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Reliability Estimation of Lomax Distribution with Fuzziness

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ABSTRACT: This paper considers the problem of estimating the reliability function for Lomax distribution with the presence of fuzziness through two procedures. The first procedure depends on fuzzy reliability definition that uses the composite trapezoidal rule in order to find the numerical integration and the second is Bayesian procedure which includes different cases depends on sample data and hyper-parameters of prior gamma distribution with squared error as a symmetric loss function and precautionary as asymmetric loss function. In the Bayesian procedure, we proposed to consider three cases to estimate the fuzzy reliability with fuzzy observations, precise observations with fuzzy hyper-parameter, and fuzzy observations with fuzzy hyper-parameter.

Key Words: Fuzzy Reliability, Lomax Distribution, Gamma Distribution, composite trapezoidal rule.

Contents

1. Introduction

The reliability function is the most commonly used functions in lifetime data analysis. It gives the probability of a system " or unit or component " operating for a specified period of time without failure. The reliability of a system can be determined on the basis of the acquisition of operative data. However due to the uncertainty and inaccuracy of this data the estimation of precise values of probabilities is very difficult in many systems. For this reason, the fuzzy reliability concept has been introduced and formulated in the context of a fuzzy measure. The theory of the fuzzy set was pioneered by Zadeh (1965) who presented the concept of fuzzy set and fuzzy logic. Later, the theory and the mathematics of fuzzy sets were applied in many research fields (Singer (1990)).

In classical reliability theory, many methods and models assumed that all parameters of lifetime density function are precise "crisp". But in real situations, the lifetime data might be mixed up with the fuzziness. The present paper considers the problem of estimating the reliability function for Lomax distribution in the presence of fuzziness.

The rest of this paper is structured as follows. In Sec. 2, we briefly introduce Lomax distribution. In Sec. 3, The theory of fuzzy reliability is introduced. In Sec. 4, two procedures to estimate the fuzzy reliability of Lomax distribution are presented. Description of the simulation study and its results offered in Sec. 5. Finally, the conclusions are presented in Sec. 6.

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2. Lomax Distribution

For modeling business failure data, Lomax (1954) introduced and studied a new distribution called by his name as Lomax distribution "sometimes called Pareto type II or Pearson type VI". The Lomax distribution plays an important role in different applied areas especially with modeling and analyzing the lifetime data in engineering, biological and medical science (for more details see, Hassan and Al-Ghamdi (2009), Al-Zahrani and Al-Sobhi(2013), Al-Noor and Alwan (2015a,b), Kilany (2016) and Kumar et al. (2018).

If X is a Lomax non-negative random variable with shape parameter $\lambda > 0$ and scale parameter $\beta > 0$, then its cumulative distribution function (cdf) and probability density function (pdf) are respectively (Ashour and Abdelfattah (2011)) given by,

$$F\left(x;\lambda,\beta\right) = 1 - \left(1 + \frac{x}{\beta}\right)^{-\lambda} \tag{2.1}$$

$$f\left(x;\lambda,\beta\right) = \frac{\lambda}{\beta}\left(1 + \frac{x}{\beta}\right)^{-(\lambda+1)} \tag{2.2}$$

Setting $\beta = 1$ in (2.1) and (2.2), we obtain the cdf and pdf of the one parameter Lomax random variable given by,

$$F\left(x;\lambda\right) = 1 - \left(1 + x\right)^{-\lambda} \tag{2.3}$$

$$f\left(x;\lambda\right) = \lambda(1 + x)^{-(\lambda+1)} \tag{2.4}$$

Then the crisp reliability function of the one parameter Lomax random variable is given by,

$$R\left(x;\lambda\right) = 1 - F\left(x;\lambda\right) = \left(1 + x\right)^{-\lambda} \tag{2.5}$$

According to the importance of Lomax distribution, it has received extensive attention in the literature especially with the estimation of its reliability based on complete and censored data assumed that the available data are precise numbers "crisp data" (see, e.g., Okasha (2014), Panahi and Asadi (2011), Kim (2016), Mahmoud et al. (2016), Parviz (2016), Rao et al. (2016), and Yadav et al. (2019)). But, in real situations, the collected lifetime data might be not precise numbers but less or more fuzzy (see Jamkhaneh (2012) and Vishwakarma et al. (2018). So, this paper considers the problem of estimating the fuzzy reliability for Lomax distribution by two procedures. The first procedure depends on the fuzzy reliability definition and the second is the Bayesian procedure.

3. Fuzzy Reliability

The theory of fuzzy reliability was proposed and developed by several authors (see Chen and Mon (1993) and Venkatesh and Elango (2013). Let T be the continuous random variable represent the failure time of a system "or unit or component ". Then by using the formula of fuzzy probability, the fuzzy reliability can be obtained (Cheng (1996)) by,

$$\tilde{R}\left(t\right) = P\left(T \tilde{>} t\right) = \int_t^\infty \mu\left(x\right) f\left(x\right)\, dx \qquad ;\ 0 \le t \le x < \infty \tag{3.1}$$

where $\mu\left(x\right)$ is a membership function that represents, for every element of a given universe, the degree to which this element belongs to fuzzy set.

Now, assume that $\mu\left(x\right)$ is,

$$\mu\left(x\right) = \begin{cases} 0 & ; x \le t_1 \\ \frac{x - t_1}{t_2 - t_1} & ; t_1 < x < t_2 \quad , \ t_1 \ge 0 \\ 1 & ; x \ge t_2 \end{cases} \tag{3.2}$$

For $\mu\left(x\right)$, by the computational method of the function of fuzzy numbers, the lifetime $x\left(\alpha\right)$ can be obtained corresponds to a certain value of $\alpha - cut$, $\alpha \in [0, 1]$, (Cheng (1996)) as follows,

$\mu(x) = \alpha \rightarrow \frac{x-t_1}{t_2-t_1} = \alpha$, then

$$\begin{cases} x(\alpha) \leq t_1 & ;\alpha = 0 \\ x(\alpha) = t_1 + \alpha(t_2 - t_1) & ;0 < \alpha < 1 \\ x(\alpha) \geq t_2 & ;\alpha = 1 \end{cases} \tag{3.3}$$

Thus, the fuzzy reliability values can be obtained for all values of α as,

$$\tilde{R}(t)_{\alpha=0} = \int_{t_1}^{t_1} f(x)\ dx = 0 \tag{3.4}$$

$$\tilde{R}(t)_{0<\alpha<1} = \int_{t_1}^{x(\alpha)=t_1+\alpha(t_2-t_1)} f(x)\ dx \tag{3.5}$$

$$\tilde{R}(t)_{\alpha=1} = \int_{t_1}^{t_2} f(x)\ dx \tag{3.6}$$

4. Fuzzy Lomax Reliability

Here, we consider the definition of fuzzy reliability and the Bayesian procedure to estimate the fuzzy reliability of Lomax distribution. Assume that $f(x)$ in (3.1),(3.4),(3.5) and (3.6) represent the pdf of Lomax distribution as in (2.4), then,

First Procedure: According to (3.1), the fuzzy reliability definition $\tilde{R}(t)_\alpha = \int_{t_1}^{x(\alpha)} \mu(x)\,\lambda(1+x)^{-(\lambda+1)}dx$ where $\mu(x)$ as in (3.2)and $x(\alpha)$ as in (3.3), then,

$$\tilde{R}(t)_{\alpha=0} = \int_{t_1}^{x(\alpha=0)} 0\ \lambda\ (1+x)^{-(\lambda+1)}\ dx = 0 \tag{4.1}$$

$$\tilde{R}(t)\ _{0<\alpha<1} = \int_{t_1}^{x(0<\alpha<1)} \frac{x-t_1}{t_2-t_1}\lambda\ (1+x)^{-(\lambda+1)}\ dx \tag{4.2}$$

$$\tilde{R}(t)_{\alpha=1} = \int_{t_1}^{x(\alpha=1)} 1\ \lambda\ (1+x)^{-(\lambda+1)}\ dx = (1+t_1)^{-\lambda} - (1+t_2)^{-\lambda} \tag{4.3}$$

and λ represented by its maximum likelihood estimator that can be obtained from the likelihood function $L(\lambda|\underline{x})$,

$$L(\lambda|\underline{x}) = \prod_{i=1}^{n} f(x_i;\lambda) = \lambda^n\ e^{-(\lambda+1)\sum_{i=1}^{n} ln(1+x_i)} \tag{4.4}$$

Taking the natural logarithm, $L(\lambda|\underline{x})$ will be,

$$l(\lambda|\underline{x}) = ln\ L(\lambda|\underline{x}) = n\ln\lambda - (\lambda+1)\sum_{i=1}^{n}\ln(1+x_i) \tag{4.5}$$

Differentiating the natural log-likelihood function, $l(\lambda|\underline{x})$, partially with respect to λ and then equating to zero we get the maximum likelihood estimator of λ by, $\hat{\lambda} = \frac{n}{\sum_{i=1}^{n}\ln(1+x_i)}$.

The numerical integration in (4.2) obtained by using the composite trapezoidal (CT) rule (Dahlquist and Björck (2008)), so, its symbolized by $\tilde{R}(t)_{0<\alpha<1}^{CT}$.

Second Procedure: According to the Bayesian procedure, assume that λ has a gamma prior distribution, $g(\lambda)$, with hyper-parameters a and b, then the posterior distribution can be obtained as,

$$\pi\left(\lambda|\underline{x}\right)=\frac{L(\lambda|\underline{x})g(\lambda)}{\int_0^\infty L(\lambda|\underline{x})g(\lambda)d\lambda}=\frac{\lambda^n\ e^{-(\lambda+1)\sum_{i=1}^n ln(1+x_i)}\ \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-\lambda b}}{\int_0^\infty \lambda^n\ e^{-(\lambda+1)\sum_{i=1}^n ln(1+x_i)}\ \frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-\lambda b}d\lambda}$$

$$=\frac{\lambda^{n+a-1}\ e^{-\lambda\left(b+\sum_{i=1}^n ln(1+x_i)\right)}}{\int_0^\infty \lambda^{n+a-1}\ e^{-\lambda\left(b+\sum_{i=1}^n ln(1+x_i)\right)}d\lambda}$$

By using the transformation $u=\lambda\left(b+\sum_{i=1}^n ln\left(1+x_i\right)\right)\rightarrow\lambda=\frac{u}{b+\sum_{i=1}^n ln(1+x_i)}$ and $d\lambda=\frac{du}{b+\sum_{i=1}^n ln(1+x_i)}$, then,

$$\int_0^\infty \lambda^{n+a-1}\ e^{-\lambda\left(b+\sum_{i=1}^n ln(1+x_i)\right)}d\lambda=\int_0^\infty\left(\frac{u}{b+\sum_{i=1}^n \ln(1+x_i)}\right)^{n+a-1}\ e^{-u}\ \frac{du}{b+\sum_{i=1}^n \ln(1+x_i)}$$

$$=\frac{1}{\left(b+\sum_{i=1}^n \ln(1+x_i)\right)^{n+a}}\int_0^\infty u^{n+a-1}\ e^{-u}\ du$$

$$=\frac{1}{\left(b+\sum_{i=1}^n ln(1+x_i)\right)^{n+a}}\Gamma\left(n+a\right)$$

The posterior distribution of λ will be,

$$\pi\left(\lambda|\underline{x}\right)=\frac{\left(b+\sum_{i=1}^n ln\left(1+x_i\right)\right)^{n+a}}{\Gamma\left(n+a\right)}\lambda^{n+a-1}e^{-\lambda\left(b+\sum_{i=1}^n ln(1+x_i)\right)}$$

Now according to(3.4),(3.5) and (3.6), we get,

$$\tilde{R}\left(t\right)_{\alpha=0}=0 \tag{4.6}$$

$$\tilde{R}\left(t\right)_{0<\alpha<1}=\left(1+t_1\right)^{-\lambda}-\left(1+x\left(\alpha\right)\right)^{-\lambda}\ ;\ x\left(\alpha\right)=t_1+\alpha(t_2-t_1) \tag{4.7}$$

$$\tilde{R}\left(t\right)_{\alpha=1}=\left(1+t_1\right)^{-\lambda}-\left(1+t_2\right)^{-\lambda} \tag{4.8}$$

With the squared error loss as a symmetric loss function, the Bayes estimator of fuzzy reliability is obtained as follows,

$$\tilde{R}\left(t\right)_\alpha^{BS}=E\left(\tilde{R}\left(t\right)_\alpha\right)=\int_0^\infty\tilde{R}\left(t\right)_\alpha\ \pi\left(\lambda|\underline{x}\right)d\lambda$$

$$=\frac{\left(b+\sum_{i=1}^n \ln(1+x_i)\right)^{n+a}}{\Gamma(n+a)}\int_0^\infty\left[\left(1+t_1\right)^{-\lambda}-\left(1+x\left(\alpha\right)\right)^{-\lambda}\right]\lambda^{n+a-1}e^{-\lambda\left(b+\sum_{i=1}^n \ln(1+x_i)\right)}d\lambda$$

$$=\frac{\left(b+\sum_{i=1}^n \ln(1+x_i)\right)^{n+a}}{\Gamma(n+a)}\left[\int_0^\infty\lambda^{n+a-1}e^{-\lambda\left(b+\sum_{i=1}^n \ln(1+x_i)+\ln(1+t_1)\ \right)}d\lambda-\right.$$

$$\left.\int_0^\infty\lambda^{n+a-1}e^{-\lambda\left(b+\sum_{i=1}^n \ln(1+x_i)+\ln(1+x(\alpha))\ \right)}d\lambda\right]$$

Again using the transformation $u^*=\lambda\left(b+\sum_{i=1}^n \ln\left(1+x_i\right)+\ln\left(1+t_1\right)\ \right)$, we get that,

$$\int_0^\infty\lambda^{n+a-1}e^{-\lambda\left(b+\sum_{i=1}^n ln(1+x_i)+\ln(1+t_1)\ \right)}d\lambda=\frac{1}{\left(b+\sum_{i=1}^n \ln\left(1+x_i\right)+\ln\left(1+t_1\right)\right)^{n+a}}\Gamma\left(n+a\right)$$

and using the transformation $u^{**}=\lambda\left(b+\sum_{i=1}^n \ln\left(1+x_i\right)+\ln\left(1+x\left(\alpha\right)\right)\ \right)$, we get that,

$$\int_0^\infty\lambda^{n+a-1}e^{-\lambda\left(b+\sum_{i=1}^n \ln(1+x_i)+\ln(1+x(\alpha))\ \right)}d\lambda=\frac{1}{\left(b+\sum_{i=1}^n \ln\left(1+x_i\right)+\ln\left(1+x\left(\alpha\right)\right)\right)^{n+a}}\Gamma\left(n+a\right)$$

Then,

$$\tilde{R}(t)^{BS}_{0<\alpha<1} = \frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\left(b+\sum_{i=1}^{n}\ln(1+x_i)+\ln(1+t_1)\right)^{n+a}} - \frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\left(b+\sum_{i=1}^{n}\ln(1+x_i)+\ln(1+x(\alpha))\right)^{n+a}} \quad (4.9)$$

$$\tilde{R}(t)^{BS}_{\alpha=1} = \frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\left(b+\sum_{i=1}^{n}\ln(1+x_i)+\ln(1+t_1)\right)^{n+a}} - \frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\left(b+\sum_{i=1}^{n}\ln(1+x_i)+\ln(1+t_2)\right)^{n+a}} \quad (4.10)$$

With the precautionary loss as asymmetric loss function, the Bayes estimator of fuzzy reliability is obtained as follows,

$$\tilde{R}(t)^{BP}_{\alpha} = \sqrt{E\left[\left(\tilde{R}(t)_{\alpha}\right)^2\right]},$$

where,

$$E\left[\left(\tilde{R}(t)_{\alpha}\right)^2\right] = \int_0^{\infty}\left(\tilde{R}(t)_{\alpha}\right)^2 \pi(\lambda|\underline{x})\ d\lambda$$

$$= \frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\Gamma(n+a)} \int_0^{\infty}\left[(1+t_1)^{-\lambda} - (1+x(\alpha))^{-\lambda}\right]^2 \lambda^{n+a-1} e^{-\lambda\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)} d\lambda$$

$$= \frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\Gamma(n+a)} \left[\begin{array}{c} \int_0^{\infty}\lambda^{n+a-1} e^{-\lambda\left(b+\sum_{i=1}^{n}\ln(1+x_i)+2\ln(1+t_1)\right)} d\lambda \\ -2\int_0^{\infty}\lambda^{n+a-1} e^{-\lambda\left(b+\sum_{i=1}^{n}\ln(1+x_i)+\ln(1+t_1)+\ln(1+x(\alpha))\right)} d\lambda \\ +\int_0^{\infty}\lambda^{n+a-1} e^{-\lambda\left(b+\sum_{i=1}^{n}\ln(1+x_i)+2\ln(1+x(\alpha))\right)} d\lambda \end{array}\right]$$

Using the transformation $w = \lambda\left(b+\sum_{i=1}^{n}\ln(1+x_i)+2\ln(1+t_1)\right)$, we get that,

$$A\int_0^{\infty}\lambda^{n+a-1} e^{-\lambda\left(b+\sum_{i=1}^{n}\ln(1+x_i)+2\ln(1+t_1)\right)} d\lambda = \frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\left(b+\sum_{i=1}^{n}\ln(1+x_i)+2\ln(1+t_1)\right)^{n+a}}$$

Using the transformation $w^* = \lambda\left(b+\sum_{i=1}^{n}\ln(1+x_i)+\ln(1+t_1)+\ln(1+x(\alpha))\right)$, we get that,

$$A\int_0^{\infty}\lambda^{n+a-1} e^{-\lambda\left(b+\sum_{i=1}^{n}\ln(1+x_i)+\ln(1+t_1)+\ln(1+x(\alpha))\right)} d\lambda = \frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\left(b+\sum_{i=1}^{n}\ln(1+x_i)+\ln(1+t_1)+\ln(1+x(\alpha))\right)^{n+a}}$$

Using the transformation $w^{**} = \lambda\left(b+\sum_{i=1}^{n}\ln(1+x_i)+2\ln(1+x(\alpha))\right)$, we get that,

$$A\int_0^{\infty}\lambda^{n+a-1} e^{-\lambda\left(b+\sum_{i=1}^{n}\ln(1+x_i)+2\ln(1+x(\alpha))\right)} d\lambda = \frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\left(b+\sum_{i=1}^{n}\ln(1+x_i)+2\ln(1+x(\alpha))\right)^{n+a}}$$

Then,

$$\tilde{R}(t)^{BP}_{0<\alpha<1} = \left(\begin{array}{c} \frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\left(b+\sum_{i=1}^{n}\ln(1+x_i)+2\ln(1+t_1)\right)^{n+a}} + \frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\left(b+\sum_{i=1}^{n}\ln(1+x_i)+2\ln(1+x(\alpha))\right)^{n+a}} \\ -2\frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\left(b+\sum_{i=1}^{n}\ln(1+x_i)+\ln(1+t_1)+\ln(1+x(\alpha))\right)^{n+a}} \end{array}\right)^{\frac{1}{2}} \quad (4.11)$$

$$\tilde{R}(t)^{BP}_{\alpha=1} = \left(\begin{array}{c} \frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\left(b+\sum_{i=1}^{n}\ln(1+x_i)+2\ln(1+t_1)\right)^{n+a}} + \frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\left(b+\sum_{i=1}^{n}\ln(1+x_i)+2\ln(1+t_2)\right)^{n+a}} \\ -2\frac{\left(b+\sum_{i=1}^{n}\ln(1+x_i)\right)^{n+a}}{\left(b+\sum_{i=1}^{n}\ln(1+x_i)+\ln(1+t_1)+\ln(1+t_2)\right)^{n+a}} \end{array}\right)^{\frac{1}{2}} \quad (4.12)$$

For Bayesian procedure, with (4.9),(4.10),(4.11),(4.12), we consider the following three cases to estimate the fuzzy reliability with,

1. Fuzzy observations, $x_i = \mu(x_i)$,
2. Precise observations and fuzzy hyper-parameter $b = b(\alpha)$ with $\mu(b) = \alpha \to \frac{b-b_1}{b_2-b_1} = \alpha$, where b_1 and b_2 represents the lower and upper confidence interval of b , and then we have,

$$\begin{cases} b(\alpha) \leq b_1 & ;\alpha = 0 \\ b(\alpha) = b_1 + \alpha(b_2 - b_1) & ;0 < \alpha < 1 \\ b(\alpha) \geq b_2 & ;\alpha = 1 \end{cases} \tag{4.13}$$

3. Fuzzy observations, $x_i = \mu(x_i)$ and fuzzy hyper-parameter, $b = b(\alpha)$.

5. Simulation Study

We performed a simulation in order to investigate the performance of all estimates for fuzzy reliability. We generated 5000 samples of size $n = 10, 25, 50$ from the Lomax distribution through the adoption of the inverse transformation method with $\lambda < 1$ and $\lambda > 1$ where $\lambda = 0.5$ and 2. The values of t_1 and t_2 are taken to be 1 and 4 respectively. The real values of fuzzy reliability with $\lambda = 0.5$ and 2 are equal to 0.2599 and 0.2100 respectively. The hyper-parameters are chosen to be $(a,\ b) = (1,1),\ (1,3),(3,1)$ and $(3,3)$. The values of $\alpha - cut$ are taken to be 0.2, 0.6 and 1.The results of the average mean square error (MSE) values are given in Tables 1, 2, 3 and 4.

From the results, it appears that, for all estimates, the values of the MSE decreased as the $\alpha - cut$ increased. For the Bayes estimates, generally, the values of the MSE decreased as the b increased or both hyper-parameter $a,\ b$ increased together, while the values of the MSE increased when only the a increased. Also, for all estimates, the values of the MSE decreased as the value of λ increased.

Table 1: MSE values of CT estimate of fuzzy reliability

λ	**α-cut**	**n=10**	**n=25**	**n=50**
0.5	0.2	0.0672	0.0667	0.0658
	0.6	0.0655	0.0621	0.0566
	1	0.0614	0.0518	0.0376
2	0.2	0.0439	0.0436	0.0431
	0.6	0.0403	0.0344	0.0256
	1	0.0386	0.0302	0.0185

Table 2: MSE values of Bayes estimates of fuzzy reliability for n=10

λ	**a,b**	**α-cut**	**BS1**	**BS2**	**BS3**	**BP1**	**BP2**	**BP3**
0.5	1,1	0.2	0.0382	0.0239	0.0386	0.0377	0.0238	0.0381
		0.6	0.0253	0.0106	0.0247	0.0243	0.0103	0.0237
		1	0.0224	0.0102	0.0204	0.0212	0.0100	0.0193
	1,3	0.2	0.0342	0.0237	0.0352	0.0338	0.0236	0.0348
		0.6	0.0193	0.0102	0.0178	0.0185	0.0101	0.0171
		1	0.0157	0.0094	0.0114	0.0149	0.0094	0.0107
	3,1	0.2	0.0397	0.0240	0.0409	0.0392	0.0239	0.0396
		0.6	0.0276	0.0109	0.0271	0.0266	0.0106	0.0261
		1	0.0250	0.0103	0.0231	0.0238	0.0103	0.0220
	3,3	0.2	0.0357	0.0225	0.0366	0.0352	0.0224	0.0362
		0.6	0.0214	0.0104	0.0200	0.0206	0.0102	0.0193
		1	0.0181	0.0096	0.0139	0.0171	0.0095	0.0131

Table 2 continued from previous page

λ	**a,b**	**α-cut**	**BS1**	**BS2**	**BS3**	**BP1**	**BP2**	**BP3**
2	1,1	0.2	0.0285	0.0209	0.0289	0.0279	0.0208	0.0283
		0.6	0.0227	0.0086	0.0220	0.0217	0.0086	0.0211
		1	0.0216	0.0077	0.0197	0.0205	0.0077	0.0185
	1,3	0.2	0.0243	0.0199	0.0254	0.0238	0.0198	0.0248
		0.6	0.0165	0.0086	0.0150	0.0157	0.0086	0.0142
		1	0.0150	0.0078	0.0104	0.0140	0.0078	0.0097
	3,1	0.2	0.0296	0.0191	0.0300	0.0290	0.0190	0.0294
		0.6	0.0245	0.0076	0.0239	0.0234	0.0076	0.0229
		1	0.0235	0.0075	0.0218	0.0223	0.0075	0.0206
	3,3	0.2	0.0255	0.0192	0.0265	0.0250	0.0192	0.0259
		0.6	0.0183	0.0086	0.0169	0.0174	0.0086	0.0161
		1	0.0169	0.0077	0.0126	0.0159	0.0077	0.0117

Table 3: MSE values of Bayes estimates of fuzzy reliability for n=25

λ	**a,b**	**α-cut**	**BS1**	**BS2**	**BS3**	**BP1**	**BP2**	**BP3**
0.5	1,1	0.2	0.0370	0.0228	0.0377	0.0361	0.0227	0.0368
		0.6	0.0233	0.0097	0.0215	0.0217	0.0097	0.0200
		1	0.0199	0.0089	0.0150	0.0180	0.0088	0.0134
	1,3	0.2	0.0308	0.0226	0.0324	0.0303	0.0226	0.0317
		0.6	0.0142	0.0095	0.0111	0.0132	0.0093	0.0103
		1	0.0101	0.0088	0.0039	0.0090	0.0087	0.0034
	3,1	0.2	0.0396	0.0229	0.0407	0.0389	0.0228	0.0394
		0.6	0.0275	0.0098	0.0261	0.0260	0.0098	0.0244
		1	0.0249	0.0090	0.0201	0.0228	0.0090	0.0183
	3,3	0.2	0.0333	0.0219	0.0350	0.0327	0.0219	0.0343
		0.6	0.0177	0.0096	0.0145	0.0165	0.0095	0.0136
		1	0.0140	0.0089	0.0067	0.0127	0.0088	0.0060
2	1,1	0.2	0.0268	0.0194	0.0276	0.0258	0.0196	0.0265
		0.6	0.0199	0.0085	0.0180	0.0183	0.0085	0.0165
		1	0.0185	0.0072	0.0136	0.0166	0.0073	0.0119
	1,3	0.2	0.0202	0.0190	0.0219	0.0196	0.0195	0.0212
		0.6	0.0106	0.0084	0.0075	0.0095	0.0084	0.0067
		1	0.0086	0.0071	0.0028	0.0074	0.0071	0.0024
	3,1	0.2	0.0291	0.0191	0.0299	0.0281	0.0190	0.0288
		0.6	0.0233	0.0075	0.0217	0.0215	0.0075	0.0200
		1	0.0222	0.0069	0.0177	0.0202	0.0070	0.0159
	3,3	0.2	0.0224	0.0193	0.0242	0.0217	0.0193	0.0233
		0.6	0.0135	0.0075	0.0104	0.0123	0.0075	0.0095
		1	0.0117	0.0072	0.0049	0.0104	0.0071	0.0042

Table 4: MSE values of Bayes estimates of fuzzy reliability for n=50

λ	a,b	α-cut	BS1	BS2	BS3	BP1	BP2	BP3
0.5	1,1	0.2	0.0356	0.0220	0.0366	0.0340	0.0219	0.0350
		0.6	0.0203	0.0069	0.0148	0.0176	0.0067	0.0129
		1	0.0168	0.0059	0.0073	0.0137	0.0059	0.0059
	1,3	0.2	0.0278	0.0222	0.0291	0.0271	0.0222	0.0283
		0.6	0.0093	0.0053	0.0061	0.0082	0.0051	0.0056
		1	0.0061	0.0045	0.0056	0.0049	0.0043	0.0047
	3,1	0.2	0.0393	0.0226	0.0405	0.0375	0.0226	0.0381
		0.6	0.0273	0.0078	0.0247	0.0254	0.0076	0.0222
		1	0.0248	0.0061	0.0163	0.0225	0.0060	0.0141
	3,3	0.2	0.0326	0.0218	0.0346	0.0317	0.0198	0.0335
		0.6	0.0166	0.0067	0.0104	0.0149	0.0065	0.0095
		1	0.0136	0.0056	0.0047	0.0117	0.0054	0.0042
2	1,1	0.2	0.0240	0.0099	0.0250	0.0223	0.0098	0.0233
		0.6	0.0158	0.0062	0.0107	0.0131	0.0061	0.0087
		1	0.0139	0.0057	0.0057	0.0108	0.0058	0.0045
	1,3	0.2	0.0159	0.0105	0.0173	0.0152	0.0104	0.0164
		0.6	0.0058	0.0052	0.0055	0.0047	0.0050	0.0051
		1	0.0048	0.0041	0.0028	0.0038	0.0042	0.0019
	3,1	0.2	0.0291	0.0098	0.0298	0.0275	0.0097	0.0286
		0.6	0.0233	0.0063	0.0182	0.0205	0.0062	0.0159
		1	0.0219	0.0050	0.0119	0.0188	0.0051	0.0101
	3,3	0.2	0.0201	0.0099	0.0220	0.0192	0.0098	0.0209
		0.6	0.0111	0.0053	0.0062	0.0097	0.0055	0.0054
		1	0.0098	0.0050	0.0032	0.0083	0.0051	0.0031

6. Conclusions

In this paper, we have discussed two estimation procedures for the fuzzy reliability of the Lomax distribution. The first procedure depends on the fuzzy reliability definition that uses the composite trapezoidal rule in order to find the numerical integration and the second is Bayesian procedure with informative gamma prior based on squared error and precautionary ”as symmetric and asymmetric” loss functions.
In Bayesian, we have proposed to consider three different cases:

1. when the data are available in the form of fuzzy information ”fuzzy observations”,
2. precise observations with fuzzy hyper-parameter and
3. fuzzy observations and fuzzy hyper-parameter.

From the simulation study, with respect to minimum values of MSE, although it appears that the performance of Bayes estimates with case 2 ”under precautionary (BP2) and squared error (BS2)” better than the other with cases 1 and 2, one cannot say in some absolute sense that this two estimates are superior than the other due to its deal only with fuzzy hyper-parameter not fuzzy observations. In terms of overall comparison, the performance of the Bayes estimates is generally best and especially with precautionary than squared error loss.

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