,q}(x,y,z)\,d_q\sigma(z)\right)^{1/p}\left(\int_0^\infty D_{\alpha,q}(x,y,z)\,d_q\sigma(z)\right)^{1/p'}.$$

Therefore, in view of the property (4.3), we have

$$|\psi(x)|^p \le \int_0^\infty |\psi(z)|^p D_{\alpha,q}(x,y,z)\,d_q\sigma(z),$$

so that

$$\int_0^\infty |\psi(x,y)|^p d_q\sigma(x) \le \int_0^\infty |\psi(z)|^p d_q\sigma(z)\int_0^\infty D_{\alpha,q}(x,y,z)\,d_q\sigma(x).$$

Thus, we get the following boundedness property of the q-Bessel translation operator

$$\|\psi(.,y)\|_{p,\alpha,q} \le \|\psi\|_{p,\alpha,q},\ 1 \le p < \infty. \tag{6.6}$$

Now applying the above method of proof to (6.2) we find that

$$\left\|\psi^{\alpha,q}_{b,a}(x)\right\|_{p,\alpha,q} \le a^{(2\alpha+2)\left(\frac{1}{p}-1\right)}\|\psi\|_{p,\alpha,q},\ 1 \le p < \infty.$$

□

Theorem 6.2. *Let $f \in L^p_{\alpha,q}(\mathbf{R}_{q,+})$ and $\psi \in L^{p'}_{\alpha,q}(\mathbf{R}_{q,+})$ with $1 \le p, p' < \infty$ and $\frac{1}{p}+\frac{1}{p'}=1$, and $B_{\alpha,q}(b,a) = \left(B^{\alpha,q}_\psi f\right)(b,a)$be the continuous q- Bessel wavelet transform (6.4). Then*
(i) $B_{\alpha,q}(b,a)$ is continuous on $(0,\infty)\times(0,\infty)$.
(ii)$\left\|\left(B^{\alpha,q}_\psi f\right)(b,a)\right\|_{r,\alpha,q} \le a^{(2\alpha+2)/r}\|f\|_{p,\alpha,q}\|\psi\|_{p',\alpha,q}$, $\frac{1}{r}=\frac{1}{p}+\frac{1}{p'}-1$, $1 \le p, p', r < \infty$.
(iii)$\left\|\left(B^{\alpha,q}_\psi f\right)(b,a)\right\|_{\infty,\alpha,q} \le a^{(2\alpha+2)(1/p'-1)}\|f\|_{p,\alpha,q}\|\psi\|_{p',\alpha,q}$, $\frac{1}{p}+\frac{1}{p'}=1$.

Proof. (i) Let (b_0,a_0) be an arbitrary but fixed point in $(0,\infty)\times(0,\infty)$. Then by Holder's inequality,

$$|B_{\alpha,q}(b,a) - B_{\alpha,q}(b_0,a_0)|$$
$$\le a^{-2\alpha-2}\int_0^\infty\int_0^\infty |f(t)\,\psi(z)\,[D_{\alpha,q}(b/a,t/a,z) - D_{\alpha,q}(b_0/a_0,t/a_0,z)]|\,d_q\sigma(t)\,d_q\sigma(z)$$
$$\le a^{-2\alpha-2}\left(\int_0^\infty\int_0^\infty |f(t)|^p\,|D_{\alpha,q}(b/a,t/a,z) - D_{\alpha,q}(b_0/a_0,t/a_0,z)|\,d_q\sigma(t)\,d_q\sigma(z)\right)^{1/p}$$
$$\times\left(\int_0^\infty\int_0^\infty |\psi(z)|^{p'}\,|D_{\alpha,q}(b/a,t/a,z) - D_{\alpha,q}(b_0/a_0,t/a_0,z)|\,d_q\sigma(t)\,d_q\sigma(z)\right)^{1/p'}.$$

Since

$$\int_0^\infty |D_{\alpha,q}\,(b/a, t/a, z) - D_{\alpha,q}\,(b_0/a_0, t/a_0, z)|\, d_q\sigma\,(z) \leq 2,$$

by dominated convergence theorem and continuity of $D_{\alpha,q}(b/a, t/a, z)$ in the variable b and a, we have

$$\lim_{\substack{b\to b_0 \\ a\to a_0}} |B_{\alpha,q}\,(b,a) - B_{\alpha,q}\,(b_0,a_0)| = 0.$$

This prove that $B_{\alpha,q}\,(b,a)$ is continuous on $(0,\infty)\times(0,\infty)$.

$$(iii) \quad \left(B_\psi^{\alpha,q} f\right)(b,a) = a^{-2\alpha-2}\int_0^\infty\int_0^\infty f\,(t)\,\psi\,(z)\,D_{\alpha,q}\,(b/a,t/a,z)\,d_q\sigma\,(t)\,d_q\sigma\,(z)$$
$$= a^{-2\alpha-2}\int_0^\infty\int_0^\infty f\,(t)\,\psi\,(z)\,D_{\alpha,q}^{1/p}\,(b/a,t/a,z)\,D_{\alpha,q}^{1/p'}\,(b/a,t/a,z)\,d_q\sigma\,(t)\,d_q\sigma\,(z)\,.$$

Therefore, by Holder's inequality, we have

$$\begin{aligned}\left|\left(B_\psi^{\alpha,q} f\right)(b,a)\right| &\leq a^{-2\alpha-2}\left(\int_0^\infty\int_0^\infty |f\,(t)|^p\,D_{\alpha,q}\,(b/a,t/a,z)\,d_q\sigma\,(t)\,d_q\sigma\,(z)\right)^{1/p}\\ &\times\left(\int_0^\infty\int_0^\infty |\psi\,(z)|^{p'}\,D_{\alpha,q}\,(b/a,t/a,z)\,d_q\sigma\,(t)\,d_q\sigma\,(z)\right)^{1/p'}\\ &\leq a^{-2\alpha-2}\left(\int_0^\infty |f\,(t)|^p\,d_q\sigma\,(t)\int_0^\infty D_{\alpha,q}\,(b/a,t/a,z)\,d_q\sigma\,(z)\right)^{1/p}\\ &\times\left(\int_0^\infty |\psi\,(z)|^{p'}\,d_q\sigma\,(z)\int_0^\infty D_{\alpha,q}\,(b/a,t/a,z)\,d_q\sigma\,(t)\right)^{1/p'}\\ &\leq a^{(2\alpha+2)/\left(1/p'-1\right)}\left(\int_0^\infty |f\,(t)|^p\,d_q\sigma\,(t)\right)^{1/p}\left(\int_0^\infty |\psi\,(z)|^{p'}\,d_q\sigma\,(z)\right)^{1/p'}.\end{aligned}$$

Thus

$$\left|\left(B_\psi^{\alpha,q} f\right)(b,a)\right| \leq a^{(2\alpha+2)\left(1/p'-1\right)}\,\|f\|_{p,\alpha,q}\,\|\psi\|_{p',\alpha,q}\,.$$

□

This proves (iii).
The inequality (ii) follows from Theorem (5.3).

7. An Inversion formula

Theorem 7.1. *Let $\psi\in L_{\alpha,q}^2\,(\mathbf{R}_{q,+})$ be a basic wavelet which defines the continuous q - Bessel wavelet transform (6.4). Then, for*

$$C_\psi^{\alpha,q} = \int_0^\infty \omega^{-2\alpha-2}\left|\hat{\psi}\,(\omega)\right|^2 d_q\sigma\,(\omega) > 0, \tag{7.1}$$

$$\int_0^\infty\int_0^\infty \left(B_\psi^{\alpha,q} f\right)(b,a)\,\overline{\left(B_\psi^{\alpha,q} g\right)(b,a)}a^{-2\alpha-2}d_q\sigma\,(a)\,d_q\sigma\,(b) = C_\psi^{\alpha,q}\,\langle f,g\rangle_{\alpha,q}\,,\ \ \forall\ f,g\in L_{\alpha,q}^2\,(\mathbf{R}_{q,+})\,. \tag{7.2}$$

Proof. Using the representation (6.4) we have

$$\begin{aligned}\left(B_{\psi}^{\alpha,q}f\right)(b,a) &= a^{-2\alpha-2}\int_0^\infty\int_0^\infty f(t)\,\overline{\psi(z)}D_{\alpha,q}\left(\frac{b}{a},\frac{t}{a},z\right)d_q\sigma(z)\,d_q\sigma(t)\\ &= a^{-2\alpha-2}\int_0^\infty\int_0^\infty\int_0^\infty f(t)\,\overline{\psi(z)}j_\alpha\left(\frac{bx}{a};q^2\right)j_\alpha\left(\frac{tx}{a};q^2\right)j_\alpha\left(zx;q^2\right)d_q\sigma(x)\,d_q\sigma(z)\,d_q\sigma(t)\\ &= a^{-2\alpha-2}\int_0^\infty\int_0^\infty \hat{f}_{\alpha,q}\left(\frac{x}{a}\right)\overline{\psi(z)}j_\alpha\left(\frac{bx}{a};q^2\right)j_\alpha\left(zx;q^2\right)d_q\sigma(x)\,d_q\sigma(z)\\ &= a^{-2\alpha-2}\int_0^\infty \hat{f}_{\alpha,q}\left(\frac{x}{a}\right)\overline{\hat{\psi}_{\alpha,q}(x)}j_\alpha\left(\frac{bx}{a};q^2\right)d_q\sigma(x)\\ &= \int_0^\infty \hat{f}(\xi)\,\overline{\hat{\psi}_{\alpha,q}(a\xi)}j_\alpha\left(b\xi;q^2\right)d_q\sigma(\xi)\\ &= \left(\hat{f}_{\alpha,q}(\xi)\,\overline{\hat{\psi}_{\alpha,q}(a\xi)}\right)^\wedge(b).\end{aligned}$$

Applying Parseval identity for q-Bessel Fourier transform, we have

$$\begin{aligned}&\int_0^\infty\left[\left(B_{\psi}^{\alpha,q}f\right)(b,a)\,\overline{\left(B_{\psi}^{\alpha,q}g\right)(b,a)}\right]d_q\sigma(b)\\ &= \int_0^\infty\left(\hat{f}_{\alpha,q}(\xi)\,\hat{\psi}_{\alpha,q}(a\xi)\right)^\wedge(b)\,\overline{\left(\hat{g}_{\alpha,q}(\xi)\,\hat{\psi}_{\alpha,q}(a\xi)\right)^\wedge(b)}d_q\sigma(b)\\ &= \int_0^\infty \hat{f}_{\alpha,q}(\xi)\,\overline{\hat{\psi}_{\alpha,q}(a\xi)}\overline{\hat{g}_{\alpha,q}(\xi)\,\overline{\hat{\psi}_{\alpha,q}(a\xi)}}d_q\sigma(\xi).\end{aligned}$$

Now multiplying by $a^{-2\alpha-2}d_q\sigma(a)$ and integrating, we get

$$\begin{aligned}&\int_0^\infty\int_0^\infty\left[\left(B_{\psi}^{\alpha,q}f\right)(b,a)\,\overline{\left(B_{\psi}^{\alpha,q}g\right)(b,a)}\right]a^{-2\alpha-2}d_q\sigma(a)\,d_q\sigma(b)\\ &= \int_0^\infty\left[\int_0^\infty \hat{f}_{\alpha,q}(\xi)\,\overline{\hat{\psi}_{\alpha,q}(a\xi)}\overline{\hat{g}_{\alpha,q}(\xi)\,\overline{\hat{\psi}_{\alpha,q}(a\xi)}}d_q\sigma(\xi)\right]a^{-2\alpha-2}d_q\sigma(a)\\ &= \int_0^\infty \hat{f}_{\alpha,q}(\xi)\,\overline{\hat{g}_{\alpha,q}(\xi)}d_q\sigma(\xi)\int_0^\infty \hat{\psi}_{\alpha,q}(a\xi)\,\overline{\hat{\psi}_{\alpha,q}(a\xi)}a^{-2\alpha-2}d_q\sigma(a)\\ &= \int_0^\infty \hat{f}_{\alpha,q}(\xi)\,\overline{\hat{g}_{\alpha,q}(\xi)}d_q\sigma(\xi)\int_0^\infty \left|\hat{\psi}_{\alpha,q}(a\xi)\right|^2 a^{-2\alpha-2}d_q\sigma(a)\\ &= \int_0^\infty \hat{f}_{\alpha,q}(\xi)\,\overline{\hat{g}_{\alpha,q}(\xi)}d_q\sigma(\xi)\int_0^\infty \left|\hat{\psi}_{\alpha,q}(\omega)\right|^2 \omega^{-2\alpha-2}d_q\sigma(\omega)\\ &= C_{\psi}^{\alpha,q}\langle f,g\rangle_{\alpha,q}.\end{aligned}$$

□

8. Discrete q-Bessel wavelet transform

In this section we assume that $\psi\in L^2_{\alpha,q}(\mathbf{R}_{q,+})$ satisfies the so called stability condition

$$P\le\sum_{m=-\infty}^{\infty}\left|\hat{\psi}\left(2^{-m}\xi\right)\right|^2\le Q\text{ a.e..}\tag{8.1}$$

for certain positive constants P and Q, $0<P\le Q<\infty$. Here $\hat{\psi}$ denotes the q-Bessel Fourier transform of ψ. The $\psi\in L^2_{\alpha,q}(\mathbf{R}_{q,+})$ satisfying (8.1) is called dyadic wavelet.
We define the semi-discrete q -Bessel wavelet transform by

$$\left(B_m^{\alpha,q,\psi}f\right)(b):=(2^m)^{2\alpha+2}\left(B_{\psi}^{\alpha,q}f\right)\left(b,\frac{1}{2^m}\right)\tag{8.2}$$

$$= \left(2^{m}\right)^{2\alpha+2} \int_0^\infty f(t)\, \overline{\psi^{\alpha,q}_{b,2^{-m}}(t)} d_q\sigma(t) \tag{8.3}$$

$$= 2^{m(2\alpha+2)} \left(f *_{\alpha,q} \psi_m\right)_{m\in\mathbf{Z}}. \tag{8.4}$$

Now, using the Parseval identity stability condition (8.1) yields the following

$$P\left\|f\right\|^2_{2,\alpha,q} \le \sum_{m=-\infty}^{\infty} \left\|B^{\alpha,q,\psi}_m f\right\|^2_{2,\alpha,q} \le Q\left\|f\right\|^2_2,\ f\in L^2\left(\mathbf{R}_+\right), \tag{8.5}$$

for the some constants P and Q.

Theorem 8.1. *Assume that the semi-discrete q-Bessel wavelet transform of any $f\in L^2_{\alpha,q}\left(\mathbf{R}_{q,+}\right)$ is defined by (8.3). Let us define another wavelet ψ^* by means of its q- Bessel Fourier transform:*

$$\hat{\psi}^*_{\alpha,q}(\xi) = \frac{\hat{\psi}_{\alpha,q}(\xi)}{\sum_{k=-\infty}^{\infty}\left|\hat{\psi}_{\alpha,q}\left(2^{-k}\xi\right)\right|^2}. \tag{8.6}$$

then

$$f(t) = \sum_{m=-\infty}^{\infty}\int_0^\infty \left(B^{\alpha,q,\psi}_m f\right)(b)\left(\hat{\psi}^*_{\alpha,q}\left(2^{-m}\xi\right) j_\alpha\left(tu;q^2\right)\right)^{\wedge}{}_{\alpha,q}(b)\, d_q\sigma(b). \tag{8.7}$$

Proof. In view of (8.1) and (8.3), for any $f\in L^2_{\alpha,q}\left(\mathbf{R}_{q,+}\right)$, we have

$$\sum_{m=-\infty}^{\infty}\int_0^\infty \left(B^{\alpha,q,\psi}_m f\right)(b)\left(\hat{\psi}^*_{\alpha,q}\left(2^{-m}\xi\right) j_\alpha\left(t\xi;q^2\right)\right)^{\wedge}{}_{\alpha,q}(b)\, d_q\sigma(b)$$

$$\begin{aligned}
&= \sum_{m=-\infty}^{\infty}\int_0^\infty \left(B^{\alpha,q,\psi}_m f\right)^{\wedge}{}_{\alpha,q}(\eta)\left(\hat{\psi}^*_{\alpha,q}\left(2^{-m}\eta\right) j_\alpha\left(t\xi;q^2\right)\right) j_\alpha\left(t\eta;q^2\right) d_q\sigma(\eta)\\
&= \sum_{m=-\infty}^{\infty}\int_0^\infty \left(\hat{f}_{\alpha,q}(\eta)\right)\overline{\left(\hat{\psi}^*_{\alpha,q}\left(2^{-m}\eta\right)\right)}\hat{\psi}^*_{\alpha,q}\left(2^{-m}\eta\right) j_\alpha\left(t\eta;q^2\right) d_q\sigma(\eta)\\
&= \sum_{m=-\infty}^{\infty}\int_0^\infty \left(\hat{f}_{\alpha,q}(\eta)\right)\overline{\left(\hat{\psi}^*_{\alpha,q}\left(2^{-m}\eta\right)\right)}\frac{\hat{\psi}_{\alpha,q}\left(2^{-m}\eta\right)}{\sum_{k=-\infty}^{\infty}\left|\hat{\psi}_{\alpha,q}\left(2^{-k}2^{-m}\eta\right)\right|^2} j_\alpha\left(t\eta;q^2\right) d_q\sigma(\eta)\\
&= \int_0^\infty \hat{f}_{\alpha,q}(\eta)\, j_\alpha\left(t\eta;q^2\right) d_q\sigma(\eta)\\
&= f(t).
\end{aligned}$$

The above theorem leads to the following definition of dyadic dual.

□

Definition 8.2. *A function $\tilde{\psi}\in L^2_{\alpha,q}\left(\mathbf{R}_{q,+}\right)$ is called a dyadic dual of a dyadic wavelet ψ if every $f\in L^2_{\alpha,q}\left(\mathbf{R}_{q,+}\right)$ can be expressed as*

$$f(t) = \sum_{m=-\infty}^{\infty}\int_0^\infty \left(B^{\alpha,q,\psi}_m f\right)(b)\left(\tilde{\psi}\left(2^{-m}\xi\right) j_\alpha\left(t\xi;q^2\right)^{\wedge}{}_{\alpha,q}(b)\right) d_q\sigma(b). \tag{8.8}$$

So far we have considered semi-discrete Bessel wavelet transform of any $f\in L^2_{\alpha,q}\left(\mathbf{R}_{q,+}\right)$ discretising only variable a. Now, we discretise the translation parameter b also by restricting it to the discrete set of points

$$b_{m,n} := \frac{n}{2^m}b_0,\ m\in\mathbf{Z},\ n\in\mathbf{N}_0. \tag{8.9}$$

where $b_0 > 0$ is a fixed constant.
We write

$$\psi_{b_0;m,n}^{\alpha,q}(t) := \psi_{b_{m,n},a_m}^{\alpha,q}(t) = 2^{m(2\alpha+2)}\psi_{\alpha,q}(nb_0, 2^m t). \tag{8.10}$$

Then the discrete Bessel wavelet transform of any$f \in L^2_{\alpha,q}(\mathbf{R}_+)$ can be written as

$$\left(B_\psi^{\alpha,q} f\right)(b_{m,n}, a_m) = \left\langle f, \psi_{b_0;m,n}^{\alpha,q}\right\rangle_{\alpha,q}, \ m \in \mathbf{Z}, \ n \in \mathbf{N}_0. \tag{8.11}$$

The stability condition for this reconstruction takes the form

$$P\|f\|^2_{2,\alpha,q} \le \sum_{\substack{m\in\mathbf{Z}\\ n\in\mathbf{N}_0}} \left|\left\langle f, \psi_{b_0;m,n}^{\alpha,q}\right\rangle_{\alpha,q}\right|^2 \le Q\|f\|^2_{2,\alpha,q}, \ f \in L^2_{\alpha,q}(\mathbf{R}_{q,+}), \tag{8.12}$$

for certain positive constants P and Q satisfying $0 < P \le Q < \infty$.

Theorem 8.3. *Assume that the discrete q-Bessel wavelet transform of any $f \in L^2_{\alpha,q}(\mathbf{R}_{q,+})$ is defined by (8.12) holds. Let T be a linear operator on $L^2_{\alpha,q}(\mathbf{R}_{q,+})$ defined by*

$$Tf = \sum_{\substack{m\in\mathbf{Z}\\ n\in\mathbf{N}_0}} \left\langle f, \psi_{b_0;m,n}^{\alpha,q}\right\rangle_{\alpha,q} \psi_{b_0;m,n}^{\alpha,q}, \tag{8.13}$$

then

$$f = \sum_{\substack{m\in\mathbf{Z}\\ n\in\mathbf{N}_0}} \left\langle f, \psi_{b_0;m,n}^{\alpha,q}\right\rangle_{\alpha,q} \psi^{m,n}_{\alpha,q,b_0}, \tag{8.14}$$

where

$$\psi^{m,n}_{\alpha,q,b_0} = T^{-1}\psi_{b_0;m,n}^{\alpha,q}, \ m \in \mathbf{Z}. \tag{8.15}$$

Proof. From the stability condition (8.12) it follows that defined by (8.13) is a one-one bounded linear operator.
Set

$$g = Tf, \ f \in L^2_{\alpha,q}(\mathbf{R}_{q,+}). \tag{8.16}$$

Then we have

$$\langle Tf, f\rangle_{\alpha,q} = \sum_{\substack{m\in\mathbf{Z}\\ n\in\mathbf{N}_0}} \left|\left\langle f, \psi_{b_0;m,n}^{\alpha,q}\right\rangle_{\alpha,q}\right|^2. \tag{8.17}$$

Therefore,

$$\begin{aligned} P\left\|T^{-1}g\right\|^2_{2,\alpha,q} &= P\|f\|^2_{2,\alpha,q}\langle Tf, f\rangle_{\alpha,q} \\ &= \left\langle g, T^{-1}g\right\rangle_{\alpha,q} \\ &\le \|g\|_{2,\alpha,q}\left\|T^{-1}g\right\|_{2,\alpha,q}, \end{aligned}$$

so that

$$\left\|T^{-1}g\right\|_{\alpha,q} \le \frac{1}{P}\|g\|_{2,\alpha,q}. \tag{8.18}$$

Hence, every $f \in L^2_{\alpha,q}(\mathbf{R}_{q,+})$ can be reconstructed from its discrete q -Bessel wavelet transform values given by (8.11).
Thus

$$f = T^{-1}Tf = \sum_{\substack{m\in\mathbf{Z}\\ n\in\mathbf{N}_0}} \left\langle f, \psi_{b_0;m,n}^{\alpha,q}\right\rangle_{\alpha,q} T^{-1}\psi_{b_0;m,n}^{\alpha,q}. \tag{8.19}$$

Finally, set

$$\psi_{\alpha,q,b_0}^{m,n} = T^{-1}\psi_{b_0;m,n}^{\alpha,q},\ m \in \mathbf{Z},\ n \in \mathbf{N}_0. \tag{8.20}$$

Then the reconstruction formula (8.19) can be expressed as follows:

$$f = \sum_{\substack{m\in\mathbf{Z}\\ n\in\mathbf{N}_0}} \left\langle f, \psi_{b_0;m,n}^{\alpha,q} \right\rangle_{\alpha,q} \psi_{\alpha,q,b_0}^{m,n}.$$

□

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