Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2023 (41)** : 1–12.
ISSN-0037-8712 in press
doi:10.5269/bspm.52192

Further Results on Strong λ-statistical Convergence of Sequences in Probabilistic Metric Spaces

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ABSTRACT: In this paper we study some basic properties of strong λ-statistical convergence of sequences in probabilistic metric spaces. Also introducing the concept of strong λ-statistically Cauchy sequences we study its relationship with strong λ-statistical convergence in a probabilistic metric space. Further introducing the notions of strong λ-statistical limit point and strong λ-statistical cluster point of a sequence in a probabilistic metric space we examine their interrelationship.

Key Words: Probabilistic metric space, λ-density, strong λ-statistical convergence, strong λ-statistical Cauchyness, strong λ-statistical limit point, strong λ-statistical cluster point.

Contents

1. Introduction

The concept of probabilistic metric (PM) space was introduced and studied by Menger [12] under the name of "statistical metric space" by taking the distance between two points a and b as a distribution function F_{ab} instead of a non-negative real number and the value of the function F_{ab} at any $t > 0$ i.e. $F_{ab}(t)$ is interpreted as the probability that the distance between the points a and b is $\leq t$. After Menger, works of several mathematicians such as Schwiezer and Sklar [18,19,20,21], Tardiff [25], Thorp [26] and many others, developed the theory of probabilistic metric spaces. A through development of probabilistic metric spaces can be found in the famous book of Schwiezer and Sklar [22]. Several topologies are defined on a PM space but strong topology is one of them, which received most attention to date and it is the main tool of our paper.

As a generalization of the usual notion of convergence of sequences of real numbers, the notion of statistical convergence was introduced and studied independently by Fast [7] and Schoenberg [17] based on the notion of natural density of subsets of $\mathbb{N}$, the set of all natural numbers. A subset $\mathcal{M}$ of $\mathbb{N}$ is said to have natural density or asymptotic density $d(\mathcal{M})$ if

$$d(\mathcal{M}) = \lim_{n\to\infty} \frac{|\mathcal{M}(n)|}{n}$$

where $\mathcal{M}(n) = \{j \in \mathcal{M} : j \leq n\}$ and $|\mathcal{M}(n)|$ represents the number of elements in $\mathcal{M}(n)$.

A sequence $\{x_k\}_{k\in\mathbb{N}}$ of reals is said to be statistically convergent to $\xi \in \mathbb{R}$ if for every $\epsilon > 0$, $d(A(\epsilon)) = 0$, where $A(\epsilon) = \{k \in \mathbb{N} : |x_k - \xi| \geq \epsilon\}$.

After the great works of Salat [15] and Fridy [8], many works have been done in this area of summability theory. More primary work on this convergence notion can be found from [1,2,3,9,10,14,24].

The notion of natural density of subsets of $\mathbb{N}$ was further generalized to the notion of λ-density by Mursaleen [13] and with the help of λ-density he generalized the notion of statistical convergence of real sequences to the notion of λ-statistical convergence. If $\lambda = \{\lambda_n\}_{n\in\mathbb{N}}$ is a non-decreasing sequence of

2010 *Mathematics Subject Classification:* 54E70, 40A99.
Submitted February 14, 2020. Published November 19, 2020

positive real numbers tending to ∞ such that $\lambda_1 = 1,\ \lambda_{n+1} \le \lambda_n + 1,\ n \in \mathbb{N}$, then any subset $\mathcal{M}$ of $\mathbb{N}$ is said to have λ-density $d_\lambda(\mathcal{M})$ if

$$d_\lambda(\mathcal{M}) = \lim_{n\to\infty} \frac{|\{k \in I_n : k \in \mathcal{M}\}|}{\lambda_n},$$

where $I_n = [n - \lambda_n + 1, n]$. It is clear that if $A, B \subset \mathbb{N}$ and $d_\lambda(A) = 0$, $d_\lambda(B) = 0$ then $d_\lambda(A^c) = 1 = d_\lambda(B^c)$, $d_\lambda(A \cup B) = 0$. Also if $A, B \subset \mathbb{N}$, $A \subset B$ and $d_\lambda(B) = 0$, then $d_\lambda(A) = 0$. The collection of all such sequences λ is denoted by Δ_∞. Throughout the paper λ stands for such a sequence.

If a sequence $x = \{x_k\}_{k\in\mathbb{N}}$ satisfies a property $\mathfrak{P}$ for each k except for a set of λ-density zero, then we say that the sequence x satisfies the property $\mathfrak{P}$ for "λ-almost all k" or in short "λ-*a.a.k.*".

A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ of real numbers is said to be λ-statistically convergent or S_λ-convergent to $\mathcal{L} \in \mathbb{R}$ if, for every $\epsilon > 0$, $d_\lambda(M(\epsilon)) = 0$, where $M(\epsilon) = \{k \in \mathbb{N} : |x_k - \mathcal{L}| \ge \epsilon\}$.

If $\lambda_n = n$, $\forall n \in \mathbb{N}$, then the notions of λ-density and λ-statistical convergence coincide with the notions of natural density and statistical convergence respectively.

Because of immense importance of probabilistic metric space in mathematics, the notion of strong statistical convergence was introduced by Şençimen et al. [23] in a PM space and this was further generalized to the notion of strong λ-statistical convergence by Das et al. [4].

Following the line of Şençimen et al. [23] and also that of Das et al. [4] in this paper we study some basic properties of strong λ-statistical convergence of sequences in probabilistic metric spaces not done earlier. We also introduce the notion of strong λ-statistically Cauchy sequences and study some of its basic properties including its relationship with strong λ-statistical convergence in a probabilistic metric space. Further in section 4 of this paper we introduce and study the notions of strong λ-statistical limit points and strong λ-statistical cluster points of a sequence in a probabilistic metric space including their interrelationship.

2. Basic Definitions and Notations

In this section we recall some basic definitions and results related to probabilistic metric spaces (or PM spaces) (see [18,19,20,21,22] for more details).

Definition 2.1. *A nondecreasing function $f : [-\infty, \infty] \to [0, 1]$ with $f(-\infty) = 0$ and $f(\infty) = 1$, is called a distribution function.*

We denote the set of all distribution functions left continuous over $(-\infty, \infty)$ by $\mathcal{D}$.

We consider a relation $\le$ on $\mathcal{D}$ defined by $g \le f$ if and only if $g(x) \le f(x)$ for all $x \in [-\infty, \infty]$. Clearly the relation '$\le$' is a partial order on $\mathcal{D}$.

Definition 2.2. *For any $q \in [-\infty, \infty]$ the unit step at q is denoted by ϵ_q and is defined to be a function in $\mathcal{D}$ given by*

$$\begin{aligned} \epsilon_q(x) &= 0, \text{ if } -\infty \le x \le q \\ &= 1, \text{ if } q < x \le \infty. \end{aligned}$$

Definition 2.3. *A sequence $\{f_n\}_{n\in\mathbb{N}}$ of distribution functions is said to converge weakly to a distribution function f, if the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ converges to $f(x)$ at each continuity point x of f. We write $f_n \xrightarrow{w} f$.*

Definition 2.4. *The distance between f and g in $\mathcal{D}$ is denoted by $d_L(f, g)$ and is defined by the infimum of all numbers $w \in (0, 1]$ such that the inequalities*

$$f(x - w) - w \le g(x) \le f(x + w) + w$$
$$\text{and } g(x - w) - w \le f(x) \le g(x + w) + w$$

hold for every $x \in (-\frac{1}{w}, \frac{1}{w})$.

It is known that d_L is a metric on $\mathcal{D}$ and for any sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{D}$ and $f \in \mathcal{D}$, we have

$$f_n \xrightarrow{w} f \text{ if and only if } d_L(f_n, f) \to 0.$$

Definition 2.5. *A nondecreasing real valued function h defined on $[0,\infty]$ that satisfies $h(0) = 0$ and $h(\infty) = 1$ and is left continuous on $(0,\infty)$ is called a distance distribution function.*

The set of all distance distribution functions is denoted by $\mathcal{D}^+$. The function d_L is clearly a metric on $\mathcal{D}^+$. The metric space $(\mathcal{D}^+, d_L)$ is compact and hence complete.

Theorem 2.6. *Let $f \in \mathcal{D}^+$ be given . Then for any $t > 0$, $f(t) > 1 - t$ if and only if $d_L(f, \epsilon_0) < t$.*

Definition 2.7. *A triangle function is a binary operation τ on $\mathcal{D}^+$ which is commutative, nondecreasing, associative in each place and ϵ_0 is the identity.*

Definition 2.8. *A probabilistic metric space, briefly PM space, is a triplet $(X, \mathcal{F}, \tau)$, where X is a nonempty set whose elements are the points of the space, $\mathcal{F}$ is a function from $X \times X$ into $\mathcal{D}^+$, τ is a triangle function and the following conditions are satisfied for all $x, y, z \in X$:*

1. $\mathcal{F}(x, x) = \epsilon_0$
2. $\mathcal{F}(x, y) \neq \epsilon_0$ *if* $x \neq y$
3. $\mathcal{F}(x, y) = \mathcal{F}(y, x)$
4. $\mathcal{F}(x, z) \geq \tau(\mathcal{F}(x, y), \mathcal{F}(y, z))$.

From now on we will denote $\mathcal{F}(x, y)$ by $\mathcal{F}_{xy}$ and its value at b by $\mathcal{F}_{xy}(b)$.

Definition 2.9. *Let $(X, \mathcal{F}, \tau)$ be a PM space. For $x \in X$ and $r > 0$, the strong r-neighborhood of x is denoted by $\mathcal{N}_x(r)$ and is defined by*

$$\mathcal{N}_x(r) = \{y \in X : \mathcal{F}_{xy}(r) > 1 - r\}.$$

The collection $\mathfrak{N}_x = \{\mathcal{N}_x(r) : r > 0\}$ is called the strong neighborhood system at x and the union $\mathfrak{N} = \bigcup_{x \in X} \mathfrak{N}_x$ is called the strong neighborhood system for X.

From Theorem 2.6, we can write $\mathcal{N}_x(r) = \{y \in X : d_L(\mathcal{F}_{xy}, \epsilon_0) < r\}$. If τ is continuous, then the strong neighborhood system $\mathfrak{N}$ determines a Hausdorff topology for X. This topology is called the strong topology for X and members of this topology are called strongly open sets.

Throughout the paper, in a PM space $(X, \mathcal{F}, \tau)$, we always consider that τ is continuous and X is endowed with the strong topology.

In a PM space $(X, \mathcal{F}, \tau)$ the strong closure of any subset A of X is denoted by $k(A)$ and for any nonempty subset A of X strong closure of A is defined by,

$$k(A) = \{a \in X : \text{ for any } t > 0, \ \exists\, b \in A \text{ such that } \mathcal{F}_{ab}(t) > 1 - t\}.$$

Definition 2.10. *[6] Let $(X, \mathcal{F}, \tau)$ be a PM space. Then a subset H of X is called strongly closed if its complement is a strongly open set.*

Definition 2.11. *Let $(X, \mathcal{F}, \tau)$ be a PM space and $H \neq \emptyset$ be a subset of X. Then $\xi \in X$ is said to be a strong limit point of H if for every $t > 0$,*

$$\mathcal{N}_\xi(t) \cap (H \setminus \{\xi\}) \neq \emptyset.$$

The set of all strong limit points of the set H is denoted by $L_H^{\mathcal{F}}$.

Definition 2.12. *[6] Let $(X, \mathcal{F}, \tau)$ be a PM space and H be a subset of X. Let $\mathfrak{Q}$ be a family of strongly open subsets of X such that $\mathfrak{Q}$ covers H. Then $\mathfrak{Q}$ is said to be a strong open cover for H.*

Definition 2.13. *[6] Let $(X, \mathcal{F}, \tau)$ be a PM space and H be a subset of X. Then H is called a strongly compact set if every strong open cover of H has a finite subcover.*

Definition 2.14. *[6] Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X. Then x is said to be strongly bounded if there exists a strongly compact subset E of X such that $x_k \in E$, $\forall k \in \mathbb{N}$.*

Definition 2.15. *[6] Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X. Then x is said to be strongly statistically bounded if there exists a strongly compact subset E of X such that $d(\{k \in \mathbb{N} : x_k \notin E\}) = 0$.*

Theorem 2.16. *[6] Let $(X, \mathcal{F}, \tau)$ be a PM space and H be a strongly compact subset of X. Then every strongly closed subset of H is strongly compact.*

Definition 2.17. *Let $(X, \mathcal{F}, \tau)$ be a PM space. Then for any $r > 0$, the subset $\mathfrak{V}(r)$ of $X \times X$ given by*

$$\mathfrak{V}(r) = \{(x, y) : \mathcal{F}_{xy}(r) > 1 - r\}$$

is called the strong r-vicinity.

Theorem 2.18. *Let $(X, \mathcal{F}, \tau)$ be a PM space and τ be continuous. Then for any $r > 0$, there is an $\eta > 0$ such that $\mathfrak{V}(\eta) \circ \mathfrak{V}(\eta) \subset \mathfrak{V}(r)$, where $\mathfrak{V}(\eta) \circ \mathfrak{V}(\eta) = \{(x, z) :$ for some y, (x, y) and $(y, z) \in \mathfrak{V}(\eta)\}$.*

Note 1. *From the hypothesis of Theorem 2.18, we can say that for any $r > 0$, there is an $\eta > 0$ such that $\mathcal{F}_{ab}(r) > 1 - r$ whenever $\mathcal{F}_{ac}(\eta) > 1 - \eta$ and $\mathcal{F}_{cb}(\eta) > 1 - \eta$. Equivalently it can be written as: for any $r > 0$, there is an $\eta > 0$ such that $d_L(\mathcal{F}_{ab}, \epsilon_0) < r$ whenever $d_L(\mathcal{F}_{ac}, \epsilon_0) < \eta$ and $d_L(\mathcal{F}_{cb}, \epsilon_0) < \eta$.*

Definition 2.19. *[23] Let $(X, \mathcal{F}, \tau)$ be a PM space. A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X is said to be strongly convergent to $\mathcal{L} \in X$ if for every $t > 0$, $\exists$ a natural number k_0 such that*

$$x_k \in \mathcal{N}_{\mathcal{L}}(t), \qquad \text{whenever } k \geq k_0.$$

In this case, we write $\mathcal{F}\text{-}\lim_{k\to\infty} x_k = \mathcal{L}$ or $x_k \xrightarrow{\mathcal{F}} \mathcal{L}$.

Definition 2.20. *[22] Let $(X, \mathcal{F}, \tau)$ be a PM space. A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X is said to be strong Cauchy if for every $t > 0$, $\exists$ a natural number k_0 such that*

$$(x_j, x_k) \in \mathfrak{U}(t), \qquad \text{whenever } j, k \geq k_0.$$

Definition 2.21. *[23] Let $(X, \mathcal{F}, \tau)$ be a PM space. A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X is said to be strongly statistically convergent to $\xi \in X$ if for any $t > 0$*

$$d(\{k \in \mathbb{N} : \mathcal{F}_{x_k\xi}(t) \leq 1 - t\}) = 0 \text{ or } d(\{k \in \mathbb{N} : x_k \notin \mathcal{N}_\xi(t)\}) = 0.$$

In this case we write $st^{\mathcal{F}}\text{-}\lim_{k\to\infty} x_k = \xi$.

Definition 2.22. *[23] Let $(X, \mathcal{F}, \tau)$ be a PM space. A sequence $x = \{x_k\}_{k\in\mathbb{N}}$ in X is said to be strong statistically Cauchy if for any $t > 0$, $\exists$ a natural number $N_0 = N_0(t)$ such that*

$$d(\{k \in \mathbb{N} : \mathcal{F}_{x_k x_{N_0}}(t) \leq 1 - t\}) = 0.$$

3. Strong λ-Statistical Convergence and Strong λ-Statistical Cauchyness

In this section we first study some basic properties of strong λ-statistical convergence in a PM space and then introducing the notion of strong λ-statistical Cauchyness we study its relationship with strong λ-statistical convergence.

Definition 3.1. *[4] Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. Then x is said to be strongly λ-statistically convergent to $\mathcal{L} \in X$, if for every $t > 0$,*

$$d_\lambda(\{j \in \mathbb{N} : \mathcal{F}_{x_j\mathcal{L}}(t) \leq 1 - t\}) = 0.$$

or

$$d_\lambda(\{j \in \mathbb{N} : x_j \notin \mathcal{N}_{\mathcal{L}}(t)\}) = 0.$$

In this case we write, $st_\lambda^{\mathcal{F}}\text{-}\lim_{k\to\infty} x_k = \mathcal{L}$ or simply as $x_k \xrightarrow{st_\lambda^{\mathcal{F}}} \mathcal{L}$ and $\mathcal{L}$ is called strong λ-statistical limit of x.

Remark 3.2. *From Theorem 2.6 and Definition 3.1 we see that the following statements are equivalent:*

1. $x_k \xrightarrow{st_\lambda^{\mathcal{F}}} \mathcal{L}$
2. *For each* $t > 0, d_\lambda(\{k \in \mathbb{N} : d_L(\mathcal{F}_{x_k\mathcal{L}}, \epsilon_0) \geq t\}) = 0$
3. $st_\lambda^{\mathcal{F}}\text{-}\lim\limits_{k\to\infty} d_L(\mathcal{F}_{x_k\mathcal{L}}, \epsilon_0) = 0.$

Theorem 3.3. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. If the sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is strongly λ-statistically convergent in X, then the strong λ-statistical limit of x is unique.*

Proof. Let the sequence $x = \{x_k\}_{k\in\mathbb{N}}$ be strongly λ-statistically convergent in X. If possible, let $st_\lambda^{\mathcal{F}}\text{-}\lim\limits_{k\to\infty} x_k = \xi_1$ and $st_\lambda^{\mathcal{F}}\text{-}\lim\limits_{k\to\infty} x_k = \xi_2$ with $\xi_1 \neq \xi_2$. Since $\xi_1 \neq \xi_2$, $\mathcal{F}_{\xi_1\xi_2} \neq \epsilon_0$. Then there is a $t > 0$ such that $d_L(\mathcal{F}_{\xi_1\xi_2}, \epsilon_0) = t$. We choose $\gamma > 0$ so that $d_L(\mathcal{F}_{uv}, \epsilon_0) < \gamma$ and $d_L(\mathcal{F}_{vw}, \epsilon_0) < \gamma$ imply that $d_L(\mathcal{F}_{uw}, \epsilon_0) < t$. Since $st_\lambda^{\mathcal{F}}\text{-}\lim\limits_{k\to\infty} x_k = \xi_1$ and $st_\lambda^{\mathcal{F}}\text{-}\lim\limits_{k\to\infty} x_k = \xi_2$, so $d_\lambda(A_1(\gamma)) = 0$ and $d_\lambda(A_2(\gamma)) = 0$, where

$$A_1(\gamma) = \{k \in \mathbb{N} : \mathcal{F}_{x_k\xi_1}(\gamma) \leq 1 - \gamma\}$$

and

$$A_2(\gamma) = \{k \in \mathbb{N} : \mathcal{F}_{x_k\xi_2}(\gamma) \leq 1 - \gamma\}.$$

Now let $A_3(\gamma) = A_1(\gamma) \cup A_2(\gamma)$. Then $d_\lambda(A_3(\gamma)) = 0$ and this gives $d_\lambda(A_3^c(\gamma)) = 1$. Let $k \in A_3^c(\gamma)$. Then $d_L(\mathcal{F}_{x_k\xi_1}, \epsilon_0) < \gamma$ and $d_L(\mathcal{F}_{\xi_2 x_k}, \epsilon_0) < \gamma$ and so $d_L(\mathcal{F}_{\xi_1\xi_2}, \epsilon_0) < t$, this gives a contradiction. Hence strong λ-statistical limit of a strongly λ-statistically convergent sequence in a PM space is unique. □

Theorem 3.4. *Let $(X, \mathcal{F}, \tau)$ be a PM space and $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ be two sequences in X such that $x_n \xrightarrow{st_\lambda^{\mathcal{F}}} l \in X$ and $y_n \xrightarrow{st_\lambda^{\mathcal{F}}} m \in X$. Then*

$$st_\lambda^{\mathcal{F}}\text{-}\lim_{n\to\infty} d_L(\mathcal{F}_{x_ny_n}, \mathcal{F}_{lm}) = 0.$$

Proof. Since τ is continuous and X is endowed with the strong topology, so $\mathcal{F}$ is uniformly continuous. So for any $t > 0$ there exists $\eta(t) > 0$ such that $d_L(\mathcal{F}_{lm}, \mathcal{F}_{l_1m_1}) < t$, whenever $l_1 \in \mathcal{N}_l(\eta)$ and $m_1 \in \mathcal{N}_m(\eta)$. Then by the given condition, for any $t > 0$

$$\{n \in \mathbb{N} : d_L(\mathcal{F}_{x_ny_n}, \mathcal{F}_{lm}) \geq t\} \subset \{n \in \mathbb{N} : x_n \notin \mathcal{N}_l(\eta)\} \cup \{n \in \mathbb{N} : y_n \notin \mathcal{N}_m(\eta)\}.$$

This gives

$$\{n \in I_k : d_L(\mathcal{F}_{x_ny_n}, \mathcal{F}_{lm}) \geq t\} \subset \{n \in I_k : x_n \notin \mathcal{N}_l(\eta)\} \cup \{n \in I_k : y_n \notin \mathcal{N}_m(\eta)\}.$$

Thus,

$$\begin{aligned} & d_\lambda(\{n \in \mathbb{N} : d_L(\mathcal{F}_{x_ny_n}, \mathcal{F}_{lm}) \geq t\}) \\ \leq \quad & d_\lambda(\{n \in \mathbb{N} : x_n \notin \mathcal{N}_l(\eta)\} \cup \{n \in \mathbb{N} : y_n \notin \mathcal{N}_m(\eta)\}). \end{aligned}$$

As $x_n \xrightarrow{st_\lambda^{\mathcal{F}}} l$ and $y_n \xrightarrow{st_\lambda^{\mathcal{F}}} m$, so right hand side of the above inequality is zero and so

$$d_\lambda(\{n \in \mathbb{N} : d_L(\mathcal{F}_{x_ny_n}, \mathcal{F}_{lm}) \geq t\}) = 0.$$

Hence $st_\lambda^{\mathcal{F}}\text{-}\lim\limits_{n\to\infty} d_L(\mathcal{F}_{x_ny_n}, \mathcal{F}_{lm}) = 0.$ □

Theorem 3.5. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. If the sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is strongly convergent to $\mathcal{L} \in X$, then $st_\lambda^{\mathcal{F}}\text{-}\lim\limits_{k\to\infty} x_k = \mathcal{L}$.*

Proof. Let the sequence $x = \{x_k\}_{k\in\mathbb{N}}$ be strongly convergent to $\mathcal{L}$. So, for $t > 0$, there is a natural number N_0 such that $\mathcal{F}_{x_k\mathcal{L}}(t) > 1 - t$ for all $k \geq N_0$. Thus $d_\lambda(\{k \in \mathbb{N} : \mathcal{F}_{x_k\mathcal{L}}(t) \leq 1 - t\}) = 0$ and so $st_\lambda^{\mathcal{F}}\text{-}\lim_{k\to\infty} x_k = \mathcal{L}$. □

Theorem 3.6. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. Then $st_\lambda^{\mathcal{F}}\text{-}\lim_{k\to\infty} x_k = \mathcal{L}$ if and only if there is a subset $G = \{q_1 < q_2 < ...\}$ of $\mathbb{N}$ such that $d_\lambda(G) = 1$ and $\mathcal{F}\text{-}\lim_{n\to\infty} x_{q_n} = \mathcal{L}$.*

Proof. Let us assume that $st_\lambda^{\mathcal{F}}\text{-}\lim_{k\to\infty} x_k = \mathcal{L}$. Then for each $t \in \mathbb{N}$, let

$$E_t = \{q \in \mathbb{N} : d_L(\mathcal{F}_{x_q\mathcal{L}}, \epsilon_0) \geq \frac{1}{t}\}$$

and

$$G_t = \{q \in \mathbb{N} : d_L(\mathcal{F}_{x_q\mathcal{L}}, \epsilon_0) < \frac{1}{t}\}.$$

Then from Remark 3.2, we get $d_\lambda(E_t) = 0$. Also by construction of G_t for each $t \in \mathbb{N}$ we have $G_1 \supset G_2 \supset G_3 \supset ... \supset G_m \supset G_{m+1} \supset ...$ with $d_\lambda(G_t) = 1$ for each $t \in \mathbb{N}$.

Let $u_1 \in G_1$. As $d_\lambda(G_2) = 1$, so $\exists\ u_2 \in G_2$ with $u_2 > u_1$ such that for each $n \geq u_2$, $\frac{|G_2(n)|}{\lambda_n} > \frac{1}{2}$ where $G_t(n) = \{k \in I_n : k \in G_t\}$ and $|G_t(n)|$ is the number of element in the set $G_t(n)$ for each $t \in \mathbb{N}$.

Again, as $d_\lambda(G_3) = 1$, so $\exists\ u_3 \in G_3$ with $u_3 > u_2$ such that for each $n \geq u_3$, $\frac{|G_3(n)|}{\lambda_n} > \frac{2}{3}$.

Thus we set a strictly increasing sequence $\{u_t\}_{t\in\mathbb{N}}$ of positive integers such that $u_t \in G_t$ for each $t \in \mathbb{N}$ and

$$\frac{|G_t(n)|}{\lambda_n} > \frac{t-1}{t} \qquad \text{for each } n \geq u_t, t \in \mathbb{N}.$$

We now define the set G as follows

$$G = \Big\{k \in \mathbb{N} : k \in [1, u_1]\Big\} \bigcup \Big\{\bigcup_{t\in\mathbb{N}} \{k \in \mathbb{N} : k \in [u_t, u_{t+1}] \text{ and } k \in G_t\}\Big\}.$$

Then, for each n, $u_t \leq n < u_{t+1}$, we have

$$\frac{|G(n)|}{\lambda_n} \geq \frac{|G_t(n)|}{\lambda_n} > \frac{t-1}{t}.$$

Therefore $d_\lambda(G) = 1$.

Let $\eta > 0$. We choose $l \in \mathbb{N}$ such that $\frac{1}{l} < \eta$. Let $n \geq u_l$, $n \in G$. Then $\exists$ a natural number $r \geq l$ such that $u_r \leq n < u_{r+1}$. Then by the construction of G, $n \in G_r$. So,

$$d_L(\mathcal{F}_{x_n\mathcal{L}}, \epsilon_0) < \frac{1}{r} \leq \frac{1}{l} < \eta.$$

Thus $d_L(\mathcal{F}_{x_n\mathcal{L}}, \epsilon_0) < \eta$ for each $n \in G$, $n \geq u_l$. Hence $\mathcal{F}\text{-}\lim_{\substack{k\to\infty \\ k\in G}} x_k = \mathcal{L}$. Writing $G = \{q_1 < q_2 < ...\}$ we have $d_\lambda(G) = 1$ and $\mathcal{F}\text{-}\lim_{n\to\infty} x_{q_n} = \mathcal{L}$.

Conversely, let there exists a subset $G = \{q_1 < q_2 < ...\}$ of $\mathbb{N}$ such that $d_\lambda(G) = 1$ and $\mathcal{F}\text{-}\lim_{n\to\infty} x_{q_n} = \mathcal{L}(\in X)$. Then for each $t > 0$, there is an $N_0 \in \mathbb{N}$ so that

$$\mathcal{F}_{x_{q_n}\mathcal{L}}(t) > 1 - t, \qquad \forall\ n \geq N_0,$$

i.e.,

$$d_L(\mathcal{F}_{x_{q_n}\mathcal{L}}, \epsilon_0) < t, \qquad \forall\ n \geq N_0.$$

Let $r > 0$ be a real number and $E_r = \{n \in \mathbb{N} : d_L(\mathcal{F}_{x_{q_n}\mathcal{L}}, \epsilon_0) \geq r\}$. Then $E_r \subset \mathbb{N} \setminus \{q_{N_0+1}, q_{N_0+2}, ...\}$. Now $d_\lambda(\mathbb{N} \setminus \{q_{N_0+1}, q_{N_0+2}, ...\}) = 0$ and so $d_\lambda(E_r) = 0$. Therefore, $st_\lambda^{\mathcal{F}}\text{-}\lim_{k\to\infty} x_k = \mathcal{L}$. □

Theorem 3.7. *Let $(X,\mathcal{F},\tau)$ be a PM space, $x=\{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda\in\Delta_\infty$. Then $x_k \xrightarrow{st_\lambda^{\mathcal{F}}} \mathcal{L}$ if and only if there exists a sequence $\{g_k\}_{k\in\mathbb{N}}$ such that $x_k=g_k$ for λ-a.a.k. and $g_k \xrightarrow{\mathcal{F}} \mathcal{L}$.*

Proof. Let $x_k \xrightarrow{st_\lambda^{\mathcal{F}}} \mathcal{L}$. Then we have

$$st_\lambda^{\mathcal{F}}\text{-}\lim_{k\to\infty} d_L(\mathcal{F}_{x_k\mathcal{L}},\epsilon_0)=0.$$

So by Theorem 3.6, there is a set $G=\{q_1<q_2<...<q_n<...\}\subset\mathbb{N}$ such that $d_\lambda(G)=1$ and $\mathcal{F}\text{-}\lim_{n\to\infty} d_L(\mathcal{F}_{x_{q_n}\mathcal{L}},\epsilon_0)=0$.

We now define a sequence $\{g_k\}_{k\in\mathbb{N}}$ as follows:

$$g_k=\begin{cases} x_k, & \text{if } k\in G\\ \mathcal{L}, & \text{if } k\notin G.\end{cases}$$

Then clearly, $g_k \xrightarrow{\mathcal{F}} \mathcal{L}$ and $x_k=g_k$ for λ-*a.a.k.*

Conversely, let $x_k=g_k$ for λ-*a.a.k.* and $g_k \xrightarrow{\mathcal{F}} \mathcal{L}$. Let $t>0$. Then for each $n\in\mathbb{N}$, we get

$$\{k\in I_n : x_k\notin\mathcal{N}_\mathcal{L}(t)\}\subset\{k\in I_n : x_k\neq g_k\}\cup\{k\in I_n : g_k\notin\mathcal{N}_\mathcal{L}(t)\}.$$

As $\{g_k\}_{k\in\mathbb{N}}$ is strongly convergent to $\mathcal{L}$, so the set $\{k\in\mathbb{N}: x_k\notin\mathcal{N}_\mathcal{L}(t)\}$ is finite and so $d_\lambda(\{k\in\mathbb{N}: g_k\notin\mathcal{N}_\mathcal{L}(t)\})=0$.
Thus,

$$\begin{aligned}&d_\lambda(\{k\in\mathbb{N}: x_k\notin\mathcal{N}_\mathcal{L}(t)\})\\ &\leq d_\lambda(\{k\in\mathbb{N}: x_k\neq g_k\})+d_\lambda(\{k\in\mathbb{N}: g_k\notin\mathcal{N}_\mathcal{L}(t)\})=0.\end{aligned}$$

Therefore, $d_\lambda(\{k\in\mathbb{N}: x_k\notin\mathcal{N}_\mathcal{L}(t)\})=0$ for each $t>0$ i.e., the sequence $\{x_k\}_{k\in\mathbb{N}}$ is strongly λ-statistically convergent to $\mathcal{L}$. □

Definition 3.8. *Let (X,ρ) be a metric space and $x=\{x_k\}_{k\in\mathbb{N}}$ be a sequence in X. Then x is said to be λ-statistically Cauchy in X if for every $\eta>0$, there exists a natural number N_0 such that*

$$d_\lambda(\{k\in\mathbb{N}:\rho(x_k,x_{N_0})\geq\eta\})=0.$$

Now as a consequence of the proposition 4., of [5], we get the following lemma.

Lemma 3.9. *Let (X,ρ) be a metric space and $x=\{x_k\}_{k\in\mathbb{N}}$ be a sequence in X. Then the following statements are equivalent:*

1. *x is a λ-statistically Cauchy sequence.*
2. *For all $\eta>0$, there is a set $G\subset\mathbb{N}$ such that $d_\lambda(G)=0$ and $\rho(x_m,x_n)<\eta$ for all $m,n\notin G$.*
3. *For every $\eta>0$, $d_\lambda(\{j\in\mathbb{N}: d_\lambda(D_j)\neq 0\})=0$, where $D_j(\eta)=\{k\in\mathbb{N}:\rho(x_k,x_j)\geq\eta\}$, $j\in\mathbb{N}$.*

Proof. (1) $\Rightarrow$ (2): Let (1) hold. Let $\eta>0$. Since x is λ-statistically Cauchy, so there exists $j\in\mathbb{N}$ depending on $\frac{\eta}{2}$ such that $d_\lambda(\{k\in\mathbb{N}:\rho(x_k,x_j)\geq\frac{\eta}{2}\})=0$. Let $G=\{k\in\mathbb{N}:\rho(x_k,x_j)\geq\frac{\eta}{2}\}$. Then $d_\lambda(G)=0$. Now if $p,q\notin G$ then $\rho(x_p,x_j)<\frac{\eta}{2}$ and $\rho(x_q,x_j)<\frac{\eta}{2}$ and so $\rho(x_p,x_q)<\eta$.

(2) $\Rightarrow$ (3):

Let $\eta>0$ and G be chosen according to condition (2). We show that the set $\{j\in\mathbb{N}: d_\lambda(D_j(\eta))\neq 0\}\subset G$, where $D_j(\eta)=\{k\in\mathbb{N}:\rho(x_k,x_j)\geq\eta\}$. Let $P=\{j\in\mathbb{N}: d_\lambda(D_j(\eta))\neq 0\}$. Let $j\in P$. If possible, let $j\notin G$. Then we can choose $q\in D_j(\eta)\setminus G$. Then $\rho(x_q,x_j)\geq\eta$. But $j,q\notin G$ implies $\rho(x_q,x_j)<\eta$ (by (2)), which is a contradiction. So $\{j\in\mathbb{N}: d_\lambda(D_j(\eta))\neq 0\}\subset G$ and since $d_\lambda(G)=0$, so $d_\lambda(\{j\in\mathbb{N}: d_\lambda(D_j(\eta))\neq 0\})=0$.

(3) $\Rightarrow$ (1): Let for every $\eta>0$, $d_\lambda(\{j\in\mathbb{N}: d_\lambda(D_j(\eta))\neq 0\})=0$, where $D_j(\eta)=\{k\in\mathbb{N}:\rho(x_k,x_j)\geq\eta\}$, $j\in\mathbb{N}$. So, there exists $j\in\mathbb{N}$ such that $d_\lambda(D_j(\eta))=0$ and so x is λ-statistically Cauchy. □

Definition 3.10. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. Then x is said to be strong λ-statistically Cauchy sequence if for every $t > 0$, $\exists$ a natural number N_0 depending on t such that*

$$d_\lambda(\{k \in \mathbb{N} : \mathcal{F}_{x_k x_{N_0}}(t) \leq 1 - t\}) = 0.$$

Theorem 3.11. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. If x is strongly λ-statistically convergent, then x is strong λ-statistically Cauchy.*

Proof. Let $x_k \xrightarrow{st_\lambda^{\mathcal{F}}} \mathcal{L}$. Let $t > 0$ be given. Then $\exists$ a real number $\gamma = \gamma(t) > 0$ such that $d_L(\mathcal{F}_{uv}, \epsilon_0) < \gamma$ and $d_L(\mathcal{F}_{vw}, \epsilon_0) < \gamma$ implies $d_L(\mathcal{F}_{uw}, \epsilon_0) < t$. Since $x_k \xrightarrow{st_\lambda^{\mathcal{F}}} \mathcal{L}$, so for the above $\gamma > 0$ we get $d_\lambda(\{k \in \mathbb{N} : x_k \notin \mathcal{N}_{\mathcal{L}}(\gamma)\}) = 0$. Then $d_\lambda(A) = 1$, where $A = \{k \in \mathbb{N} : x_k \in \mathcal{N}_{\mathcal{L}}(\gamma)\}$. We choose $N_0 = N_0(\gamma = \gamma(t)) = N_0(t)$ such that $x_{N_0} \in \mathcal{N}_{\mathcal{L}}(\gamma)$ i.e., $d_L(\mathcal{F}_{x_{N_0}\mathcal{L}}, \epsilon_0) < \gamma$. Let $k \in A$, then $d_L(\mathcal{F}_{x_k\mathcal{L},\epsilon_0}) < \gamma$. Now, $d_L(\mathcal{F}_{x_k\mathcal{L},\epsilon_0}) < \gamma$ and $d_L(\mathcal{F}_{x_{N_0}\mathcal{L}}, \epsilon_0) < \gamma$ implies $d_L(\mathcal{F}_{x_k x_{N_0},\epsilon_0}) < t$.

Therefore,

$$\begin{aligned} & x_k \in \mathcal{N}_{x_{N_0}}(t) \\ \Rightarrow\ & k \in \{j \in \mathbb{N} : x_j \in \mathcal{N}_{x_{N_0}}(t)\} \\ \Rightarrow\ & k \in \{j \in \mathbb{N} : d_L(\mathcal{F}_{x_j x_{N_0}}, \epsilon_0) < t\} \\ \Rightarrow\ & k \in \{j \in \mathbb{N} : \mathcal{F}_{x_j x_{N_0}}(t) > 1 - t\}. \end{aligned}$$

So, we get $A \subset \{j \in \mathbb{N} : \mathcal{F}_{x_j x_{N_0}}(t) > 1 - t\}$.
Hence,

$$\begin{aligned} & d_\lambda(\{j \in \mathbb{N} : \mathcal{F}_{x_j x_{N_0}}(t) > 1 - t\}) = 1 \\ \text{or,}\quad & d_\lambda(\{j \in \mathbb{N} : \mathcal{F}_{x_j x_{N_0}}(t) \leq 1 - t\}) = 0 \end{aligned}$$

So, the given sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is strong λ-statistically Cauchy. □

Theorem 3.12. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. If the sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is strong λ-statistically Cauchy, then for each $t > 0$, there is a set $H_t \subset \mathbb{N}$ with $d_\lambda(H_t) = 0$ such that $\mathcal{F}_{x_k x_j}(t) > 1 - t$ for any $k, j \notin H_t$.*

Proof. Let $x = \{x_k\}_{k\in\mathbb{N}}$ be strong λ-statistically Cauchy. Let $t > 0$. Then by Note 1, there is a $\gamma = \gamma(t) > 0$ such that,

$$\mathcal{F}_{\mathcal{L}r}(t) > 1 - t \text{ whenever } \mathcal{F}_{\mathcal{L}j}(\gamma) > 1 - \gamma \text{ and } \mathcal{F}_{jr}(\gamma) > 1 - \gamma.$$

As the sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is strong λ-statistically Cauchy, so there is an $N_0 = N_0(\gamma) \in \mathbb{N}$ such that

$$d_\lambda(\{k \in \mathbb{N} : \mathcal{F}_{x_k x_{N_0}}(\gamma) \leq 1 - \gamma\}) = 0.$$

Let $H_t = \{k \in \mathbb{N} : \mathcal{F}_{x_k x_{N_0}}(\gamma) \leq 1 - \gamma\}$. Then $d_\lambda(H_t) = 0$ and $\mathcal{F}_{x_k x_{N_0}}(\gamma) > 1 - \gamma$ and $\mathcal{F}_{x_j x_{N_0}}(\gamma) > 1 - \gamma$ for $k, j \notin H_t$. Hence for every $t > 0$, there is a set $H_t \subset \mathbb{N}$ with $d_\lambda(H_t) = 0$ such that $\mathcal{F}_{x_k x_j}(t) > 1 - t$ for every $k, j \notin H_t$. □

Corollary 3.13. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. If the sequence $x = \{x_k\}_{k\in\mathbb{N}}$ is strong λ-statistically Cauchy, then for each $t > 0$, there is a set $G_t \subset \mathbb{N}$ with $d_\lambda(G_t) = 1$ such that $\mathcal{F}_{x_k x_j}(t) > 1 - t$ for any $k, j \in G_t$.*

Theorem 3.14. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$, $g = \{g_k\}_{k\in\mathbb{N}}$ be two strong λ-statistically Cauchy sequences in X and $\lambda \in \Delta_\infty$. Then $\{\mathcal{F}_{x_k g_k}\}_{k\in\mathbb{N}}$ is a λ-statistically Cauchy sequence in $(\mathcal{D}^+, d_L)$.*

Proof. As $x = \{x_k\}_{k\in\mathbb{N}}$ and $g = \{g_k\}_{k\in\mathbb{N}}$ are strong λ-statistically Cauchy sequences, so by corollary 3.13, for every $\gamma > 0$ there are $U_\gamma, V_\gamma \subset \mathbb{N}$ with $d_\lambda(U_\gamma) = d_\lambda(V_\gamma) = 1$ so that $\mathcal{F}_{x_q x_j}(\gamma) > 1-\gamma$ holds for any $q, j \in U_\gamma$ and $\mathcal{F}_{g_s g_t}(\gamma) > 1-\gamma$ holds for any $s, t \in V_\gamma$. Let $W_\gamma = U_\gamma \cap V_\gamma$. Then $d_\lambda(W_\gamma) = 1$. So, for every $\gamma > 0$, there is a set $W_\gamma \subset \mathbb{N}$ with $d_\lambda(W_\gamma) = 1$ so that $\mathcal{F}_{x_p x_r}(\gamma) > 1-\gamma$ and $\mathcal{F}_{g_p g_r}(\gamma) > 1-\gamma$ for any $p, r \in W_\gamma$. Now let $t > 0$. Then there exists a $\gamma(t)$ and hence a set $W_\gamma = W_t \subset \mathbb{N}$ with $d_\lambda(W_t) = 1$ so that $d_L(\mathcal{F}_{x_p g_p}, \mathcal{F}_{x_r g_r}) < t$ for any $p, r \in W_t$, as $\mathcal{F}$ is uniformly continuous. Then the result follows from Lemma 3.9. □

4. Strong λ-Statistical Limit Points and Strong λ-Statistical Cluster Points

In this section we introduce the notions of strong λ-statistical limit points and strong λ-statistical cluster points of a sequence in a PM space and study some of their properties.

Definition 4.1. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. If $\{x_{q_k}\}_{k\in\mathbb{N}}$ is a subsequence of the sequence x and $\mathcal{B} = \{q_k : k \in \mathbb{N}\}$ then we denote $\{x_{q_k}\}_{k\in\mathbb{N}}$ by $\{x\}_\mathcal{B}$. Now if $d_\lambda(\mathcal{B}) = 0$, then $\{x\}_\mathcal{B}$ is called a λ-thin subsequence of x. On the other hand, $\{x\}_\mathcal{B}$ is called a λ-nonthin subsequence of x, if $\mathcal{B}$ does not have λ-density zero i.e., if either $d_\lambda(\mathcal{B})$ is a positive number or $\mathcal{B}$ fails to have λ-density.*

Definition 4.2. *[23] Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. An element $l \in X$ is said to be a strong limit point of the sequence x, if there exists a subsequence of x that strongly converges to l.*

To denote the set of all strong limit points of any sequence x in a PM space $(X, \mathcal{F}, \tau)$ we use the notation $L_x^\mathcal{F}$.

Definition 4.3. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. An element $\mathcal{L} \in X$ is said to be a strong λ-statistical limit point of the sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if there exists a λ-nonthin subsequence of x that strongly converges to $\mathcal{L}$.*

To denote the set of all strong λ-statistical limit points of any sequence x in a PM space $(X, \mathcal{F}, \tau)$ we use the notation $\Lambda_x^{st}(\lambda)_s^\mathcal{F}$.

Definition 4.4. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. An element $\mathcal{Y} \in X$ is said to be a strong λ-statistical cluster point of the sequence $x = \{x_k\}_{k\in\mathbb{N}}$, if for every $t > 0$, the set $\{k \in \mathbb{N} : \mathcal{F}_{x_k\mathcal{Y}}(t) > 1-t\}$ does not have λ-density zero.*

To denote the set of all strong λ-statistical cluster points of any sequence x in a PM space $(X, \mathcal{F}, \tau)$ we use the notation $\Gamma_x^{st}(\lambda)_s^\mathcal{F}$.

Note 2. *If we choose $\lambda_n = n$ for all $n \in \mathbb{N}$, then strong λ-statistical limit points and strong λ-statistical cluster points coincide with strong statistical limit points and strong statistical cluster points of a sequence respectively in a PM space as introduced in [23].*

Theorem 4.5. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. Then $\Lambda_x^{st}(\lambda)_s^\mathcal{F} \subset \Gamma_x^{st}(\lambda)_s^\mathcal{F} \subset L_x^\mathcal{F}$.*

Proof. Let $\xi \in \Lambda_x^{st}(\lambda)_s^\mathcal{F}$. Then we get a subsequence $\{x_{k_n}\}_{n\in\mathbb{N}}$ of the sequence x such that $\mathcal{F}\text{-}\lim_{n\to\infty} x_{k_n} = \xi$ and $d_\lambda(\mathcal{M}) \neq 0$, where $\mathcal{M} = \{k_n \in \mathbb{N} : n \in \mathbb{N}\}$. Suppose $t > 0$ be arbitrary. Since $\mathcal{F}\text{-}\lim_{n\to\infty} x_{k_n} = \xi$, so $\exists\, p_0 \in \mathbb{N}$ such that $\mathcal{F}_{x_{k_n}\xi}(t) > 1-t$ whenever $n \geq p_0$. Let $\mathcal{B} = \{k_1, k_2, ..., k_{p_0-1}\}$. Then,

$$\begin{aligned} &\{k \in \mathbb{N} : \mathcal{F}_{x_k\xi}(t) > 1-t\} \supset \{k_n \in \mathbb{N} : n \in \mathbb{N}\}\backslash\mathcal{B} \\ \Rightarrow\quad &\mathcal{M} = \{k_n \in \mathbb{N} : n \in \mathbb{N}\} \subset \{k : \mathcal{F}_{x_k\xi}(t) > 1-t\} \cup \mathcal{B}. \end{aligned}$$

Now if $d_\lambda(\{k \in \mathbb{N} : \mathcal{F}_{x_{k_n}\xi}(t) > 1-t\}) = 0$, then we get $d_\lambda(\mathcal{M}) = 0$, a contradiction. Hence ξ is a strong λ-statistical cluster point of x. Since $\xi \in \Lambda_x^{st}(\lambda)_s^\mathcal{F}$ is arbitrary, so $\Lambda_x^{st}(\lambda)_s^\mathcal{F} \subset \Gamma_x^{st}(\lambda)_s^\mathcal{F}$.

Now let $\alpha \in \Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$. Then λ-density of the set

$$\{k \in \mathbb{N} : \mathcal{F}_{x_k\alpha}(t) > 1 - t\}$$

is not zero, for every $t > 0$. So there exists a subsequence $\{x\}_{\mathcal{K}}$ of x that strongly converges to α. So, $\alpha \in L_x^{\mathcal{F}}$.

Therefore, $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}} \subset L_x^{\mathcal{F}}$. □

Theorem 4.6. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. If $st_\lambda^{\mathcal{F}}\text{-}\lim_{k\to\infty} x_k = \alpha$, then $\Lambda_x^{st}(\lambda)_s^{\mathcal{F}} = \Gamma_x^{st}(\lambda)_s^{\mathcal{F}} = \{\alpha\}$.*

Proof. Let $st_\lambda^{\mathcal{F}}\text{-}\lim_{k\to\infty} x_k = \alpha$. So for every $t > 0$, $d_\lambda(\{k \in \mathbb{N} : \mathcal{F}_{x_k\alpha}(t) > 1 - t\}) = 1$. Therefore, $\alpha \in \Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$. Now assume that there exists at least one $\beta \in \Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$ such that $\alpha \neq \beta$. Then $\mathcal{F}_{\alpha\beta} \neq \epsilon_0$. Then there is a $t_1 > 0$ such that $d_L(\mathcal{F}_{\alpha\beta}, \epsilon_0) = t_1$. Then there exists $t > 0$ such that $d_L(\mathcal{F}_{uv}, \epsilon_0) < t$ and $d_L(\mathcal{F}_{vw}, \epsilon_0) < t$ imply that $d_L(\mathcal{F}_{uw}, \epsilon_0) < t_1$. Now since $\alpha, \beta \in \Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$, for that $t > 0$, $d_\lambda(G) \neq 0$ and $d_\lambda(H) \neq 0$, where $G = \{k \in \mathbb{N} : \mathcal{F}_{x_k\alpha}(t) > 1 - t\}$ and $H = \{k \in \mathbb{N} : \mathcal{F}_{x_k\beta}(t) > 1 - t\}$. As, $\alpha \neq \beta$, so $G \cap H = \emptyset$ and so $H \subset G^c$. Since $st_\lambda^{\mathcal{F}}\text{-}\lim_{k\to\infty} x_k = \alpha$ so $d_\lambda(G^c) = 0$. Then $d_\lambda(H) = 0$, which is a contradiction.

Therefore, $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}} = \{\alpha\}$.

As $st_\lambda^{\mathcal{F}}\text{-}\lim_{k\to\infty} x_k = \alpha$, so from Theorem 3.7, we have $\alpha \in \Lambda_x^{st}(\lambda)_s^{\mathcal{F}}$. Now by Theorem 4.5, we get $\Lambda_x^{st}(\lambda)_s^{\mathcal{F}} = \Gamma_x^{st}(\lambda)_s^{\mathcal{F}} = \{\alpha\}$. □

Theorem 4.7. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $\lambda \in \Delta_\infty$ and $x = \{x_k\}_{k\in\mathbb{N}}$, $y = \{y_k\}_{k\in\mathbb{N}}$ be two sequences in X such that $d_\lambda(\{k \in \mathbb{N} : x_k \neq y_k\}) = 0$. Then $\Lambda_x^{st}(\lambda)_s^{\mathcal{F}} = \Lambda_y^{st}(\lambda)_s^{\mathcal{F}}$ and $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}} = \Gamma_y^{st}(\lambda)_s^{\mathcal{F}}$.*

Proof. Let $\xi \in \Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$ and $\epsilon > 0$ be given. Let $\mathcal{B} = \{k \in \mathbb{N} : x_k = y_k\}$. Since $d_\lambda(\mathcal{B}) = 1$, so $d_\lambda(\{k \in \mathbb{N} : \mathcal{F}_{x_k\xi}(t) > 1 - t\} \cap \mathcal{B})$ is not zero. This gives $d_\lambda(\{k \in \mathbb{N} : \mathcal{F}_{y_k\xi}(t) > 1 - t\})$ is not zero and so $\xi \in \Gamma_y^{st}(\lambda)_s^{\mathcal{F}}$. Since $\xi \in \Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$ is arbitrary, so $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}} \subset \Gamma_y^{st}(\lambda)_s^{\mathcal{F}}$. Similarly, we get $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}} \supset \Gamma_y^{st}(\lambda)_s^{\mathcal{F}}$. Hence $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}} = \Gamma_y^{st}(\lambda)_s^{\mathcal{F}}$.

Now let $\eta \in \Lambda_y^{st}(\lambda)_s^{\mathcal{F}}$. Then y has a λ-nonthin subsequence $\{y_{k_n}\}_{n\in\mathbb{N}}$ that strongly converges to η. Let $\mathcal{Z} = \{k_n \in \mathbb{N} : y_{k_n} = x_{k_n}\}$. Since $d_\lambda(\{k_n \in \mathbb{N} : y_{k_n} \neq x_{k_n}\}) = 0$ and $\{y_{k_n}\}_{n\in\mathbb{N}}$ is a λ-nonthin subsequence of y so $d_\lambda(\mathcal{Z}) \neq 0$. Using the set $\mathcal{Z}$ we get a λ-nonthin subsequence $\{x\}_{\mathcal{Z}}$ of the sequence x that strongly converges to η. Thus $\eta \in \Lambda_x^{st}(\lambda)_s^{\mathcal{F}}$. Since $\eta \in \Lambda_y^{st}(\lambda)_s^{\mathcal{F}}$ is arbitrary, so $\Lambda_y^{st}(\lambda)_s^{\mathcal{F}} \subset \Lambda_x^{st}(\lambda)_s^{\mathcal{F}}$. By similar argument, we get $\Lambda_x^{st}(\lambda)_s^{\mathcal{F}} \subset \Lambda_y^{st}(\lambda)_s^{\mathcal{F}}$. Hence $\Lambda_x^{st}(\lambda)_s^{\mathcal{F}} = \Lambda_y^{st}(\lambda)_s^{\mathcal{F}}$. □

Theorem 4.8. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. Then the set $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$ is a strongly closed set.*

Proof. To show that $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$ is a strongly closed, let ξ be a strong limit point of the set $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$. Then for every $t > 0$ we have $\mathcal{N}_\xi(t) \cap (\Gamma_x^{st}(\lambda)_s^{\mathcal{F}}) \setminus \{\xi\}) \neq \emptyset$. Let $\beta \in \mathcal{N}_\xi(t) \cap (\Gamma_x^{st}(\lambda)_s^{\mathcal{F}}) \setminus \{\xi\})$. Now we can choose $t_1 > 0$ such that $\mathcal{N}_\beta(t_1) \subset \mathcal{N}_\xi(t)$. Since $\beta \in \Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$ so

$$d_\lambda(\{k \in \mathbb{N} : \mathcal{F}_{x_k\beta}(t_1) > 1 - t_1\}) \neq 0$$

$$\Rightarrow d_\lambda(\{k \in \mathbb{N} : \mathcal{F}_{x_k\xi}(t) > 1 - t\}) \neq 0.$$

Hence $\xi \in \Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$. □

Theorem 4.9. *Let $(X, \mathcal{F}, \tau)$ be a PM space, $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X and $\lambda \in \Delta_\infty$. Let C be a strongly compact subset of X such that $C \cap \Gamma_x^{st}(\lambda)_s^{\mathcal{F}} = \emptyset$. Then $d_\lambda(G) = 0$, where $G = \{k \in \mathbb{N} : x_k \in C\}$.*

Proof. As $C \cap \Gamma_x^{st}(\lambda)_s^{\mathcal{F}} = \emptyset$, so for all $\beta \in C$, there exists a real number $t = t(\beta) > 0$ so that $d_\lambda(\{k \in \mathbb{N} : \mathcal{F}_{x_k \beta}(t) > 1 - t\}) = 0$. Let $B_\beta(t) = \{a \in X : \mathcal{F}_{a\beta}(t) > 1 - t\}$. Then the family of strongly open sets $\mathfrak{Q} = \{B_\beta(t) : \beta \in C\}$ forms a strong open cover of C. As C is a strongly compact set, so there exists a finite subcover $\{B_{\beta_1}(t_1), B_{\beta_2}(t_2), ..., B_{\beta_m}(t_m)\}$ of the strong open cover $\mathfrak{Q}$. Then $C \subset \bigcup_{j=1}^{m} B_{\beta_j}(t_j)$ and also for each $j = 1, 2, ..., m$ we have $d_\lambda(\{k \in \mathbb{N} : \mathcal{F}_{x_k \beta_j}(t_j) > 1 - t_j\}) = 0$. So,

$$|\{k \in \mathbb{N} : x_k \in C\}| \leq \sum_{j=1}^{m} \left|\{k \in \mathbb{N} : \mathcal{F}_{x_k \beta_j}(t_j) > 1 - t_j\}\right|.$$

Then,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : x_k \in C\}| \leq \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{j=1}^{m} \left|\{k \in I_n : \mathcal{F}_{x_k \beta_j}(t_j) > 1 - t_j\}\right| = 0.$$

This gives $d_\lambda(G) = 0$, where $G = \{k \in \mathbb{N} : x_k \in C\}$. □

Theorem 4.10. *Let $(X, \mathcal{F}, \tau)$ be a PM space and $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X. If x has a strongly bounded λ-nonthin subsequence, then the set $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$ is nonempty and strongly closed.*

Proof. Let $\{x\}_{\mathcal{B}}$ be a strongly bounded λ-nonthin subsequence of x. So $d_\lambda(\mathcal{B}) \neq 0$ and there exists a strongly compact subset C of X such that $x_k \in C$ for all $k \in \mathcal{B}$. If $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}} = \emptyset$ then $C \cap \Gamma_x^{st}(\lambda)_s^{\mathcal{F}} = \emptyset$ and then by Theorem 4.9, we get $d_\lambda(G) = 0$, where $G = \{k \in \mathbb{N} : x_k \in C\}$. But $|\{k \in I_n : k \in \mathcal{B}\}| \leq |\{k \in I_n : x_k \in C\}|$, which gives $d_\lambda(\mathcal{B}) = 0$, which contradicts our assumption. Hence $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$ is nonempty and also by Theorem 4.8, $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$ is strongly closed. □

Definition 4.11. *Let $(X, \mathcal{F}, \tau)$ be a PM space and $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X. Then x is said to be strongly λ-statistically bounded if there exists a strongly compact subset C of X such that $d_\lambda(\{k \in \mathbb{N} : x_k \notin C\}) = 0$.*

Theorem 4.12. *Let $(X, \mathcal{F}, \tau)$ be a PM space and $x = \{x_k\}_{k\in\mathbb{N}}$ be a sequence in X. If x is strongly λ-statistically bounded then the set $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$ is nonempty and strongly compact.*

Proof. Let x be strongly λ-statistically bounded. Let C be a strongly compact set with $d_\lambda(E) = 0$, where $E = \{k \in \mathbb{N} : x_k \notin C\}$. Then $d_\lambda(E^c) = 1 \neq 0$ and so C contains a λ- nonthin subsequence of x. So, by Theorem 4.10, $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$ is nonempty and strongly closed. We now prove that $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$ is strongly compact. For this we only show that $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}} \subset C$. If possible, let $\eta \in \Gamma_x^{st}(\lambda)_s^{\mathcal{F}} \setminus C$. As C is strongly compact so there is a $q > 0$ such that $\mathcal{N}_\eta(q) \cap C = \emptyset$. So we get $\{k \in \mathbb{N} : \mathcal{F}_{x_k \eta}(q) > 1 - q\} \subset \{k \in \mathbb{N} : x_k \notin C\}$ which implies that $d_\lambda(\{k \in \mathbb{N} : \mathcal{F}_{x_k \eta}(q) > 1 - q\}) = 0$, which contradicts that $\eta \in \Gamma_x^{st}(\lambda)_s^{\mathcal{F}}$. So, $\Gamma_x^{st}(\lambda)_s^{\mathcal{F}} \subset C$. □

Acknowledgments

The second author is grateful to Council of Scientific and Industrial Research, India for his fellowships funding under CSIR-JRF scheme during the preparation of this paper.

References

1. Connor, J., Fridy, J., Kline, J., *Statistically pre-Cauchy Sequences*, Analysis 14, 311-317, (1994).

2. Connor, J., *R-type summability methods, Cauchy criteria, P-sets and Statistical convergence*, Proc. Amer. Math. Soc. 115, 319-327, (1992).

3. Connor, J., *The statistical and strong P-Cesaro convergence of sequences*, Analysis 8, 47-63, (1988).

4. Das, P., Dutta, K., Karakaya, V., Ghosal, S., *On some further generalizations of strong convergence in probabilistic metric spaces using ideals*, Abstract and App. Anal. DOI: 10.1155/2013/765060, (2013).

5. Dems, K., *On $\mathcal{I}$-Cauchy sequences*, Real Analysis Exchange 30(1), 123-128, (2004).

6. Dutta, K., Malik, P., Maity, M., *Statistical Convergence of Double Sequences in Probabilistic Metric Spaces*, Selçuk J. Appl. Math. 14(1), 57-70, (2013).

7. Fast, H., *Sur la convergence statistique*, Colloq. Math 2, 241-244, (1951).

8. Fridy, J. A., *On statistical convergence*, Analysis 5, 301-313, (1985).

9. Fridy, J. A., *Statistical limit points*, Proc. Amer. Math. Soc. 118(4), 1187-1192, (1993).

10. Fridy, J. A., Orhan, C., *Statistical limit superior and limit inferior*, Proc. Amer. Math. Soc. 125, 3625-3631, (1997).

11. Karakus, S., *Statistical convergence on probabilistic normed spaces*, Math. Commun. 12, 11-23, (2007).

12. Menger, K., *Statistical metrics*, Proc. Nat. Acad. Sci. USA 28, 535-537, (1942).

13. Mursaleen, M., *λ-statistical convergence*, Mathematica Slovaca 50(1), 111-115, (2000).

14. Pehlivan, S., Güncan, A., Mamedov, M. A., *Statistical cluster points of sequences in finite dimensional spaces*, Czechoslovak Mathematical Journal 54(129), 95-102, (2004).

15. Sálat, T., *On statistically convergent sequences of real numbers*, Math. Slovaca 30, 139-150, (1980).

16. Savas, E., Das, P., *A generalized statistical convergence via ideals*, Appl. Math. Lett. 24, 826-830, (2011).

17. Schoenberg, I. J., *The integrability of certain functions and related summability methods*, Amer. Math. Monthly 66, 361-375, (1959).

18. Schweizer, B., Sklar, A., *Statistical metric spaces*, Pacific J. Math. 10, 314-334, (1960).

19. Schweizer, B., Sklar, A., Thorp, E., *The metrization of statistical metric spaces*, Pacific J. Math. 10, 673-675, (1960).

20. Schweizer, B., Sklar, A., *Statistical metric spaces arising from sets of random variables in Euclidean n-space*, Theory of probability and its Applications 7, 447-456, (1962).

21. Schweizer, B., Sklar, A., *Tringle inequalities in a class of statistical metric spaces*, J. London Math. Soc. 38, 401-406,(1963).

22. Schweizer, B., Sklar, A., *Probabilistic Metric Spaces*, North Holland: New York, Amsterdam, Oxford (1983).

23. Şençimen, C., Pehlivan, S., *Strong statistical convergence in probabilistic metric space*, Stoch. Anal. Appl. 26, 651-664, (2008).

24. Steinhus, H., *Sur la convergence ordinatre et la convergence asymptotique*, Colloq. Math. 2, 73-74, (1951).

25. Tardiff, R. M., *Topologies for Probabilistic Metric spaces*, Pacific J. Math. 65, 233-251, (1976).

26. Thorp, E., *Generalized topologies for statistical metric spaces*, Fundamenta Mathematicae 51, 9-21, (1962).

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